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The University Tutorial Series

THE TUTORIAL ARITHMETIC

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PREFACE.

The present treatise on Arithmetic, though written so as to be as far as possible complete in itself, is intended primarily for those who have already received some grounding in the subject. For this reason the order of treatment is rather the logical order of development than that which a beginner would usually follow. For instance, the Four Rules and Fractions are treated first with Abstract then with Concrete Numbers, while in actual teaching there can be no doubt that these should be intermingled. So, again, the Unitary method is dealt with under Ratio and Proportion, but most teachers would in practice prefer to take it shortly after Compound Division. The sections on Vulgar and Decimal Fractions are so written that either may be commenced before the other.

Unusual prominence is given to questions of Theory. Formal completeness is perhaps beyond the reach of an English author whose soul is vexed perforce with Weights and Measures and the barbarous mechanism by which they are manipulated, who is condemned also by the tyranny of an examination system to waste the space at his disposal in elaboration of the details of Complex Fractions, Abstract and Concrete, and other subjects of similar inelegance and uselessness. Moreover, further to militate against completeness, there exists an extraordinary prejudice against the use of literal symbols which are supposed to be the inalienable property of Algebra. The consequence is that

while nobody doubts that the work in Ex. 2, p. 89, for instance, is Arithmetic, yet it becomes Algebra in the eyes of many if we replace "cost of clock" by *c*, *i.e.* use one letter instead of eleven to denote the quantity in question. The absurdity of this is manifest. A really scientific treatise on Arithmetic is impossible so long as this divorce between Arithmetic and Algebra continues. I have endeavoured to bring about a better understanding between the two, as may be seen in Chapter XXXV., and elsewhere. Other novelties will be found in § 29, in the sections on Factorisation (§§ 89-93), on Multiplication of Fractions (§§ 115-118), and on recurring Decimals (ch. xvii), in the treatment of Compound Proportion (§ 181), in the explanation of Shares before Stocks, and throughout Section X.

The Examples, which are as numerous as in most text-books, have been in part borrowed from the papers set at the various public examinations, especially those for the Civil Service, a few have been contributed by friends, but probably about half the questions are original. Section XI. will be found to contain, I believe, the hardest collection of arithmetical problems in existence. It will serve as a playground for those who imagine that they have nothing more to learn. The examples (except in Section XI.) have been for the most part reworked from the proof-sheets, and it is hoped that not many errors remain undetected. Both publishers and authors would be glad to receive intimations of any such, or suggestions for the improvement of the book.

Examples for mental work have not, usually, been added, not because I undervalue their importance but because I believe every teacher prefers to construct his own. An asterisk is used generally to denote a question of unusual

difficulty which is best deferred to a second reading of the book. Questions so marked are not suited for class work, and the teacher should beware of attempting them before a class without some previous preparation. This remark applies particularly to Section XI. An asterisk is occasionally used with other significance which is explained when it occurs.

There remains the pleasant task of expressing my indebtedness to my colleague, Mr. R. H. Chope, without whose co-operation the completion of this work, under the pressure of other duties, would have been impossible. Mr. Chope has contributed Sections VII. and VIII. (except Chapter XXVIII.), Chapters VIII. to XI., and other paragraphs here and there. In addition to this each of us has read and revised the work of the other, and Mr. Chope's skill as a teacher has repeatedly left its trace. My thanks are due also to Mr. R. W. D. Christie for permission to use the brilliant problems which he has contributed to the *Educational Times* and a large collection of others which he placed at my disposal, and also to Mr. W. Watson for considerable help in the verification of the Examples. If I name no treatises here it is because there are few of recent date from which I have not gathered help.

W. P. W.

February, 1902.

NOTE TO THE SECOND EDITION.

In the present edition the mistakes of the first edition have been, it is hoped, for the most part corrected, the chapter on Approximations has been enlarged and revised, and a section has been added on Alligation, and numerous smaller changes have been made.

July 1902.

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SECTION I.

THE FOUR RULES.

CHAPTER I.

NUMERATION AND NOTATION.

1. To **count** a group of objects is to find how many objects there are in the group.

A **number** expresses the result of counting, and is written either in words or in shorthand symbols, which take the place of words in the writing of numbers, and are called **figures,** or **numerals.***

A number may be **concrete*** when applied to definite objects, or **abstract*** when thought of apart from any object.

Ex.—If we speak of *two shillings*, or *three pence*, the numbers *two* and *three* are *concrete*, because they are thought of in connexion with particular objects, shillings and pence. If the coins are together, and we speak of *five coins*, the number *five* is again concrete, because thought of in connexion with coins. But, if we write down the word *five*, and completely forget to what objects it referred, the number has become *abstract* to us. It expresses simply that there were five. The same word would be used whether the objects were coins or cats.

Numeration is the art of expressing a number in words.

Notation is the art of expressing a number in figures.

Arithmetic (Gk. ἀριθμός, "number") is the science which treats of the relations of numbers to one another.

---

* No attempt is made to obtain philosophic completeness in these definitions. *Number* is really an elementary concept not much easier to define than *time*. Some writers hold that *number* is always abstract, and that the phrase "concrete number" is a contradiction for which they would substitute "concrete expression." Again, it is argued that number *must* be associated with definite objects, and that the contradiction therefore occurs in the phrase "abstract number."

2. The shorthand symbols employed for the representation of numbers by the Greeks were the letters of their alphabet; by the Romans certain signs which gradually came to look like letters of the alphabet, but were, perhaps, originally pictures of the hand. The Greek system is never used now, and we shall not describe it. The Roman shorthand was used almost everywhere in Europe until the time of Elizabeth, and is still frequently seen, particularly in dates and in the headings of chapters of books. We shall explain it later (§ 8). It has been displaced by a system which the Arabs derived from the Hindus, and brought into Europe at some time before A.D. 1200. The symbols used in this **Arabic system** of shorthand are, in order,

1, 2, 3, 4, 5, 6, 7, 8, 9,

together with another symbol 0, which expresses that there are none of the objects spoken of.

Ex.—Thus, if we lay 3 pennies down and take 2 up again, we say that there is 1 left, if we take up 1 more, we say that there is 0 left.

This last symbol is called *naught*, *nought*, *cipher*, or *zero*.

It should never be called *aught*, or *ought*. The spelling *naught* is perhaps slightly better than *nought*, but both are admissible. The spelling *cypher* is wrong. *Cipher* and *zero* are from the same root, an Arabic word meaning "empty" The invention of this symbol is the greatest step ever taken in the history of Arithmetic, and one of the greatest steps in the history of human thought. So important is it that it gave its name in German and French to every other figure, and in English to the science which was for long called the *art of ciphering*. We still say *to cipher* when we mean "to reckon." The author of this great invention is unknown.

The symbol 1 is called *one*, or *unity*, or *a unit*.

A number is often described as made up of so many units: thus 7 is made up of seven units.

Counting was originally done by everybody on the fingers, as it still is by savages and little children, and hence the shorthand signs which represent the numbers which the fingers can give are called **digits** (Lat. *digitus*, "finger").

N.B.—This name was originally, and is still occasionally confined to the signs 1, 2, ... 9. But it is usually understood to include 0. Ten is *not* a digit.

3. Since we have ten fingers, counting is done by tens. A

system of counting by tens is called a **decimal system** (Lat. *decem*, "ten").

But other systems are possible. There is considerable evidence of nations having once counted by twenties, *i.e.* having used their toes as well as their hands. Thus the Romans expressed "eighteen" as "two from twenty" (*duodeviginti*. The form *octodecim* is late). The French express "eighty" as "four twenties" (*quatrevingts*), and "ninety" as "four-twenty ten" (*quatrevingtdix*). There is a trace of it in the English *score*, in such phrases as "three score and ten." (Cajori.) We shall study other systems in Chap. xxxiv.

The English words for numbers show that our system of counting is a decimal system. We have:—

(*eleven**), (*twelve**), *thirteen* (*i.e.* three-ten), *fourteen* (*i.e.* four-ten), ... till we get to *nineteen* (*i.e.* nine-ten).

twenty, *i.e.* twain-ten, or two tens, the ending -ty being an old Teutonic form for "ten," which still survives in the Danish *ti* for ten.

twenty-one, *i.e.* two-tens-one, &c.

thirty, *i.e.* three-tens, &c.

We shall now explain the two most important shorthand systems for writing numbers down.

I. The Arabic System.

4. Take a slate, or a piece of paper, and rule two lines down it so as to divide it into three columns. Write at the head of these columns the words "hundreds," "tens," "units." Then we can express any small number, less than a thousand, on our slate, by writing in the first column how many complete hundreds it has in it; then in the second column how many tens are left; then in the third column how many units are left. Thus we should write:—

Hundreds.	Tens.	Units.
	1	7
	7	1
1		7
1	7	
7	1	
7	2	6

Seventeen, *i.e.* seven-ten as 1 ten 7 units.

Seventy-one, *i.e.* seven-tens-one as 7 tens 1 unit.

One hundred and seven as 1 hundred 7 units, the tens' place being left blank.

* The words "eleven" and "twelve" are not really exceptions, but are differently formed. The O.H.G. forms show this distinctly, *einlif*, *zwelif*, where we have the numerals *ein* ("one"), *zwei* ("two") combined with a suffix *-lif*, meaning "ten," so that the forms are exactly parallel to the others, and mean "one-ten," "two-ten." *Cf.* Lat. *undecim*, *duodecim*.

One hundred and seventy, *i.e.* 1 hundred and 7 tens, as 1 hundred 7 tens, the units being left blank.

Seven hundred and ten, *seven hundred and twenty-six*, as shown.

So far the Roman schoolboy did just the same on his calculating-slate, which he called an *abacus*, except that, instead of writing the figure 7, he put seven pebbles in its place.

Hence the word *calculate* (Lat. *calculus*, "pebble"). As we shall see later, the Roman schoolboy would head his columns with the letters C, X, I.

5. The next great step was taken by the Hindus. The Romans failed to discover it. The Hindus explained it to the Arabs, and the Arabs brought the secret into Europe. It was seen that you might omit the ruling on the slate if you wrote your numbers very carefully. But there was clearly a great chance of confusing the 17 of the first line with the 1 7 of the third line, and even more with the 17 of the fourth line. It was not then safe to omit the ruling unless we had some means of telling when a column was blank. Hence the sign 0 was invented to tell when there is no other figure in the column for that number, and the numbers were written as shown, so that there was no longer any chance of confusion. The figure at the right of a number now tells you how many units there are, the next to it how many tens, and the third figure, at the left, how many hundreds.

```
17
71
1 7
17
71
726

17
71
107
170
710
726
```

It is not necessary (but not incorrect) to write 017, because 17 itself tells you that there are no hundreds. It *is* necessary to write 107 for 1 7, because without the 0 there is so much chance of mistake when the number is written or read quickly.

Clearly, too, 7 means something very different when it stands in the first column of our slate from what it means when it is in the second, or in the third, *i.e.* 7 in 710, 7 in 170, and 7 in 107 are very different things. In the first case 7 means 7 hundreds, in the second 7 tens, in the third 7 units. Thus a figure has now a **local value,** and changes its meaning when you change its place. It has also a **digit value,** in virtue of which 7 always means 7 somethings, whether units or thousands.

The right-hand column of the slate, or the right-hand place

in a number, is called the **units' place.** The figure that stands in it indicates how many units there are. The next column, or place, going towards the left, is called the **tens' place;** the next the **hundreds' place.**

6. To write a 0 at the right (*i.e.* the *end*) of a number, *i.e.* to make 17 into 170 tells you to leave the units' place blank, to count that 7 which was in the units' place in the tens' place, and the 1 which was in the tens' place in the hundreds' place.

We may sum up our results in the following important statements:—

To write a 0 at the beginning (left) of a number makes no difference to it.
To write a 0 anywhere in the middle of a number changes it completely.
To write a 0 at the end changes units into tens, tens into hundreds, &c.

7. We can now extend this system to the writing of much larger numbers. Suppose the slate ruled not into three, but into six, columns. The right-hand column would still be units; the next (to the left) tens; the next tens-of-tens, *i.e.* hundreds, as before; the next, the first new column, tens-of-hundreds, *i.e.* thousands; the next tens-of-thousands; the next hundreds-of-thousands.

Thus 234576 would be—

6 units,	*i.e.* six.
7 tens,	*i.e.* seven-ty,
5 hundreds,	*i.e.* five hundred,
4 thousands,	*i.e.* four thousand,
3 tens-of-thousands,	*i.e.* thir-ty thousand,
2 hundreds-of-thousands,	*i.e.* two hundred thousand:

and, since in English we name all numbers (except 13, 14, . . . 19) with the greater part named first, we read it "two hundred and thirty-four thousand five hundred and seventy-six."

So 400705 would be "four hundred thousand seven hundred and five."

For higher numbers still we need other words. A seventh column on our slate would be used for "tens-of-hundred thousands," *i.e.* "thousands-of-thousands." A thousand thou-

sand is called a million and written 1000000. A million million is called a billion.* Billions would be entered in the thirteenth place (counting from the right), so that one billion is written 1000000000000. Beyond these we have the terms, not often used, of—

trillion,* *i.e.* million billion, or 1000000000000000000;

quadrillion,* *i.e.* billion billion, &c.

It will be noted that the names used occur in threes:—units, tens, hundreds; thousands, tens-of-thousands, hundreds-of-thousands; millions. It is convenient, therefore, to mark the figures off into groups of threes, and we often write, not 1000000, but 1,000,000. Sometimes, with very large numbers, we mark them off into groups of six, thus 1,000000. Remember that this marking into groups *must* be made from right to left, and not from left to right.

We will now give some examples of the reading of high numbers. You will do well, at first, to use a divided slate or paper, and write at the head of it the following:—

Thousands-of-millions.
Hundreds } -of-millions.
Tens
Units
Hundreds } -of-thousands.
Tens
Units
Hundreds }
Tens
Units

Ex.—1,734,567 is one million, seven hundred and thirty-four thousand, five hundred and sixty-seven.
100,004,001 is one hundred million, four thousand, and one.
1,001,109,091,100 is one billion, one thousand one hundred and nine million, ninety-one thousand, one hundred.

Note that this system of notation is a decimal system, *i.e.* the local value (see § 5) becomes ten times as great when we

* These terms have a different meaning in France and the United States, where 1 billion is 1000000000, 1 trillion is 1000000000000, &c. Some English books have followed the French usage. The meaning given in the text is the original one.

move a figure one place from right to left. We have, therefore, both a decimal system of numeration (see § 3) and a decimal system of notation.

EXAMPLES I.

Write in words the numbers— **1.** 137, 317, 731, 713, 703.

2. 9876, 9067, 6009, 6900, 6970. **3.** 12345, 21987, 37981, 41897, 61003.

4. 786543, 865432, 657321, 500703, 900001.

5. In 1897 the number of elementary schools in England was 19958, the number of children on the registers was 5507039, and the average attendance was 4488543. Rewrite this statement in words.

6. In 1896 the number of elementary schools in England was 19848, the number of children on the registers was 5422989, and the average attendance was 4422911. Rewrite this statement in words.

7. The total area of the British Isles is 121115 square miles, and the population in 1891 was 37740283, an increase of 2855435 in ten years, and 604182 had emigrated. Write these numbers in words.

8. The entire population of England and Wales in 1891 was 29002525. 836913 poor were receiving relief in 1898, as against 836674 in 1897. Write these numbers in words.

Write in words— **9.** 123456789, 987654321, 100200304, 100000203.

10. 321654987, 789456123, 400300201, 300000102.

11. 175632987976001. **12.** 716523899700067.

13. Write in figures one million two hundred and thirty-five thousand and three.

14. Write in figures three million ten thousand three hundred and five.

15. Write in figures seven billion two hundred and sixty-seven thousand, and thirty six million one thousand and seven.

16. Write in figures nine billion twenty seven thousand six hundred million and one.

17. What is the greatest number which can be formed with the figures 1, 3, 9, 0? What is the least?

18. What is the greatest number which can be formed with the figures 2, 3, 0, 5? What is the least?

19. What is the greatest number which begins with 9 and ends with 0 and has five figures altogether? What is the least?

20. What is the least number of six figures which begins with 2 and ends with 3? What is the greatest?

II. The Roman System.

8. The symbols employed in the Roman system of number-shorthand are as follows:—

I for 1,	M for 1000,	L for 50,
X „ 10,	V „ 5,	D „ 500.
C „ 100,		

The rules, by which the Romans managed to express all small numbers by means of these symbols, are as follows:—

(A) When a symbol is repeated the number is repeated also.

Thus II means (*not* 11) but "one and one," *i.e.* 2. So XX is 20, CC is 200, III is 3, CCC is 300.

There is no need for VV (10), LL (100), DD (1000), and they are never used.

A symbol is not usually repeated more than three times. IIII is allowable for 4, but is unusual. Such forms are better written by (C) below.

(B) When a symbol is placed *to the right* of a greater symbol the values of the two are added.

Thus in VI the I is to the right of V and is to be added to it, so that VI means 6. So XII is 12, XXIII is 23, CCXII is 212, MDCCCXXIII is 1823.

(C) When a *single* symbol from the first column is placed *to the left* of a greater symbol its value is to be subtracted from that of the greater.

Thus in IV the I is to the left of V and is to be subtracted from it, so that IV means "one less than V," *i.e.* 4. So XC is 90, XCIX is 99.

Not more than one symbol is ever to be subtracted from a greater symbol. Thus IIV is not a correct form for "two less than 5," or 3.

(D) A horizontal line over a symbol increases its value a thousand times.

Thus $\overline{\text{V}}$ means 5000. But this device is rarely used.

Ex.—(1) A Roman writes the first ten numbers thus:—

I, II, III, IV (IIII), V, VI, VII, VIII, IX (VIIII), X.*

* The teacher will notice how the Roman arrangement I ... V, VI ... X, XI ... XV, XVI ... contains traces of the use of a *quinary scale* of notation (Cajori). Also that while the Roman system of numeration is a decimal system, that of notation is not. Also that in the Roman system the figures have only a limited "local value," according as they stand to the right or left of other figures. It is not correct to say that they have *no* local value.

After these ten numbers he arranges the others as far as possible on the same model :—

XI, XII, XIII, XIV, XV.

Thus he writes XIX in preference to IXX.

The tens are :—

X, XX, XXX, XL (or XXXX), L, LX, LXX, LXXX, XC, C, &c.

For the reason just named he writes XCIX for 99, rather than IC.

Following the analogy of XIX, the correct sign for 1900 is MCM rather than MDCCCC. The latter form, however, is the one employed in inscriptions.

(2) CDXXXIV is 400 (CD), 30 (XXX), and 4 (IV), *i.e.* 434; CMXLIX is 949 ; MDCCCXCIX is 1899.

EXAMPLES II.

Write in Roman figures—

1. 34, 39, 79, 97, 110.

2. 43, 78, 85, 93, 101.

3. 256, 512, 1024, 2048, 4096.

4. 162, 324, 648, 1296, 2592.

Write in Arabic figures—

5. CCII, CCCIX, DCVII, MDCXIX, MCCCVIII.

6. CCXX, CCCIV, CDVIII, MCCIV, MDCCXVIII.

7. MDCCCXXXVII, MMDCCXXII, MMMDCCCXXXIII.

8. MCCXV, MMDCXCIX, MMDCXLVIII.

9. What is the largest number which can be formed with the symbols X, C, I? What is the smallest? Write also the largest and smallest numbers which can be formed with the symbols X, X, C, I.

10. What is the largest number which can be formed with the symbols V, X, L? What is the smallest? Write also the largest and smallest numbers which can be formed with the symbols X, L, I, V.

CHAPTER II.

ADDITION

9. **Addition** is the process of finding a single number which is just as large as two or more numbers put together.

Such an operation can, of course, be applied only to numbers of the same kind. It is nonsense to talk of adding £5 to 10 feet.

Such a number is called the **sum, total, sum-total,** or **amount** of the given numbers.

The **sign of addition, +** (**plus**), is used to denote the operation. Thus $5+7$ means 7 added to 5.

The **sign of equality, =** (**"equals"**), is used to denote the result of an operation. Thus $5+7=12$ means that the result of adding 5 and 7 is 12. It is read: 5 plus 7 equals 12.

$\therefore$ means "therefore," and is used to draw attention to the steps of an argument.

Addition of Units.

10. You must first be able to say at once, *without a moment's hesitation*, what is the sum of any two digits. Thus $5+7=12$.

You should then practise the following:—

Count by 2's: 2, 4, 6, 8, 10, 12, ...

Count by 3's: 3, 6, 9, 12, 15, ...

&c.

You must continue this until you can count by 9's without hesitation and as fast as you can say the numbers. In this rule, speed is of the utmost importance.

You should then practise the following, and make up others of the same kind for yourself:—

$$5+3+9+1+8+7+6+2+2=$$
$$1+4+2+8+5+7+3+1+3=$$
$$4+7+7+1+2+1+3+9+8=$$
$$6+3+1+1+3+9+3+2+4=$$

You must not rest satisfied until you can do four such sums in between 15 and 20 seconds after writing them.

You must accustom yourself to adding the figures horizontally, *i.e.* straight across, as well as vertically, *i.e.* straight up or down.

Hints.—(1) *Never* work thus: "5 and 3 are 8, 8 and 9 are 17, 17 and 1 are 18, ..."; (2) Nor thus, though this is better: 5 and 3, 8, and 9, 17, and 1, 18, ..."

(3) The *only* figures to be used mentally are: "5, 8, 17, 18, 26, 33, 39, 41, 43." Of these, the first may be omitted, but is useful to many.

(4) Form the habit of casting the eye ahead and picking out the pairs or triplets which make 10. Thus, in the first sum, the eye notes that $9 + 1 = 10$ and $6 + 2 + 2 = 10$. The figures used now are "5, 8, 18, 26, 33, 43." In the second they would be "1, 5, 15, 20, 30, 34."

Addition of Larger Numbers.

11. The most convenient way of adding numbers is to place them under one another, taking care that the units are under the units, the tens under the tens, and so on. Thus, to add 395, 1234, 62, and 876, we write:—

	Thousands.	Hundreds.	Tens.	Units.
3 9 5 *i.e.*		3	9	5
1 2 3 4	1	2	3	4
6 2			6	2
8 7 6		8	7	6
	1	13	25	17
	1	3	5	7
	1	2	1	
Ans.	2	5	6	7

Explanation.—The 17 units mean 1 ten and 7 units. Put down 7 and add the 1 to the tens. So the 25 tens in the second writing of the sum means 2 hundreds and 5 tens, and are so written in the third. In practice, this is conducted as follows (See Hint 3 above):—

Units: 6, 8, 12, 17. Write 7 and "carry" 1.

Tens: 1 (carried), 8, 14, 17, 26. Write 6 and carry 2.

Hundreds: 2 (carried), 10, 12, 15. Write 5 and carry 1.

Thousands: 1, 2. *Ans.* 2567.

Rapid Addition.

12. **Hints.**—(1) Extend the practice of counting by 2's, &c., thus:

Count by 13's: 1, 14, 27, 40, ...
2, 15, 28, 41, ...
Count by 17's: 1, 18, 35, 52, ...

and so on until you can count by 50's or even higher numbers.*

To test your work, count on until you have a number above 200, then divide by the number you are dealing with, 13 or 17, &c. The remainder is the number you began with.

This must be done until you can do it *without a moment's hesitation.*

(2) When you are successful in this, it is easy to add two columns at once. Thus the mental work in the sum given (§ 11) would run:—

First two columns: 76, 138, 172, 267. Write 67 and carry 2.
Second two columns: 2, 10, 22, 25. *Ans.* 2567.

(3) Practise such as the following, and make others for yourself:—

$$94 + 16 + 77 + 83 + 12 + 45 + 51 =$$

Four such questions ought to take about 30 to 40 seconds.

Tests of Accuracy.

13. Test all your work by one of the following methods:—

(1) Adding not only from bottom to top (or right to left), but also from top to bottom (or left to right).

(2) Proving by *Nines.* (See § 28.)

EXAMPLES III.†

The following examples may be read either from top to bottom or from left to right. The thicker numbers refer to the Answers, which, however, should not be written on the printed page itself. Exs. 17, 34, 49, 62, and 75 are, of course, much longer than the others.

1.	**2.**	**3.**	**4.**	**5.**	**6.**	**7.**	**8.**	
380	629	877	1125	1373	1621	1869	2117 = ...	**9.**
2861	3109	3357	3605	4101	4349	4845	5093 = ...	**10.**
7819	8067	543	5492	59	3508	4008	7100 = ...	**11.**
7278	3822	7358	151	4000	1840	5242	4013 = ...	**12.**
6874	1570	1042	2403	1153	6991	796	8654 = ...	**13.**
5803	2001	397	981	1307	1230	1500	5926 = ...	**14.**
956	837	69	3967	98	117	89	799 = ...	**15.**
7897	7311	6312	8716	724	765	1922	103 = ...	**16.**
.....	.....	.....	.....	.....	.....	.....	..... = ...	**17.**

* If this subject is taught in class, this Exercise is at once valuable and interesting taken *viva voce.*

† For footnote, see next page.

18.	19.	20.	21.	22.	23.	24.	25.		
8530	3400	719	1505	99	1212	27	72	=	**26.**
3380	17	9	297	2416	9779	3602	3206	=	**27.**
956	2139	5842	6004	3527	4896	8000	800	=	**28.**
8224	4258	6574	5368	4638	653	575	5057	=	**29.**
645	708	7653	1430	5749	3472	4283	2438	=	**30.**
3208	1404	3976	111	1011	8641	3646	6364	=	**31.**
7897	3727	699	2209	234	5090	1289	2198	=	**32.**
3931	9354	9110	88	9852	1977	952	9025	=	**33.**
.....	.....	.....	.....	.....	.....	.....	.....	=	**34.**

35.	36.	37.	38.	39.	40.	41.		
60067	71037	61824	62990	11168	61811	69847	=	**42.**
88345	234	4040	189	49	72928	6003	=	**43.**
123	59650	82679	66643	51762	734	48271	=	**44.**
576	87511	18392	14400	25338	8693	19354	=	**45.**
12345	40030	17600	4879	43579	54927	672	=	**46.**
31	61004	5743	981	6954	15571	87410	=	**47.**
735	5899	29927	28706	40793	4180	49	=	**48.**
.....	.....	.....	.....	.....	.....	.....	=	**49.**

50.	51.	52.	53.	54.	55.		
689042	687082	192028	27	135	176543	=	**56.**
123456	249	511329	922003	820456	300602	=	**57.**
27	38675	763817	753962	200	985673	=	**58.**
3458	841214	689101	666547	59677	40127	=	**59.**
14359	158492	143028	859100	968438	7009	=	**60.**
987654	99886	795412	1177	411253	769245	=	**61.**
.....	.....	.....	.....	.....	.....	=	**62.**

63.	64.	65.	66.		
18137445	97531357	10214365	21765431	=	**67.**
1009	28910	1604	54006729	=	**68.**
12567231	19724865	45296583	70653	=	**69.**
99999989	43152736	706	88110792	=	**70.**
14567	5610	72805679	478	=	**71.**
723456	60790042	87394563	33627547	=	**72.**
1253788	54908639	92548617	464872	=	**73.**
71625344	91058	6001055	95743526	=	**74.**
.....	.....	.....	.....	=	**75.**

† In order to practise his class in addition, the teacher can readily construct addition sums of any length, to which he himself can write the answers at once, by the following method:—Write down, underneath one another, say four numbers of seven figures each. Write next the "complements" (§ 81) of all these numbers *except one, i.e.* the numbers which when added to them turn them into the same number of nines. In the example given the complements are lettered alike. Then the sum can be written at once from the number whose complement is *not* written.

7465332	*a*
1234567	*b*
8001213	*c*
9764135	
1998786	*c*
2534667	*a*
8765432	*b*
39764132	

CHAPTER III.

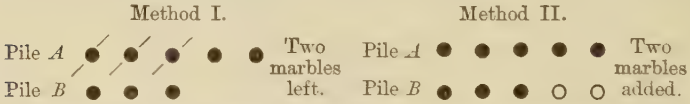
SUBTRACTION.

14. If we had two piles of marbles, A and B, and wished to know how many more there were in one pile (A) than in the other (B), we might proceed in two ways :—

(1) We might withdraw (*subtract*, from the Latin *subtraho*, "withdraw") marbles from A until we had taken away as many from A as there were in B. The numbers of marbles left (*remainder*) would be the number required.

(2) We might add marbles to B until the pile was the same size as A. The number of marbles added would clearly be the *difference* in number of the two piles.

These two methods must lead to the same result. If this is not clear, the following diagram will make it so :—



Either of these two methods might be employed in Arithmetic. Until recently Method I. has always been used, and the terms *subtract*, subtract *from*, *remainder*, are all appropriate to Method I. But Method II. is much easier to learn, for it shows that subtraction is only another way of looking at addition, and it alone will be explained here. If you have learnt Method I., you are earnestly advised to give it up. Rapid calculation is only possible when Method II. is used.

15. **Subtraction** is the process of finding the **difference** of two numbers, *i.e.*, of finding that number which must be added to the smaller to make the larger. Thus, to subtract 262 from 785 is to find the difference of 262 and 785, *i.e.*, to find that number (namely, 523) which must be added to 262 to make 785. This process is sometimes called **complementary addition.** The difference of the two numbers is necessarily also the **remainder** when the smaller is taken away from the larger.

The number which has to be subtracted, the smaller number, is sometimes called the **subtrahend,** and the number from which it is to be subtracted, the larger number which is to be di*min*ished, is called the **minuend.** These terms are not much used.

The **sign of subtraction,** – **(minus),** is used to denote the operation.

Thus $785-262=523$ is read 785 minus 262 equals 523, and means: (1) That 523 must be added to 262 to make 785. The *difference* is 523.

Or (2) That, if 262 be taken from 785, 523 is left. The *remainder* is 523.

Subtraction of Units.

16. You must be able to fill up the gaps in the following *without a moment's hesitation.*

3 and	= 10,	5 and	= 9,	9 and	= 17,
7 and	= 11,	2 and	= 7,	1 and	= 10,
6 +	= 11,	9 +	= 12,	7 +	= 13.

Ans. 7, 4, 8, 4, 5, 9, 5, 3, 6.

You should make other examples of this kind for yourself until you are absolutely perfect in them.

Subtraction of Larger Numbers.

17. Suppose, now, you are required to subtract 2345 from 4796. Write these numbers as we did in addition:—

Thousands.	Hundreds.	Tens.	Units.	
4	7	9	6	
2	3	4	5	
2	4	5	1	*Ans.*

Explanation.—We have to make the second line up to the first. The second line is short in units, short in tens, short in hundreds, and short in thousands. We have to find how much short.

We reckon thus:—

5 *and one* make 6, 1 unit is wanting
4 *and five* ,, 9, 5 tens are wanting.
3 *and four* ,, 7, 4 hundreds ,,
2 *and two* ,, 4, 2 thousands ,,

Altogether 2451 is wanting, and this is the *difference* or answer required.

The actual work is "five and *one*" a slight mental stress on the word *one* which is now written down), "four and *five*, three and *four*, two and *two*." No other words are necessary.

EXAMPLES IV.

1.	2.	3.	4.	5.	6.	7.	8.	9.
4796	3456	9876	6543	2345	1234	4231	4444	9999
1234	1234	6543	4443	1213	112	2211	1233	2345

10.	11.	12.	13.	14.	15.
142857	758241	3141592	9988776	87654321	123456789876
30103	142130	2031451	1234575	76432100	12121212215

SUBTRACTION: GENERAL CASE.

18. If we have to find the difference between, say, 18 and 42, a difficulty arises. Arranging it as before, we see that the lower line is not short in units, but has actually more than the upper. This difficulty is met by a method sometimes called the **method of equal additions,** the principle of which is that the difference between two quantities is not altered if the same quantity be added to each.

42
18
—

Thus, in the illustration used on page 14, the difference of the numbers of marbles in the two heaps is unaltered if 5 marbles be added to each heap.

Thus, again, the difference of the ages of two boys is always the same, although their ages are constantly increasing.

If, therefore, we add 10 units to the first line, and 1 ten (*i.e.*, another 10 units) to the second, the effect is

that

Tens.	Units.
4	2
1	8

becomes

Tens.	Units.
4	12
1+1 = 2	8

,

and the difference is 24—4 units and 2 tens, for the units are 4 short and the tens are 2 short.

Mental work.—8 and *four*, 12; 1 and 1 (from the *12*), 2, and *two*, 4. A slight stress is laid on the words in italics, and the figures thus marked are written down at once.

The same method applies to every case. When the tens are in excess in the smaller number, we add 10 tens to the greater, and 1 hundred (*i.e.*, another 10 tens) to the smaller, and so on. Thus, to find the difference of 12345 and 4748—

Ten Thousands.	Thousands.	Hundreds.	Tens.	Units.
1	2	3	4	5
	4	7	4	8

becomes

Ten Thousands.	Thousands.	Hundreds.	Tens.	Units.
1	12	13	14	15
1	5	8	5	8
	7	5	9	7

and the actual work is:

```
12345
 4748
-----
 7597
```

Mental work is as follows: "8 and *seven*, fifteen; 4 and 1, 5, and *nine*, fourteen; 7 and 1, 8, and *five*, thirteen; 4 and 1, 5, and *seven*, twelve; 1 and *nought*, 1." As the stress falls on the words *seven*, *nine*, *five*, *seven*, we write the corresponding figures down in the answer.

This process of adding the 1 is sometimes spoken of as *carrying*. The 1 is often spoken of as *borrowed*. The latter term should not be used. It arises from another way of looking at the subject.

19. **Test** your work by the following method:

Add the difference to the smaller number. The result should be the larger number.

Hint.—In the very frequent operation of finding the difference between any number and a larger number which ends in a series of zeros much time will be saved if both numbers be mentally lessened by 1 before commencing. Thus, work

```
10000000        9999999
 7891234   as    7891233
```

The sum should not actually be rewritten. This change should be made mentally.

EXAMPLES V.

1.	2.	3.	4.	5.
9445	9317	91234	96206	234567
5255	3787	70793	70793	145628

6.	7.	8.	9.	10.
345678	1000000	1000000	10101010	123454321
256782	789123	897321	2639465	67898765

In Exx. 11–22 you should endeavour to do the subtraction without arranging one number underneath the other.

11. 1602996 less 1590438.
12. 3702814 less 2917632.
13. 42964714 less 35773996.
14. 90269445 less 76438062.
15. 893443925 less 99999999.
16. 123456789 less 88888888.
17. 9518230 minus 6666666.
18. 5918320 minus 4444444.
19. 8710362958 minus 7108623985.
20. 7634280624 minus 6342806247.
21. 7690079561 − 5563025314.
22. 6350062000 − 5364432888.

Fill up the missing lines in the following addition sums, Exx. 23–26. These must not be rearranged.

23.	**24.**	**25.**	**26.**
1602996	6029691	5918320	7612345
.....	.....	.....	.....
8973214	7068763	10000000	10000000

27. The sum of two numbers is 8970, and one of them is 1234: what is the other?

28. The sum of two numbers is 9708, and one of them is 1394: what is the other?

29. Add the sum of 1234, 2345, 3456 to the difference of 4567 and 7654.

30. Add the sum of 2345, 3456, 4561 to the difference of 5647 and 7564.

Combined Addition and Subtraction.

20. It is easy to add together any number of quantities and to subtract their sum from any greater number in one operation. An example will make the method clear. Suppose it is required to add together 78976, 98723, 10204 and to subtract the sum from 1234567.

1234567
———
78976
98723
10204
———
1046664 *Ans.*

Explanation.—We work thus;

4, 7, 13 and *four*, seventeen. Carry 1.
1, 3, 10 and *six*, sixteen. Carry 1.
3, 10, 19 and *six*, twenty-five. Carry 2.
2, 10, 18 and *six*, twenty-four. Carry 2.
3, 12, 19 and *four*, twenty-three. Carry 2.
2 and *nought*, two. Carry 0.
0 and *one*, one.

The *italicised* numbers are written as the stress of the voice falls on them.

Test.—The answer, added to all the subtrahends, should give the minuend.

It should be clear also that this answer gives the result of subtracting the subtrahends separately from the minuend. *i.e.* of subtracting 78976 from 1234567, then subtracting 98723 from the result, then again subtracting 10204 from the result. The question may clearly therefore be written

$$1234567-78976-98723-10204.$$

The ability to perform these operations together will save a great deal of time in "casting up" accounts, as will be seen later. You should practise Examples VI. until you can do them rapidly and without hesitation.

Hint.—It will be a great help to accustom yourself to subtract with the figures in any position without troubling to re-write the less underneath the greater. Thus, it is as easy to work

$$\begin{array}{r} 12345 \\ 786321 \\ \hline \end{array} \quad \text{as} \quad \begin{array}{r} 786321 \\ 12345 \\ \hline \end{array}$$

and it is not much harder to work 786321 − 12345.

EXAMPLES VI.

In Exx. 1–6 the second and following numbers are to be added together and subtracted from the first.

1.	**2.**	**3.**	**4.**	**5.**	**6.**
1575	5175	6018	8016	459146	549641
23	32	1641	4161	109212	102921
345	435	1129	1291	44751	47415
99	96	23	32	23245	24325
126	129	345	354	17	71

In Exx. 7–10 you are given an incomplete addition sum. One line is missing and you are required to find it.

7.	**8.**	**9.**	**10.**
16411711	11118971	111189	189111
23442414	43244124	714324	741342
30463818	40638381	412440	421404
.....	.....	638	836
		.....	.....
87654321	123456789	2345678	2436587

In Exx. 11–16 the questions should not be re-arranged.

11. Subtract the sum of 1147, 4117, 4711 from 75812.

12. Subtract the sum of 1247, 1151, 1274 from 25617.

13. 78613 − 384 − 1123 − 13579.

14. 87631 − 348 − 1321 − 15397.

15. 4871213 − 3010300 − 142857.

16. 4782131 − 3030010 − 124578.

CHAPTER IV.

MULTIPLICATION.

21. **Multiplication** is the process of finding the sum of a given number of repetitions of the same number.

Thus multiplication is simply another form of addition. All multiplication sums might, indeed, be done by addition; but, if we were required to find the sum of, say, 4800 repetitions of the number 12345, our addition sum, printed as those on p. 13 are printed, would be about 15 yards long, and the labour of doing it would be enormous, and the chance of a mistake very great indeed. Most calculating machines perform their multiplication, however, by addition.

The number which is to be repeated, or multiplied, is the **multiplicand.**

The number which expresses how often the multiplicand is repeated is the **multiplier.**

The sum of the repetitions is the **product.**

The multiplier and multiplicand are **factors** of the product.

The **sign of multiplication** is $\times$, which is read "multiplied by" or "times."

Thus $6 \times 7 = 42$ is read "seven times six $\substack{\text{equals}\\\text{is}}$ 42, or "six multiplied by 7 $\substack{\text{equals}\\\text{is}}$ 42."

The . is also used as the sign of multiplication, but is not recommended at first. Thus we may write 6 . 7. *Be careful not to write* 6·7, which means something very different. (See Decimals, Chap. XVI.)

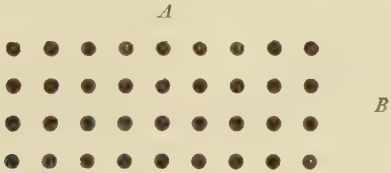
Laws of Arithmetic.

22. All Arithmetic is founded upon three laws, which should be stated here:—

Law I. The Commutative Law.—*Additions and multiplications may be performed in any order, i.e.,* $9+4=4+9$, and $9\times4=4\times9$, or, more generally, if a and b stand for any numbers whatever, $a+b=b+a$ and $a\times b=b\times a$.

Explanation.—(1) As applied to **addition :** We are required to find the total number of units in two groups, one of which contains 9 (or a), the other 4 (or b) units. We may write the 9 units (or a units) as dots running from left to right, and then continue the line with 4 units (or b units), also shown by dots. The total number of dots is $9+4$ (or $a+b$), if we count from left to right, and $4+9$ (or $b+a$) if we count from right to left. These must, therefore, be the same.

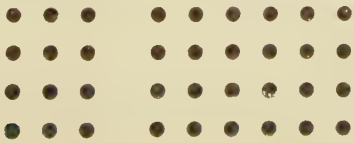
(2) As applied to **multiplication** :



Suppose that a squad of boys is being drilled in the playground. If you stand at A, with all the boys facing you, the squad is four nines, or 9×4. If the command is given, "Right Turn," and you stand at B, the squad becomes nine fours, or 4×9. But the number in the squad is just the same. Therefore $4\times9=9\times4$. This argument holds however large the squad is, and we express it generally by saying that $a\times b=b\times a$.

Law II. The Distributive Law.—*If a number is to be multiplied by the sum of two numbers, the same result is obtained if the multiplicand be multiplied by the numbers separately and the products added.*

Suppose that when you are standing at B in the illustration last employed, the command is given, "First six ranks one pace forward." The squad is now split into two squads :—



The squad nearest you has 6 fours, the other has 3 fours. Thus $4 \times 9 = 4 \times 6 + 4 \times 3$. If only the first four ranks had moved, you would have $4 \times 9 = 4 \times 4 + 4 \times 5$; if only the first two ranks, $4 \times 9 = 4 \times 2 + 4 \times 7$. In fact, generally, if a, b, and c be any numbers whatever,

c multiplied by the sum of $a + b = c \times a + c \times b$,
= also sum of $a + b$ multiplied by c [Law I.].

This is the Distributive Law.

Law III. The Factor Law. — *If a number is to be multiplied by a number which is the product of two factors, the same result is obtained if the multiplicand is multiplied by one of the factors and this product then multiplied by the other factor.*

Suppose, again, that the squad is being drilled in the playground and has got into the second position shown. The command is given, "First three ranks one pace forward." The squad is now split into three squads:

B

and the 4×9 boys have become three groups of 4×3 each; in other words, $4 \times 9 = 4 \times 3 \times 3$.

So, generally, $a \times$ the product of $bc = a \times b \times c$.

You should convince yourself by similar diagrams of the truth of the following :—

$$96 = 12 \times 8 = 4 \times 3 \times 8 = 3 \times 4 \times 8 = 3 \times 2 \times 16 = 6 \times 4 \times 4 = 2 \times 3 \times 2 \times 8,$$

and generally* that $a \times b \times c = a \times c \times b = b \times c \times a$.

23. When you were doing Addition (p. 10) you were recommended to "count by 2's," "count by 3's," &c. If you put all these results down one underneath another, you obtain what is called the **Multiplication Table,** which must now be thoroughly learnt.

---

* The teacher must carefully note that these "proofs" are valid only for whole numbers. It is a very serious mistake, only too commonly committed, to suppose that illustrations of this kind *prove* the laws universally.

THE MULTIPLICATION TABLE.

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

In counting by threes, for instance, when you have counted three numbers you have 3 threes, and the Table shows you that this is 9; so 9 eights is 72, and to find the product of any two numbers up to 12, you find one of the numbers among the black figures at the head standing at the top of a column, the other among the black figures at the left standing at the left of a row, and their product is found where this column and this row meet.

The sum of any number of 0's is 0, therefore 0 multiplied by any number is 0.

Note.—The Table must be learnt first by Addition, as suggested in § 10. But you must know it so thoroughly that you have not to pause a moment before giving the product of any two numbers named. Then practise yourself in giving the factors when the number is named. For instance, write down 36, then write all the

ways in which 36 can be found in the Table, 3×12, 4×9, 6×6. Then write down a long row of any figures whatever and practise yourself in the Table by giving as rapidly as ever you can the product of each figure by the one that follows it.

MULTIPLICATION (Easy Cases).

24. Suppose, now, we have to multiply 235 by 7.

The 235 is 2 hundreds, 3 tens, and 5 units.

$\therefore$ 235×7 is 2×7 hundreds, 3×7 tens, and 5×7 units [Law II.],

i.e. 14 hundreds, 21 tens, 35 units,

i.e. 1 thousand, $4+2 = 6$ hundreds, $1+3 = 4$ tens, 5 units,

i.e. 1645.

This may be arranged thus—

Thousands.	Hundreds.	Tens.	Units.
	2	3	5 7
	14	21	35

i.e.

Thousands.	Hundreds.	Tens.	Units.
	2	3	5 7
1	2 4	3 1	5
1	6	4	5

i.e.

$$\begin{array}{r} 235 \\ 7 \\ \hline 1645 \end{array}$$

Mental Work.—"7 fives, thirty-*five*." Write the **5** down and carry the 3 on. "7 threes, 21, and 3, twenty-*four*." Write the **4** down and carry the 2. "7 twos, 14, and 2, *sixteen*." Write the **16** down.

If the same process is applied to a multiplication by 10, it will be seen easily that:—

To multiply by 10 is to change units into tens, leaving the units place blank; to change tens into ten-tens, *i.e.* hundreds,

and so on. This results in the following rule:—*To multiply by* 10, *place a zero at the right.*

Similarly, to multiply by 100, *place two zeros at the right; for to multiply by* 100 *changes the units into hundreds, the tens into thousands, and so on.*

EXAMPLES VII.*

1. 13579 multiplied by 2, 3, 5, 7, 9, 11.
2. 31597 multiplied by 2, 3, 5, 7, 9, 11.
3. 345678 multiplied by 4, 5, 6, 7, 9, 12.
4. 436587 multiplied by 4, 5, 6, 7, 9, 12.
5. 9898987 multiplied by 3, 4, 7, 8, 10, 11.
6. 8989896 multiplied by 3, 4, 7, 8, 10, 11.
7. 98798797 multiplied by 2, 3, 5, 6, 7, 12.
8. 78978979 multiplied by 2, 3, 5, 6, 7, 12.

Multiplication by Factors.

25 By the Factor Law explained in § 22, it is clear that, if we have to multiply a number by 56, which is the product of 7 and 8, we may obtain the result either by multiplying first by 7 and then by 8, or by multiplying first by 8, and then by 7. Here are three multiplication sums worked in this way:—

14857×56	14857×56	41857×54
7	8	6
103999	118856	251142
8	7	9
831992 *Ans.*	831992 *Ans.*	2260278 *Ans.*

A particular case of this method is of great importance. Suppose we had to multiply a number by 40. We can write this as 10×4. But we have seen that to multiply by 10 changes units into tens, tens into hundreds, &c., and is effected by simply placing a zero at the end. Hence, *to multiply by* 40, *multiply by* 4, *and place a zero at the right.*

* These may all be done mentally by an average class if the multiplicand be written up in *large figures* on the blackboard. Otherwise the difficulty of retaining the multiplicand is very serious at first. The pupil must be allowed to write his answer as he gets it, but should not be allowed to correct that answer in mental work.

Similarly, *to multiply by* 400, *multiply by* 4, *and place two zeros at the right.* So in all similar cases.

EXAMPLES VIII.

1. Multiply 5643 by 15, 16, 36, 54, 99, and 108.
2. Multiply 6534 by 12, 18, 35, 56, 96, and 132.
3. Multiply 618765 by 22, 32, 42, 72, and 144.
4. Multiply 816756 by 24, 44, 64, 84, and 132.
5. Multiply 719675 by 20, 300, 4000, 770, 6600, and 55000.
6. Multiply 917765 by 30, 200, 7000, 440, 5500, and 66000.

MULTIPLICATION: GENERAL CASE.

26. Suppose that you are asked to multiply 34561 by 342. 342 is $300+40+2$; therefore, by the Distributive Law, 34561×342 is the same as $34561 \times 300 + 34561 \times 40 + 34561 \times 2$. We can arrange the work thus :—

34561		
342		
10368300	This line is 34561×300	These lines are called **Partial Products.**
1382440	This line is 34561×40	
69122	This line is 34561×2	
11819862	This line is 34561×342.	

It is usual to omit the 0's in the partial products. Great care is needed in this case to get the figures into their right places. Adopt the following **Rule** * :—

Write the units figure of multiplier directly underneath that of multiplicand.

Write the right-hand figure in each partial product directly underneath that figure of the multiplier which makes the partial product.

Thus, the last 2 of 69122 is under the 2 of 342; the last 4 of 138244 is under the 4 of 342; and the last 3 of 103683 is under the 3 of 342.

* Teachers are advised to see that this rule is obeyed. Arithmetic, which ought, at all events at first, always to be done on squared paper (*i.e.* paper divided into squares), or on slates similarly divided, is too often done in a haphazard fashion on any scrap, and the answer alone deemed to be of importance. Neglect of the rule given above is almost a necessary consequence. Hence all the difficulties with decimals.

27. We add two other examples :—

```
     5200903                    257813
       10309                      2481
  ----------                  ---------
  5200903                     515626
   15602709                   1031252
     46808127                  2062504
  ----------                    257813
  53616109027  Ans.           ---------
                              639634053  Ans.
```

The first of these examples is to illustrate the importance of the rule for position of the partial products. In the second a little time may be saved by noting that, since $4 = 2 \times 2$, the second partial product is double the first, and the third double the second.

EXAMPLES IX.

1. Multiply 234 by 17, 23, 46, 59, 65, and 97.

2. Multiply 243 by 19, 26, 43, 57, 69, and 95.

3. Multiply 7641 by 104, 138, 247, 358, 469, and 572.

4. Multiply 6714 by 105, 147, 238, 369, 458, and 725.

5. Multiply 67459 by 106, 138, 247, 358, 469, and 572.

6. Multiply 64795 by 106, 147, 238, 369, 458, and 725.

7. Multiply 67459 by 1007, 7138, 6247, 5358, 4469, and 3572.

8. Multiply 64795 by 7001, 7147, 6238, 5369, 4458, and 3725.

9. Multiply 142857 by 123, 1234, 2341, 3412, 4123, and 1020304.

10. Multiply 428571 by 234, 2345, 3452, 4523, 5234, and 2030405.

Tests of Accuracy.

28. I. **Proving by Nines.**—There is an interesting method of testing the accuracy of multiplication results* which may be stated here, although its full explanation must be deferred. It is known as the *Method of Casting out the Nines.*

The process of casting out the nines from a number is as follows :—

Add up the digits from right to left (or from left to right), rejecting all nines and all digits whose sum makes nine, and subtracting nine whenever the sum is greater than nine. A single digit, which we call the remainder, is left.

* Applicable also, with modifications, to addition, subtraction, and division.

Thus, take the numbers in the first of the two multiplications given above—

5200903.	We add 3, 0, 9 (rejected), 0, 0, 2, 5. Total 10. Subtract 9. *Remainder* **1**. The actual work is: 3, 5, 10, **1**.
10309.	Reject the 9. $3+1=$ **4**. *Remainder* **4**. The actual work is: 3, **4**.
53616109027.	The actual work is 7, 9 ($=7+2$), 1, 7, 8, 14 $=9+5$. Reject the 9. 5, 8, $13=9+4$. Reject the 9. *Remainder* **4**.

When this process is thoroughly understood, we may test any addition or multiplication sum as follows:—

(1) **Addition.**—*Cast out the nines from all the quantities to be added, treating them as if they all formed one continuous number. The remainder must be the same as that obtained by casting out the nines from the sum of the quantities.*

(2) **Multiplication.**—*Cast out the nines from multiplicand, multiplier, and product. Multiply the first two remainders. If necessary, cast out the nines again. The result must be the third remainder.*

Ex. 1.—	5200903	Remainder **1** (see above).
	10309	Remainder **4** (see above).
	53616109027	Remainder **4** (see above).

This **4** is the same as 4×1, which comes by multiplying the first two remainders.

Ex. 2.—	257813	Remainder **8**	$6 \times 8 = 48$. Remainder **3**.
	2481	Remainder **6**	
	639634053	.....	Remainder **3**.

Hence the answer is probably right.

Note.—You are strongly advised to apply this method to *every* multiplication sum you do; but you must be warned that this test will not detect your mistake if the amount of the error happens itself to be a multiple of 9. In particular, if *you happen to put one of the partial products in the wrong place* before you add up, the test will not show your error, because the amount of the error is then always a multiple (see § 81) of 9. So also, if you have written down the right figures at any stage, but happen to have put two or more in their wrong places—for example, if you write 132 instead of 123, here, again, the error is a multiple of 9.

EXAMPLES X.

Cast out the nines from the numbers given in Exx. 35–41, 50–55, 63–66 of Examples III., reading the numbers in each sum vertically downwards. Thus, for Ex. 35 there will be 7 answers.

29. II. **Proving by Elevens.**—This method is much less subject to error than that of *Proving by Nines*, but is a little harder to acquire. It is well worth the trouble. To find the remainder after dividing by 11, *add together* ALTERNATE *digits, beginning with the units digit, and, when you have reached the left of the number, return, subtracting from the sum you have obtained the digits which still remain. If, when you have again reached the right of the number, your result is less than* 11, *it is the remainder after dividing by* 11. *If it is greater than* 11, *subtract elevens until it becomes less. Elevens may be rejected or added at any stage.*

Ex.—	The arrows indicate the direction of the operation.
4 7 8 1 2 1 3	The working is 3, 5, 13, 17, *10* (you begin subtracting), 9, **8**. *Ans.*
191919.	The work is 9, 18, 27, 26, 25, 24, reject 22, **2**. *Ans.*
257813.	The work is 3, 11, 16. 14, 7, **6**. *Ans.*
30698275.	The work is 5, 7, 16, 3, 7, 18 (11 is added), 10, **3**. *Ans.*

The rule for Proving by Elevens is now exactly the same in Multiplication as that for Proving by Nines.

Note.—If one of the partial products of a multiplication be displaced one place only, this test *will* usually detect the mistake. It will not detect it, however, if it be displaced two places; for the error, in this case, is a multiple of 11.

EXAMPLES XI.

Cast out the elevens from the numbers given in Exx. 63-66 of Examples III.

Combined Multiplication and Addition.

30. It is easy to perform multiplication by single figures and addition, or subtraction, in what is practically one operation, and it is very important that the art of doing this should be acquired. The method will be clear from some examples.

Ex. 1.—*Multiply* 4771213 *by* 3, *and add the result to* 8450980.

Explanation.—The process is as follows :—

8450980	3 threes 9, and 0 = **9**.	
4771213 × 3	3 ones 3, and 8 = 1**1**.	Carry 1.
22764619 *Ans.*	3 twos 6, and 1, 7, and 9 = 1**6**.	Carry 1.
	3 ones 3, and 1, 4, and 0 = **4**.	
	3 sevens 21, and 5 = 2**6**.	Carry 2.
	3 sevens 21, and 2, 23, and 4 = 2**7**.	Carry 2.
	3 fours 12, and 2, 14, and 8 = **22**.	

Ex. 2.—*Multiply* 4771213 *by* 4, *and subtract the result from* 28450980.

Explanation.—The process is as follows :—

28450980	4 threes 12, and **8** = 20.	Carry 2.
4771213 × 4		
9366128	4 ones 4, and 2, 6, and **2** = 8.	
	4 twos 8, and **1** = 9.	
	4 ones 4, and **6** = 10.	Carry 1.
	4 sevens 28, and 1, 29, and **6** = 35.	Carry 3.
	4 sevens 28, and 3, 31, and **3** = 34.	Carry 3.
	4 fours 16, and 3, 19, and **9** = 28.	

The advanced student may do several of these operations at once, as in Ex. 3.

Ex. 3.—*Multiply* 4771213 *by* 5, 2304489 *by* 2, 8260748 *by* 3. *Add* 2833012 *to the sum of the products, and subtract the sum-total from* 60000000.

60000000

2833012
8260748 × 3
2304489 × 2
4771213 × 5

3919701 *Ans.*

Explanation.—The process is as follows :—

15 (= 3 × 5), and 18 (= 9 × 2), 33, and 24 (= 8 × 3), 57, and 2, 59, and **1**, 60. Carry **6**.
6 and 5 (= 5 × 1), 11, and 16, 27, and 12, 39, and 1, 40, and **0**, 40. Carry **4**.
4 and 10 (= 2 × 5), &c., &c.

EXAMPLES XII.

1. Multiply 1023 by 6, and add the result to 3021.
2. Multiply 1032 by 6, and add the result to 2013.
3. Multiply 3021 by 5, and subtract the result from 23456.
4. Multiply 3012 by 5, and subtract the result from 32546.

There are 5 answers to each of Questions 5 to 10 inclusive.

5. $987654 + 3 \times 14312$, $987654 + 4 \times 25861$, $987654 + 5 \times 34713$, $987654 + 6 \times 43827$, $987654 + 7 \times 91254$.

6. $896745 + 3 \times 41132$, $896745 + 4 \times 52681$, $896745 + 5 \times 43173$, $896745 + 6 \times 34287$, $896745 + 7 \times 19542$.

7. $1112625 + 5 \times 370279$, $1112625 + 6 \times 7301360$, $1112625 + 7 \times 8120439$, $1112625 + 8 \times 9470414$, $1112625 + 9 \times 8314638$.

8. $1261152 + 5 \times 307729$, $1261152 + 6 \times 7033106$, $1261152 + 7 \times 8214093$, $1261152 + 8 \times 9740144$, $1261152 + 9 \times 8136488$.

9. $10000000 - 3 \times 3010300$, $10000000 - 4 \times 2057455$, $10000000 - 5 \times 663259$, $10000000 - 6 \times 1532049$, $10000000 - 7 \times 999912$.

10. $10000000 - 3 \times 1030003$, $10000000 - 4 \times 2075554$, $10000000 - 5 \times 636592$, $10000000 - 6 \times 1520394$, $10000000 - 7 \times 919929$.

11.* $7041505 + 8909796 + 2 \times 8870544 + 3 \times 8780445$.

12.* $4071055 + 8097969 + 3 \times 8780454 + 2 \times 8580447$.

13.* $20712345 - 7654321 - 2 \times 13456 - 3 \times 43978 - 4 \times 34789$.

14.* $27021435 - 7563412 - 2 \times 31546 - 3 \times 34798 - 4 \times 37498$.

Miscellaneous Notes.

31. When the product of two factors is multiplied by a third quantity, the result is called the **continued product** of the three factors. Thus, 24 is the continued product of 2, 3, and 4. This product may again be multiplied by a fourth factor, and so on. Thus, 120 is the continued product of 2, 3, 4, 5.

The continued product of a number with itself is called a **power** of the number. Thus, 123×123 is the **second power** or **square** of 123; $123 \times 123 \times 123$ is the **third power** or **cube** of 123; and so on. The square of 123 is written 123^2, the cube 123^3, the fourth power 123^4, and so on. The small figure written above and to the right of the number to express how often that number is repeated to form a power is called an **index.**

Generally $a \times a$ is the *second power* or *square* of a. Written a^2.
$a \times a \times a$ is the *third power* or *cube* of a. Written a^3.
$a \times a \times a \times a$ is the *fourth power* of a. And so on.

The process of finding the value of the various powers is called **involution.** It might be done by ordinary multiplica-

* Questions marked with an asterisk are for advanced students.

tion. Other devices will be explained in the chapter on Involution.

32. For multiplication by **5**, **25**, 125, 375, 625, 875, &c., see § 44.

Multiplication by **11**. *Ex.* 4354×11. If this operation be performed in two lines, a rule is easily seen for writing down the answer. Thus, in 4354×11, the figures of the answer, beginning at the right, are 4, $4+5$, $5+3$, $3+4$, 4. There will sometimes be a figure to carry from one pair to the next.

$$\begin{array}{r} 4354 \\ 4354 \\ \hline 47894 \end{array}$$

Multiplication by **111**. *Ex.* 4356×111. The rule is similar. The figures, beginning at the right, are 6, $6+5$, $6+5+3$, $5+3+4$, $3+4$, 4, a figure being carried from one group to the next when necessary.

33. When the multiplier is very nearly a complete hundred, a modification of the ordinary process is desirable. The following example will illustrate the method :—

Ex.—*Multiply* 78926 *by* 495.

$$\begin{array}{rl} 78926 & \\ 495 & \\ \hline 39463000 & = 78926 \times 500 \\ 394630 & = 78926 \times \quad 5 \\ \hline \textit{Ans.}\ 39068370 & = 78926 \times 495 \end{array}$$

Explanation.—Here $495 = 500 - 5$; therefore 78926×495 is

$$78926 \times 500 - 78926 \times 5.$$

The advanced student may use the process of § 30 to enable him to perform such multiplications in one line.

34. The form of the multiplier may render it advisable to change the order or the method, as in the following examples :—

Ex. 1.—

$$\begin{array}{rl} 78926 & = N \\ 728 & \\ \hline 552482 & = N \times 700 \\ 2209928 & = N \times 7 \times 4 \\ \hline 57458128 & \end{array}$$

or thus

$$\begin{array}{rl} 78926 & = N \\ 728 & \\ \hline 631408 & = N \times 8 \\ 5682672 & = N \times 80 \times 9 \\ \hline 57458128 & \end{array}$$

Explanation.—We may *either* regard 728 as $700 + 28$, and then find $N \times 28$ by aid of the $N \times 7$ which we have found as the first partial product; *or* we may regard it as $720 + 8$, and find $N \times 8$ first. Multiplying this by 9, we get $N \times 8 \times 9$, *i.e.* $N \times 72$, and, shifting it one place, we have $N \times 720$.

Ex. 2.—*Multiply* 78926 *by* 72864, *using only* 3 *partial products.*

78926	$= N$	
72864		
631408	$= N \times 800$	$= N \times 800$
5682672	$= N \times 800 \times 90$	$= N \times 72000$
5051264	$= N \times 8 \times 8$	$= N \times 64$
5750864064		$= N \times 72864$

Explanation.—The 8-product is taken first. Its first figure must be placed under the 8. Multiplying by 9, and shifting one place to left, we have the 72-product. Multiplying the first line by 8, and shifting two places to the right, we have the 64-product.

A similar method applies to such multipliers as 81972 (3 lines), 819729 (3 lines for $729 = 9 \times 81$), 497343 (3 lines). Great care must be taken to have the partial products in their right places, and the result should be tested by *elevens.*

EXAMPLES XIII.

1. Find the continued product of 123, 234, 345.

2. Find the continued product of 132, 243, 354.

3. Find the squares of 123, 234, 345, 456, and 567.

4. Find the squares of 132, 243, 354, 465, and 576.

5. Find the cubes of 231, 342, 453, and 564.

6. Find the cubes of 213, 324, 435, and 546.

7. Multiply 4771213 by 295, 396, 497, 598, and 699.

8. Multiply 7417123 by 296, 397, 498, 599, and 695.

9. Multiply 477121 by 525, 636, 749, 856, and 972, using only 2 partial products.

10. Multiply 147712 by 255, 366, 497, 568, and 729, using only 2 partial products.

11. Multiply 771213 by 49756, 54636, and 216636, using only 3 partial products.

12. Multiply 717123 by 56749, 36654, and 366216, using only 3 partial products.

CHAPTER V.

DIVISION.

35. **Division** is the process of finding how often one number is contained in another larger number.

This might be done by *adding* the smaller to itself until we get the larger, and then counting up how often we have had to repeat the smaller. Thus, Division may be regarded as a problem which might be solved by Addition, just as Multiplication and Subtraction were. Or it might be done by *subtracting* the smaller from the larger again and again until we get nothing left, and noting how often we have to subtract. Thus, Division may be regarded as shortened Subtraction. Both these methods would generally be very long indeed. Most calculating machines perform Division by repeated Subtraction.

Division may also be regarded as the process of splitting-up (or *dividing*) a larger number into a specified number of equal parts.

Thus, since $7 \times 5 = 35$, 7 is contained 5 times in 35; or 35 may be divided into 7 equal parts, and each part will contain 5.

The larger number which has to be split up, or *divided*, into smaller numbers is called the **dividend.**

The smaller number which has to be repeated to make the dividend is called the **divisor.**

The number which expresses how often the divisor must be repeated to make the dividend is called the **quotient** (Lat., *quotiens* = how often).

When we have subtracted the divisor from the dividend as often as we can, it may happen that there is a portion of the dividend *remaining*. This part is called the **remainder.**

Thus, if we have to divide 27 by 6, we may proceed thus: $27 - 6 = 21$, $21 - 6 = 15$, $15 - 6 = 9$, $9 - 6 = 3$. We cannot take 6 away again; therefore 3 is the *Remainder*. We have subtracted 6 four times; therefore 4 is the *Quotient*, 27 is the *Dividend*, 6 the *Divisor*.

It follows from the definition that

Dividend = Divisor × Quotient + Remainder.

e.g. 27 = 6 × 4 + 3

The **Sign of Division** is ÷, or /. Thus, 27 ÷ 6, or 27/6. This is read "27 divided by 6," or sometimes "27 by 6." It is often written $\frac{27}{6}$ and is then read "27 over 6."

The sign / should be little used at first. It is known as the **Solidus.**

Any multiplication result contains also a result in division.

Thus, since $25 \times 18 = 450$, it follows that 18 is contained 25 times in 450, and 25 is contained 18 times in 450; therefore $450 \div 25 = 18$, or $450/25 = 18$, or $\frac{450}{18} = 25$.

The first step in division is, therefore, to look at the Multiplication Table (§ 23) from a fresh point of view. It should teach you not only that 7×6 is 42, but also that 7 is contained 6 times in 42. This will not be difficult if you have attended to the Note in § 23, and practised yourself in giving the factors of the numbers in the Multiplication Table.*

Simple Division.

36. Suppose we have 42 boys arranged in 4 groups of 10, with 2 boys standing by themselves, and we want to make them march in threes,† how should we proceed?

Clearly the lines *A*, *B*, *C* could be marched off at once; for they form threes as they are if they face to the right. We have the lines *D* and *E* over, forming $10 + 2 = 12$ boys. But our Multiplication Table tells us that 12 is 4 threes; therefore we have altogether 10 threes (from *A*, *B*, and *C*, together), and 4 more threes—14 threes in all.

What we have done in the above illustration is this: We have regarded the 42 as 4 tens and a 2, then found how many threes of com-

* The teacher must make certain that this can be *instantaneously* done before proceeding further.

† All the principles of Division, including that most difficult point of "carrying," can be easily explained by drill in the playground.

plete tens we have, and, when we have taken these away, regarded the ten left as made up of ten 1's, which we have joined to the two 1's we had before, then divided the 12 1's into 3's by the multiplication table. We may express this operation thus—

3) 42 (10 + 4	Here the first line, 30, gives the	3) 42 (14
30	number of the 10 threes which we re-	3
—	move first. Taking the 30 away, we	—
12	have 12 left. This 12 contains 4 threes.	12
12	The working on the right omits un-	12
—	necessary figures.	—

The following is a more difficult example :—

	Here we firstly remove 7	
7) 924 (100 + 30 + 2	hundreds. We have 2 hundreds,	7) 924 (132
700	*i.e.* 20 tens, left. But we had	7 (*hundreds*)
—	2 tens before; therefore we have	—
224	a total of 22 tens. But 3 sevens	22 (*tens*)
210	are 21; therefore 30 sevens are	21 (*tens*)
—	21 tens. Taking these away, we	—
14	have 1 ten, *i.e.* 10 units. But	14 (*units*)
14	we had 4 units before; there-	14 (*units*)
—	fore 14 units in all; but this is	—
	2 sevens.	

The above process is known as **Long Division.***

When the divisor is a small number, a contracted method, known as **Short Division,** is used. In this the remainders are not written down, but mentally carried forward. Thus, in the case of 924 ÷ 7, we write only—

7) 924	We work thus: "7 into 9, **1**, and 2 over." There are 2 hundreds, *i.e.* 20 tens; therefore we have 22 tens. "7 into
132	22, **3**, and 1 over." The 1 ten becomes 10 units. "7 into 14, **2**." The **132** is written down figure by figure as obtained.

Since 924 = 92 tens + 4, or 9 hundreds + 24, it is clear that 924 divided by 10 gives 92, with 4 remainder. So, generally, **To divide by 10:** *The (last) units figure of the dividend is the remainder; the other figures form the quotient.* **To divide by 100:** *The last two figures of the dividend form the remainder, the other figures the quotient.*

Division by 20, 300, 4000, and the like, can be most easily done by Short Division.

---

* In teaching Division it is of the utmost importance to take at first small numbers which are within a child's grasp and can be represented by concrete illustrations.

EXAMPLES XIV.

[There are 5 answers each to Questions 1–8.]

1. Divide 720 by 2, 3, 4, 5, 6.
2. Divide 840 by 2, 3, 4, 5, 7.
3. Divide 6720 by 4, 5, 6, 7, 8.
4. Divide 9720 by 4, 5, 6, 8, 9.
5. Divide 44352 by 6, 7, 8, 11, 12.
6. Divide 49896 by 6, 7, 8, 11, 12.
7. Divide 4471213 by 30, 40, 500, 700, 8000.
8. Divide 4742131 by 50, 70, 300, 400, 9000.

Ordinary Long Division.

37. When the divisor is not a small number, or a small number followed by zeros, it is generally best, at first, to use the process of Long Division, which will now be more fully explained.

The advanced student is recommended to practise division by all numbers up to 20 by Short Division. It is easy to work by Short Division with divisors 21, 31, 41,... up to 91. With other divisors greater than 20 it is not recommended.

Suppose we have to divide 6847 by 41.

```
     167
41 ) 6847
     41 (hundreds)
     274 (tens)
     246
      287 (units)
      287
```

Explanation.—A glance shows us that in 6847 there cannot be any thousands of 41's. We ask next how many hundreds. Now, we have 68 hundreds and 47 units altogether. Clearly we can take 41 hundreds away (giving us *100* 41's). We have 27 hundreds left, *i.e.* 270 tens. But we had 4 tens at first; therefore we have 274 tens. 41 is contained 6 times in 274; therefore we can take 6×41 tens away (giving us *60* 41's). $6 \times 41 = 246$. Taking 246 from 274, we have 28 tens left, *i.e.* 280 units. But we had seven units before; therefore we have 287 units. These contain *7* 41's. We have altogether *100*, *60*, *7*, *i.e. 167* 41's.

Each figure of the quotient is written **above** *the corresponding figure of the dividend.*

Suppose we now have to divide 23562 by 63.

```
     374
63 ) 23562
     189
      466
      441
       252
       252
```

Explanation.—We cannot split 2 or 23 into 63 parts; therefore we begin with 235. Now, a little mental work shows us that 63 is contained 3 times in 235, but is not contained 4 times. We begin by taking away 3 63's from 235 (really 300 63's from 23562). We have 46 left. These are 46 hundreds, *i.e.* 460 tens. Adding the 6 tens we had at first, we have 466 tens. 63 is contained 7 times in this. Taking away the 7 63's (really 7 tens of 63's), *i.e.* 441 from the 466, we have 25 (*i.e.* 25 tens) left. These make 250 units, or 252 units in all. 63 is contained 4 times in this.

Note.—The one difficulty in this process is to decide how often a number is contained in a larger number which is not ten times as large. For instance, we had to ask in the above how often 63 was contained in 466. We really reasoned that, since 6 was contained 7 times in 46 (with 4 over), 60 was contained 7 times in 460; therefore 63 was contained in 466 *about* 7 times. Then we tried 7×63, and found it was not too big. Nothing but practice can enable you to answer questions of this kind at once. For instance, with $260 \div 89$, you would argue at first that, since 8 was contained 3 times in 26, 89 was contained 3 times in 260; but, when you worked out 3×89, you would find that the 9 had made 3 too large a multiplier. 86, 85, 84, 83, 82, 81, and 80 are all contained 3 times in 260, but 87, 88, and 89 only twice.

The best way, in difficult cases, of settling this matter is the following*:—

If 89 was contained in 260 3 times, 3 would be contained in 260 89 times. Divide 260 by 3. We obtain 86 +; therefore 3 is not contained as often as 89 times; therefore 89 is *not* contained as often as 3 times. This method may always be applied, and will avoid the writing of erroneous figures.

38. We add a longer and harder example of Long Division.

```
        31097
397 ) 12345677
      1191
        435
        397
         3867
         3573
          2947
          2779
           168
```

Here *Divisor* is 397.

Dividend is 12345677.

Quotient is 31097.

Remainder is 168

The result shows us that

$$12345677 = 31097 \times 397 + 168.$$

Problems.—(1) How much must be taken from 12345677 to make it divisible by 397 without remainder? *Ans.* 168.

(2) How much must be added to 12345677 to make it divisible by 397 without remainder? *Ans.* 229; for $168 + 229 = 397$.

The "Italian" Method.

39. Before reading this section, you are recommended to revise § 30. In the so-called Italian Method of division, the partial products of the divisor are omitted, and the operations of finding these partial products, and of subtracting them from the dividend, are performed in one, as shown in § 30.

---

* Mackay's *Arithmetic.*

Thus the sum just worked would be written as follows :—

```
        31097
397 ) 12345677
        435
         3867
          2947
            168 = Remainder.
```

Working.—The first step is to multiply 397 by 3, and subtract in one operation. Thus—

3 7's 21, and **3** = 24. Write the **3** and carry 2.

3 9's 27, and 2, 29, and **4** = 33. Write the **4** and carry 3.

3 3's 9, and 3, 12, and **0** = 12.

This **43** is made into 435 by bringing down 5 from the dividend.

397 is then subtracted from 435, the remainder, 38, alone being written. Thus, 7 and **8** = 15, carry 1 ; 9 and 1, 10, and **3** = 13, carry 1 ; 3 and 1, 4.

Notes.—(1) The Italian Method looks very short for such questions ; but it is much more liable to error, and an error is very difficult to detect. Again, there is often a loss of time, because, if a figure is repeated in the quotient, the multiplication by this figure has to be performed again.

(2) In very long division sums * it is preferable to make a table of the multiples of the divisor. This table should be on a loose slip of paper, and should take the following form for divisor 397,

1. 397. **2.** 794. **3.** 1191. **4.** 1588. **5.** 1985. **6.** 2382. **7.** 2779. **8.** 3176. **9.** 3573.

This slip may be placed at the right point under the dividend, and it is now quite unnecessary to write the partial products. Thus, 1000000000 ÷ 397 would run as follows :—

```
        2518891
397 ) 1000000000
        2060
          750
          3530
           3540
            3640
              670
              273   Rem.
```

This method is quite as short as the Italian Method, but not subject to its defects †

Tests of Accuracy.

40. (1) Multiply the quotient by the divisor, and add the remainder. The result should be the dividend.

---

* Such as occur, for instance, in the conversion of fractions into decimals.

† The subsidiary multiplication table is best constructed by multiplication by 2, and by addition of consecutive lines. Thus, for **3** add the **2** and **1** products, for **5** the **3** and **2** products, and so on.

(2) It is generally sufficient to test by **Nines** or **Elevens.** The rule is as follows :—

Cast out the $\frac{\textit{nines}}{\textit{elevens}}$ *from the quotient and from the divisor. Multiply the results. Cast out the* $\frac{\textit{nines}}{\textit{elevens}}$ *again if necessary, and add the result to that obtained by casting out from the remainder. Cast out again if necessary. The result must be that obtained by casting out from the dividend.*

Note.—*Casting out Nines* will not detect the commonest error in division—the omission of a zero in the quotient. *Casting out Elevens* will, unless two zeros are omitted.

Example.—We will test the sum given in § 38 by elevens. Casting out elevens, we get for *divisor* 1, *dividend* 3, *quotient* 0, *remainder* 3. Multiplying quotient and divisor results, we get 0, adding the remainder result, we get 3, which is the dividend result, as it should be.

EXAMPLES XV

In Exx. 1–10 there is no remainder. Exx. 1–4 and 11, 12 contain each five answers.

1. Divide 8108100 by 35, 63, 117, 675, and 2025.

2. Divide 8108100 by 45, 77, 143, 495, and 4725.

3. Divide 89237148 by 84, 132, 209, 299, and 2244.

4. Divide 89237148 by 91, 187, 221, 253, and 3059.

5. 97994448 ÷ 108.

6. 32470956 ÷ 556.

7. 823479885 ÷ 9897.

8. 273132329750 ÷ 875.

9. What is the quotient when 3986544 is divided by 2116?

10. How often is 8976 contained in 3877632?

11. Divide 238970196 by 135, 999, 1035, 10357, and 770531.

12. Divide 138790916 by 153, 989, 1053, 10537, and 775013.

13. 964374 ÷ 165.

14. 3624789 ÷ 252.

15. 76300417 ÷ 2041.

16. 782034163 ÷ 70907.

17. 10001001000101 ÷ 10001.

Division (Miscellaneous Notes).

41. **The Factor Method.**—We have seen that to divide a number by 35 is to split the number into 35 equal parts and to find the size of 1 part. But, if we split the number first into 7 equal parts, and then each of these parts into 5 equal parts, we shall have altogether $7 \times 5 = 35$ equal parts. Hence,

when we have to divide by 35, we may divide by 7, and then divide the result by 5.

$$\text{Generally } \frac{N}{a \times b} = \frac{N}{a} \div b.*$$

Ex. 1.—43190 ÷ 35.

7) 43190	The second line indicates that 43190 contains 6170 sevens.
5) 6170	The third line indicates that 6170 contains 1234 fives.
1234	And, therefore, that 43190 contains 1234 (7 × 5's, *i.e.*) 35's.

Care is required in finding the remainder if the division cannot be exactly made.

Ex. 2.—191103095 ÷ (2 × 12 × 12).

		Remainder.
2) 191103095		
12) 95551547	2's and 1 unit over	1
12) 7962628	24's and 11 2's over	22
663552	288's and 4 24's over	96
	Total remainder	119

From Ex. 2 it should be clear that the following rule † may be given :—

> *To find the remainder after division by factors, multiply each partial remainder by all the factors which precede it,* EXCEPT THE FACTOR WHICH PRODUCED IT, *and add the products to the remainder, if any, from the first division.*

42. **Division by Multiplication.**—If a number is equal to **25** times 37, then 4 times the number is equal to **100** times 37. Hence the result of dividing the *original number* by 25 is the same as the result of dividing *4 times the number* by 100. Hence the simple rule—

> *To divide by* 25, *multiply by* 4 *and divide by* 100.

This rule, however, does not give the *remainder* exactly, if there is one. Suppose the original number is **25** times 37 + **3**, so that the correct remainder is **3**. Then 4 times the original number is **100** times 37 + **12** : hence when we divide by 100 the remainder obtained is 4 times too large : that is we must "correct" the remainder by *dividing it by 4*.

Similar arguments can be applied to the cases of division by 5 and by 125. The following inclusive rule is now easy to understand :—

---

* The teacher must not allow this result to be assumed as a mere corollary of multiplication. He has really proved a theorem in fractions.

†Neither here nor anywhere else should the student be allowed to use the rule until he has had abundant practice in the fuller and self-explanatory methods.

To divide by $\left\{\begin{matrix}5\\25\\125\end{matrix}\right\}$, *multiply by* $\left\{\begin{matrix}2\\4\\8\end{matrix}\right\}$ *and divide by* $\left\{\begin{matrix}10\\100\\1000\end{matrix}\right\}$.

The remainder must be corrected by dividing by $\left\{\begin{matrix}2\\4\\8\end{matrix}\right\}$.

Of course, we divide by 10, 100, 1000 in the usual way (see bottom of page 36).

43. **Division by 20, 400, 5000, and similar numbers.**—We have seen (§ 36) that this may be accomplished by Short Division. We may also use the factor method.

Ex.—13579 ÷ 400. Here 400 = 4 × 100.

Hun- Un-
dreds its
4·00) 135·79

33 and 3 hundreds, with 79 units over.
Ans. 33 and 379 remainder.

To divide by 100, we mark off the last two figures as remainder.

EXAMPLES XVI.

1. Divide 46656 by 2 × 3, 3 × 4, 4 × 6, 6 × 8, and 9 × 12.

2. Divide 46872 by 2 × 4, 3 × 3, 4 × 9, 6 × 9, and 9 × 12.

3. Divide 61446337 by 2 × 3 × 4, 3 × 4 × 5, 4 × 5 × 6, 5 × 6 × 7, and 6 × 7 × 8.

4. Divide 64164373 by 2 × 3 × 4, 3 × 4 × 5, 4 × 5 × 6, 5 × 6 × 7, and 6 × 7 × 8.

5. Divide 122003404 by 300, 4000, 11000, 2200, and 32000.

6. Divide 212030440 by 400, 3000, 12000, 3300, and 36000.

44. **Multiplication by Division.**—If we have to multiply by 5, clearly we may multiply by 10 and then divide by 2.

For to multiply a number by 10 is to find a number 10 times as great as the given one, and the half of this number is 5 times as great.

Hence, *to multiply by* 5 *put* 0 *at the right and divide by* 2.

To multiply by 25 *put* 00 *at the right and divide by* 4; for 25 = 100 ÷ 4.

To multiply by 125 *put* 000 *at the right and divide by* 8; for 125 = 1000 ÷ 8.

To multiply by 75 *put* 00 *at the right and divide by* 4, *and then multiply by* 3. And so on.

45. **Division by 99, 999, &c.—**

Ex. 1.—47712 ÷ 99.

```
     477|12
       4|77
        |  4
Q.   481|93  R.
```

Explanation.—

		Hundreds		Units		Ninety-nines		Units
47712	=	477	+	12	=	477	+	477 + 12
477	=	4	+	77	=	4	+	4 + 77
								4
						Q. 481		93 R.

Ex. 2.—47712 ÷ 98.

```
     477|12
       9|54 = 2 × 477
        |18 = 2 × 9
Q.   486|84  R.
```

Explanation.—

		Hundreds		Units		Ninety-eights		Units	Units
47712	=	477	+	12	=	477	+	2 × 477	+ 12
954	=	9	+	54	=	9	+	2 × 9	+ 54
									18
						486 Q.			84 R.

Ex. 3.—47712131415 ÷ 997.

```
     47712131|415
       143136|393 = 3 × 47712131
          429|408 = 3 × 143136
            1|287 = 3 × 429
             |  3 = 3 × 1
     47855697|
            1|506
             |  3
Q.   47855698|509  R.
```

Explanation.—We get, in the first instance, 47855697 for quotient and 1506 for remainder. As 1506 is greater than 997, we repeat the division process on the 1506.

EXAMPLES XVII.

1. Multiply 99856 by 25, 125, 375, 625, and 875.

2. Multiply 98965 by 25, 125, 375, 625, and 875.

3. Divide 3626491, 3264619, 2364619, and 2366914 by (*a*) 25 and also by (*b*) 125 and by (*c*) 99.

4. Divide 3727491, 3274719, 2374719, and 2377916 by (*a*) 25 and also by (*b*) 125 and by (*c*) 99.

5. Divide 183303028 by 99, 998, 9997, 99996, and 999995.

6. Divide 180330382 by 98, 999, 9996, 99997, and 999994.

CHAPTER VI.

SIGNS, ABBREVIATIONS, BRACKETS.

46. It will be convenient to collect at this place the signs and abbreviations which we have used, and to explain some others which we shall find useful later.

Equality, $=$. Thus, $7 \times 6 = 42$. See § 9.

This sign is often badly used. It should *never* be used without something on both sides of it. Nothing is commoner than for a piece of work to commence with $=$, where the meaning is "the expression we have to simplify $=$." Use such intelligible abbreviations as Exp. (= expression), L.H. (= left-hand expression), R.H. (= right-hand expression), &c.

Addition, $+$. See § 9. Thus, $7+6=13$.

Subtraction, $-$. See § 15. Thus, $7-6=1$.

Multiplication, $\times$; or . (see § 21) *placed on the line* on which the numbers are written.

Thus, $7 \times 6 = 7 . 6 = 42$.

Division, $\div$; or ——, with the divisor written at the bottom, the dividend at the top; or /. See § 35.

Thus, $42 \div 6 = \frac{42}{6} = 42/6 = 7$.

$\therefore$ means "therefore." See § 10.

$\because$ means "because" or "since."

$>$ means "is greater than." $<$ means "is less than."

Brackets are of various shapes, ( ), { }, [ ], and ——— or ———, and indicate that the quantities contained in them are to be treated as a whole.

Thus, $(7-3) \times 5$ means 4×5. So $(42-7) \div 5$ means $35 \div 5$, and might be expressed $\frac{42-7}{5}$, or $[42-7] \div 5$, &c. It is of *great importance* to notice that the —— in $\frac{42-7}{5}$ is *really a bracket.* $(42-7) \div 5$ would not usually be expressed $\overline{42-7} \div 5$, because there is more chance of reading this incorrectly. It *might*, however, be so expressed.

Multiplication with Brackets is often expressed by writing the quantities side by side, without any sign between them.

Thus, $3[5]$ would mean 3×5, and *not* 35.
So $[5-3][7-4]$ means 2×3, *i.e.* 6.
The same method is used when a number is denoted in any other way than by the 10 digits—as, for instance, by letters. Thus, $5a$ means $5 \times$ the number a.

To prevent mistakes in dealing with these signs, the following rules have been agreed upon :—

(1) *Multiplications and divisions must be performed before additions and subtractions.*

Thus $18+15 \div 3-2$ means $18+5-2$, *i.e.* 21, the division being performed first,
not $(18+15) \div (3-2)$, *i.e.* 33,
nor $18+[15 \div (3-2)]$, *i.e.* 33 again,
nor $[(18+15) \div 3]-2$, *i.e.* $\frac{33}{3}-2$, *i.e.* 9.

(2) *Multiplications and divisions must be performed from left to right.*

Thus, $36 \div 9 \times 2$ means $(36 \div 9) \times 2$, *i.e.* 4×2, *i.e.* 8,
not $36 \div (9 \times 2)$, *i.e.* $36 \div 18$, *i.e.* 2.

Note also:

(3) *When several brackets occur one within another, we should find the value of the innermost bracket first.*

Ex.—*Simplify* $18 \div \{9-(6-3)\}+5$.

To simplify the brackets, we begin with $(6-3)$, *i.e.* 3.
$\therefore$ $9-(6-3)$ is $9-3$, *i.e.* 6.

We now have $18 \div 6+5$. By rule 1 the division takes precedence; therefore we get $3+5$, *i.e.* 8.

The above may be *written* thus: $(\ )=3$; $\therefore$ $9-(\ )=9-3=6$; $\therefore$ $\{\ \}=6$; $\therefore$ $18 \div \{\ \}=18 \div 6=3$; $\therefore$ exp. $=3+5=8$.

EXAMPLES XVIII.

Simplify the following expressions :—

1. $7 \times 6 - 4 \times 2 + 16 \div 2$.

2. $6 \times 9 + 3.5 - 27 \div 3$.

3. $7 + 4 \div 2 + 9 \div 3 + \frac{36}{4}$.

4. $9 - 8 \div 2 + 9 \times 3 - \frac{28}{4}$.

5. $36/6 + 36.6 + 36 - 6.3$.

6. $27 \div 3 + 27/3 + \frac{27}{3}$.

7. $(7+3) - (5-2) + (5+2)$.

8. $(7-3) + (5+2) + 5 + 2$.

9. $\{7 + [9 - (8-3)]\}$.

10. $[9 - \{7 - (6-5)\}]$.

11. $11 - \{2 - (1 - \overline{2-1})\}$.

12. $12 - \{2 + (11 - \overline{2-1})\}$.

13. $[\{(11-2) - 3\} - 4] - 1$.

14. $[\{(13+1) - 3\} + 2] - 1$.

15. $[(15-3) + \{15 - (10-3)\}] - [(14-7) - \{14 - (10+3)\}]$.

16. $2(17+3) - \{17 + (10-3)\} + 3.4$.

17. $2(10-9) + 3(9-7) - 4(8-6)$.

18. $2(8-6) + 3\{(9-7) + 4(8-6)\}$.

19. $2 - [2 - \{2 - (2 - \overline{2-1})\}]$.

20. $4 - [3 - \{4 - (3 - \overline{4-3})\}]$.

EXAMPLES XIX.

(Miscellaneous Exercises on the Four Rules.)

1. Add together seventeen hundred and twenty, a hundred and seventy thousand and seventy, and seventeen million eight hundred and twenty-nine thousand, and express the result in words.

2. Add together eight million one hundred thousand and five, two thousand and seventy-one, five hundred and three thousand and ninety, and express the result in words.

3. The ages of the members of a family are 45, 35, 14, 11, 9. What ages were the others when the youngest was born?

4. How much must be added to the product of 175 and 53 to give the same as the product of 177 and 53?

5. The product of two numbers is 32470956, and one of them is 6489. What is the other?

6. Which is the greater—the product of 5609 and 6427 or the product of 5906 and 6247, and by how much?

7. Which is the greater—the product of 5942 and 7685 or the product of 5492 and 7865, and by how much?

8. Find the sum of the quotient and the remainder when 16316316 is divided by 163.

9. In a certain sum the dividend is 31884740, the quotient 40930. Find the divisor and remainder.

10. When the number 582167 is divided by a certain divisor, the quotient is 762 and the remainder 761. Find the divisor.

11. What is the least whole number above 5000 which is divisible by 63?

12. Divide the product of 999999 and 1955 by the continued product of 37, 13, 17, 7, and 23.

13. If, in a division sum, the divisor be eight times, and the quotient seven times the remainder, what is the dividend when the remainder is 452?

14. In a division sum the divisor is 478, the quotient 50236, and the remainder 17. What is the dividend?

15. In a division sum the divisor is 9183, the quotient 1207, and the remainder 129. By how much must the dividend be increased in order that, with the same divisor, the quotient may be 1209 and the remainder 637?

16. The number of scholars in primary schools in England in 1893 was five million one hundred and thirty-seven thousand and two, and the number in each school was, on the average, two hundred and sixty-one. Find the number of schools.

17. What is the total number of figures required in numbering the pages of a book of 400 pages?

18. The numbers 900, 901, ... to 1200, inclusive, are all written down. How many figures are there?

19. The continued product of 2, 5, 7, 13, 19, and another number is 570570. What is the other number?

20. The number of emigrants from the United Kingdom, in ten years, was 1967579; 649742 of these were English and Welsh, 158226 were Scotch, 301983 were foreigners. How many were Irish?

21. Divide 973047 into two numbers, of which one is greater than the other by 279425.

22. In a train containing 310 passengers, the total number of first- and third-class passengers was 220, that of the second- and third-class 265. How many passengers were there of each class?

23. A box contains 400 balls altogether of three colours. There are 258 blue and red balls, and 256 blue and yellow balls. How many are there of each?

24. Arrange the figures 194678 in the six different ways in which 1, 9, 4 are the first three figures in different orders, the last three being 678, and also in the six different ways in which 6, 7, 8 are the last three figures in different orders, the first three being 194, and add the twelve arrangements together.

25. How many numbers are there with five figures? How many with not more than five figures?

26. The annual deaths in a town being 1 in 45, and in the country 1 in 50, in how many years will the number of deaths out of 18675 persons living in the town and 79250 persons living in the country amount together to 10000?

27. The *Iliad* contains 15683 lines, and the *Æneid* contains 9892 lines. How many days will it take a boy to read through both of them at the rate of 75 lines a day?

28. At a game of cricket, A, B, and C together score 137 runs; A and B together score 111 runs; while B scores 64 more than C. Find the number scored by each.

29. A gentleman, whose age is 60, has two sons and a daughter. His age equals the sum of the ages of his children. Two years since his age was double that of his eldest son; while now the sum of the ages of the father and the eldest son is seven times as great as that of the youngest son. Find the ages of the children.

[*In Exx.* 30–41 *a dot denotes a missing figure. You are required to find all missing figures.*]

30. 2982 . . divided by 456 leaves a remainder 1.

31.

```
   5378
   2.3.
  -----
  .....
  .....
  .....
   26...
  -----
 .41710..
```

32.*

```
  3..
   6.
 -----
 24..
 ..8.
 -----
 ..2..
```

33.*

```
  4..
   3.
 ----
 ....
  32..
 -----
 .... 3
```

34.*

```
  .8.
   .3
 -----
 ...5
  .96.
 -----
 .....
```

It is here given that the product is divisible by 3.

* Throughout the book questions thus marked are difficult. There are two solutions to Ex. 33; and three solutions to Ex. 36 if the *3* is not included in the data.

35.* 4.3) ..64. (4.
....
7..
...
.02

36.* 4..) ..3.. (3.
..7.
36..
....

These problems can be solved without assuming the *3* as part of the data, but are then much more difficult.

37.*

...) 3..5.. (.7..
...
.28.
1...
74.
...
...
...

38.* . . . 78418 divided by 7583 leaves a remainder 7531.

39.* . . . 55196 divided by 8437 leaves a remainder 137.

40.* What three figures must be placed after 656459 to make the result exactly divisible by 1899?

41.* What three figures must be placed in the middle of 495 . . . 816 to make the result exactly divisible by 2112?

42.* What three figures must be placed before 115088 to make the result exactly divisible by 1234?

43.* How many numbers between 1000 and 7000 can be exactly divided by 171?

44.* If the first thousand numbers be written down, and those divisible by 2 be crossed off, and then those divisible by 3, how many are left?

45.* If the first thousand numbers be written down, and those divisible by 3 be crossed off, and then those divisible by 5, how many are left?

46.* The numbers are written down in order without interval—thus, 123456789101112 . . . What is the thousandth figure? What is the millionth? What the billionth?

47.* The odd numbers (see § 81) are written down in order without interval—thus, 13579111315 . . . What is the thousandth figure? What is the millionth? What the billionth?

SECTION II.

WEIGHTS AND MEASURES.

CHAPTER VII.

TABLES OF WEIGHTS AND MEASURES.

47. Suppose that you take a foot-rule, and find that you can lay it along the edge of a table exactly twelve times: you say that the table measures 12 feet in length. This foot-rule is your **unit** or **standard.** If, however, you use a yard-measure, you will find that you can place it exactly four times along the edge of the table. You have now used a new unit or standard, and it has this advantage over your first unit—that the measure of the table is expressed by a smaller number. But this standard is not enough for all purposes. It is not much use to be told that the distance from London to Bath is 186,560 yards. The number is so big that we cannot grasp it. We need a big standard to measure it by, and so we use what is called a mile—a standard 1760 yards long; and, since 186,560 contains 1760 exactly 106 times, we say that the distance is 106 miles. But we have not, even yet, got standards enough for measuring lengths. If we use our foot-rule for measuring the heights of men, we shall need to use it for most men five times, and then we shall probably find a portion of the height left—too small to be measured by the rule, and needing a still smaller unit to enable us to describe it. This smaller unit we call an inch, and twelve of these inches put side by side would amount to a foot.

When we put down a list of the number of inches that make a foot, the number of feet that make a yard, and the number of yards that make a mile, we get what is called a **Table** of **Long Measure.** Similar tables are needed for other measures.

The following tables must be learnt thoroughly by heart. They are all the measures which are really needed; though a great many more are in use, and will be mentioned later. Fuller information will be given about each table in the paragraph which is referred to at the top of the table.

48. TABLES.*

MONEY. (§ 49.)

4 **farthings** (*f*) = 1 **penny** (*d.*)
12 **pence** = 1 **shilling** (*s.*)
20 **shillings** = 1 **pound** (£.)
or 1 **sovereign**

WEIGHT ("*Avoirdupois*"). (§ 50.)

16 *drams* (*dr.*) = 1 *ounce* (*oz.*)
16 **ounces** = 1 **pound** (*lb.*)
14 **pounds** = 1 **stone** (*st.*)
2 **stones**
= 28 **pounds** = 1 **quarter** (*qr.*,
4 **quarters** = 1 **hundred-weight** (*cwt.*)
20 **hundred-weights** = 1 **ton** (*ton*)

LENGTH ("*Long Measure*") (§ 51.)

12 **inches** (*in.*) = 1 **foot** (*ft.*)
3 **feet** = 1 **yard** (*yd.*)
220 **yards** = 1 **furlong** (*furl.*)
8 **furlongs**
= 1760 **yards** = 1 **mile** (*ml.*)

$5\frac{1}{2}$ *yards* = 1 *pole* (rod or perch)
40 *poles* = 1 *furlong*

AREA ("*Square Measure*"). (§ 52.)

144 **square inches** (*sq.in.*)
= 1 **square foot** (*sq.ft.*)
9 **square feet**
= 1 **square yard** (*sq.yd.*)
4840 **square yards**
= 1 **acre** (*ac.*)
640 **acres**
= 1 **square mile** (*sq.ml.*)

$30\frac{1}{4}$ *square yards*
= 1 *square pole* (*sq.po.*)
40 *square poles* = 1 *rood* (*r.*)
4 *roods* = 1 *acre*

VOLUME ("*Cubic Measure*"). (§ 53.)

1728 **cubic inches** (*c.in.*)
= 1 **cubic foot** (*c.ft.*)
27 **cubic feet**
= 1 **cubic yard** (*c.yd.*)

CAPACITY ("*Liquid Measure*," "*Dry Measure*"). (§§ 50, 54.)

4 *gills* (*g.*) = 1 *pint* (*pt.*)
2 **pints** = 1 **quart** (*qt.*)
4 **quarts** = 1 **gallon** (*gall*) } Liquids.

2 **gallons** = 1 **peck** (*pk.*)
4 **pecks** = 1 **bushel** (*bush.*)
8 *bushels* = 1 *quarter* (*qr.*) } Corn, &c.

TIME. (§ 55.)

60 **seconds** (*sec.*)
= 1 **minute** (*min.*)
60 **minutes** = 1 **hour** (*hr.*)
24 **hours** = 1 **day**
7 **days** = 1 **week** (*wk.*)
365 **days** = 1 **year**
366 **days** = 1 **leap year**

ANGLES.

60 **seconds** (″) = 1 **minute** (′)
60 **minutes** = 1 **degree** (°)
90 **degrees** = 1 **right angle** (*rt.ang.*)

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* In these tables the less important measures are named in *italics*, those which should be thoroughly mastered at once are in **clarendon**. The abbreviations in brackets which follow the names of the measures are those which will be used throughout the book. The table of *Angular Measure* must, of course, be deferred until the notion of angles has been reached.

49. **Money Table.**—A sum of money is usually written in terms of the four units named in the table (*pounds, shillings, pence, farthings*). Thus, £14 16s. 11¼d. The letters £, s, d are the first letters of the Latin words *Libra, Solidus, Denarius.* "One farthing," "two farthings," and "three farthings" are written ¼d., ½d., ¾d., the meaning of which symbols will be further explained in Chap. xv. Thus, 7½d. means 7 *pence* and 2 *farthings.*

But, if we were to pay away £14 16s. 11¼d. by means of the coins (*i.e.* money units) named, we should have to use 14 separate pounds, 16 separate shillings, 11 pennies, and 1 farthing, *i.e.* 42 coins in all. To avoid this inconvenience the following other coins are *current* (*i.e.* in use) as well as those named :—

Gold Coin : **Half-sovereign** (= 10s.).
Silver Coins : **Crown** (5s.), **Double Florin** (4s.), **Half-crown** (2s. 6d.), **Florin** (2s.), **Sixpence** (6d.), **Threepence** (3d.).
Bronze Coin : **Halfpenny** (½d.).

In addition to these the following coins are authorized, but very rarely issued, and never used in business :—

Gold : Five Pounds (£5), Two Pounds (£2).
Silver : Groat (4d.), Twopence (2d.), Penny (1d.).

Sums of money are often reckoned in **Guineas** (21s.) and **Half-guineas** (10s. 6d.). This is especially the case with the fees of doctors and lawyers, the price of horses and pictures, and subscriptions to charity.

No coins of this value have been issued since 1813.

To avoid the carrying of large sums of gold **Bank Notes** are issued, chiefly by the Bank of England, for sums of £5, £10, £20, £50, £100, £200, £500, and £1000. These notes can be exchanged for gold at any time at the Bank of England or its branches. They pass from hand to hand, and serve all the purposes of coins. They are called **Paper Money.** Coined money, on the other hand, is known as **Specie.** Precious metals in bulk are known as **Bullion.**

We see now that to pay away the sum of £14 16s. 11¼d., which, in the units first named, would require 42 coins, there are needed 1 £10 note, 4 *sovereigns,* 1 *half-sovereign,* 1 *double florin,* 1 *half-crown,* 1 *threepence,* 2 *pennies,* and 1 *farthing, i.e.* 12 coins (if we call the note a coin) instead of 42. It may, of course, be paid in many other ways.

Gold is worth £3 17s. 10½d. per ounce (*troy*, see § 50), and any one taking gold to the Mint may have it converted into sovereigns at this rate, without any charge whatever. It is too soft for coins when pure, and is mixed with copper in the proportion of 22 gold to 2 copper. This copper is called *alloy*. In Australia a mixture of copper and silver is used for alloy, and the sovereigns are whiter. Silver coins are also made of a mixture of 25 parts silver to 2 copper alloy (more correctly 222 to 18). The value of silver as compared with gold is very variable, and thus the real value of a shilling, as metal, its *bullion value*, is also variable. At present (1901) it is about 4½d. The "copper" money is really made of bronze (95 parts of copper, 4 of tin, 1 of zinc). As bullion the penny is worth (1901) only about a sixth of its *face value*, *i.e.* a pennyworth of copper will make about 6 pennies. Ten half-pennies weigh exactly as much as 6 pence.

50. All articles may be sold by avoirdupois weight; but other systems of weights exist in examination papers, and, to a certain smaller extent, elsewhere. By the Weights and Measures Act of 1878 it was enacted that gold, silver, platinum, and precious stones *might* be sold by the "ounce troy"; also that drugs *might* be sold retail by "Apothecaries' Weight." The "British Pharmacopœia" of 1885 modified the older Apothecaries' Weight by ceasing to recognize the "ounce" of 480 grains, employing in its place the ounce (Av.) of 437½ grains. The tables are now

TROY WEIGHT.		APOTHECARIES' WEIGHT (WHOLESALE).	
24 *grains* (*gr.*)	= 1 *pennyweight* (*dwt.*)	437½ *grains*	= 1 *ounce*
20 *dwts.*	= 1 *ounce troy* (*oz. troy*)	16 *ounces*	= 1 *pound*
12 *ounces troy*	= 1 *pound troy*		

APOTHECARIES' MEASURE (*Fluids*).

60 *minims* (*min.*) = 1 *fluid drachm* (*fl. dr.*)
8 *drachms* = 1 *fluid ounce* (*fl. oz.*)
20 *ounces* = 1 *pint*

Troy Weight.—The ounce is the only *legal* remnant of Troy Weight.

Apothecaries' Weight.—Besides the grain and ounce, two other weights, the *scruple* and the *drachm*, are used, though not recognised in the "Pharmacopœia" of 1885. Their value is given by

20 *grains* = 1 *scruple* (*scr.*)
3 *scruples* = 1 *drachm* (*dr.*)

It must be carefully noted that this drachm (60 grs.) is *not* an eighth part of an ounce as in Apothecaries' Measure (Fluids). There is great confusion about this.

A fluid ounce of water weighs an ounce (Av.).

Note carefully—

Dram (Av.) is *not* the same as *Drachm* (Ap.).
Ounce (Av.) is *not* the same as *Ounce* (Troy).
Pound (Av.) is *not* the same as *Pound* (Troy).
But *Grain* (Ap.) *is* the same as *Grain* (Troy)
Ounce (Ap.) *is* the same as *Ounce* (Av.).

To connect—

Avoirdupois and Troy, remember that 1 lb. Av. = 7000 grains Troy;
Apothecaries' and Troy, remember that the *grains* are the same.
Apothecaries' and Avoirdupois, remember that the *ounces* and *pounds* are the same.

51. The following additional measures of length should be noted:—

100 *links* = 22 *yards* = 1 **chain**		Used in land-surveying.
10 *chains* = 1 *furlong*		
6 *feet* = 1 **fathom**		Used for measuring depth of sea.
6080 *feet* = 1 **knot***		Used for measuring rate of ships.
3 *miles* = 1 **league**		
4 *inches* = 1 *hand.*		Used for measuring horses.

52. *Square measure* is used for measuring the surface or area of anything, *e.g.* the floor, the wall, a field, a parish, &c. The tables of long measure and square measure may be conveniently presented thus:—

12 *inches* = 1 *foot*	$12 \times 12 =$	144 *sq. inches* = 1 *sq. foot*
3 *feet* = 1 *yard*	$3 \times 3 =$	9 *sq. feet* = 1 *sq. yard*
22 *yards* = 1 *chain*	$22 \times 22 =$	484 *sq. yards* = 1 *sq. chain*
80 *chains* = 1 *mile*	$80 \times 80 =$	6400 *sq. chains* = 1 *sq. mile*
10 *chains* = 1 *furlong*		10 *sq. chains* = 1 *acre*

The numbers in "square measure" as given above are the squares of the corresponding numbers in "long measure." Note that this applies also to—

$5\frac{1}{2}$ *yards* = 1 *pole* $\quad 5\frac{1}{2} \times 5\frac{1}{2} = 30\frac{1}{4}$ *sq. yards* = 1 *sq. pole.*

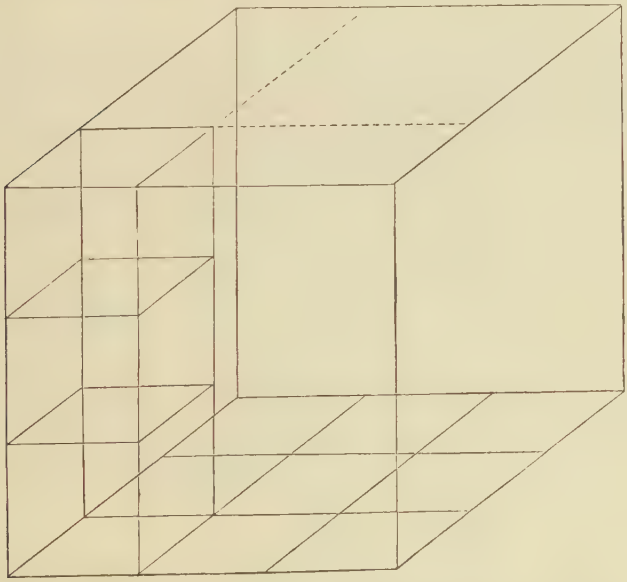
If the reason for this is not clear, take a large sheet of paper, and draw a square the length of whose sides is 1 ft. This is a "square

* To convert "knots" into "miles," *either* (1) add to the number of knots one-seventh of itself, *or*, more exactly (2) add a tenth part, and then the half of this, *i.e.* multiply by 1·15.

foot." Divide all four sides into inches, and join the points which are just opposite one another. The square is now divided into smaller squares, each of which is a "square inch." But it will be seen that there are twelve rows with twelve in each row, *i.e.* 12×12 smaller squares; therefore 1 sq. ft. = 144 sq. in. A chess-board will help you here.

Distinguish carefully "3 square miles" and "3 miles square." The latter means a square each side of which is 3 miles, and which, therefore, contains $3 \times 3 = 9$ square miles.

53. *Cubic measure* is used for measuring the volume of solids, *e.g.* stone, brickwork, wood. The numbers of the table are the cubes of the corresponding numbers in long measure. The reason for this will be seen from the diagram.



The large cube may be taken to represent a cubic yard, *i.e.* each edge of it represents a yard. The edges are divided into 3 parts, each of which is, therefore, a foot. The base on which the cube stands is seen to be divided into 9 squares, *i.e.* square feet, and, if corresponding points of sides directly opposite are joined, the whole cube is divided into

smaller cubes, each edge of which is 1 ft., and each of which is, therefore, 1 cubic foot. One vertical column of such cubes is drawn. There are 3 cubes in it. Similar columns will stand on each of the 9 squares of the base, so that the cubic yard will contain $3 \times 9 = 3 \times 3 \times 3$ (or 27) cubic feet.

For a very large body, such as the Earth, we should use cubic miles. Each cubic mile would, of course, contain 1760^3 cubic yards.

In dealing with this number 1760^3 should be left as a power, and not multiplied out as 5,451,776,000 until it is certain that this will be necessary.

For solids which are loose, such as sand, corn, &c., "measures of capacity" may be used.

54. *Capacity.*—It should be remembered that *a gallon of water weighs* 10 *lbs. av., and contains* $277\frac{1}{2}$ * *cubic inches.* This is sometimes remembered by the rhyme:

A pint of pure water
Weighs a pound and a quarter.

Distinguish carefully "quart," "quarter" = 8 bushels, and "quarter" = 28 lbs. av.

In the United States the gallon contains only 231 cubic inches, and all the other measures are smaller in the same proportion; also the pint of "liquid measure" is only about six-sevenths of the pint of "dry measure."

55. *Time.*† — The following additional details must be thoroughly known:—

(1) The "*Calendar Months*" differ in length from 28 to 31 days. This may be remembered by the rhyme—

Thirty days hath *September*,
April, *June*, and *November*.
All the rest have thirty-one,
Excepting *February* alone,
Which hath but twenty-eight days clear,
And twenty-nine in each leap year.

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* More exactly 277·463. $6\frac{1}{4}$ galls. make almost exactly a cubic foot.
† For "Problems on the Calendar" see Sect. IX., §§ 242, 243.

(2) *Rule for Leap Years.*—A year is a leap year

(*a*) When the last two figures of its number are exactly divisible by 4.

E.g. 1896 is a leap year, because 96 is divisible by 4. So is 1904.

(*b*) For the centuries, however, when the number of the *century* is exactly divisible by 4.

E.g. 2000 is a leap year, because 20 is divisible by 4; but 1900 is *not* a leap year, because 19 is not exactly divisible by 4.

(3) *Lunar Month.*—This is sometimes, incorrectly, given as 28 days, or 4 weeks. A more accurate measure is $29\frac{1}{2}$ days.

56. The following measures should be known:—

(1) 12 *units* = 1 *dozen*
12 *dozens* = 1 *gross*
20 *units* = 1 *score*

(2) For Paper.
24 *sheets* = 1 *quire*
20 *quires* = 1 *ream*

(2) *A cubic foot of water weighs nearly* 1000 *oz. av.*

A *ton* in shipping indicates a carrying capacity of 40 cubic feet. It is used as a measure of weight for heavy articles such as pig-iron, but not for lighter goods.

40 cubic feet of coal weigh about a ton.

(3) *Carats.*—When gold is described as being "22-carats fine," it means that *in* 24 *ounces* there are 22 pure gold.

Coinage gold is 22-carat. Gold chains are made of 14-18 carats, inferior jewellery of 9-carat. The "carat" is also a measure of weight for diamonds. It varies in different countries; but in England is 3·17 grains.

(4) The diameter of a halfpenny is exactly 1 in.
5 halfpennies weigh 1 oz. av.
3 pennies weigh 1 oz. av.

CHAPTER VIII.

REDUCTION OF MONEY.

57. Quantities are called **Simple** when, in naming them, one unit only is used.

Ex.—£2, 11s., 6d., 17 ft.

Quantities are called **Compound** when, in naming them, more than one unit of the same kind is used.

Ex.—£2 11s. 6d., 3 yds. 2 ft.

Reduction is the process by which we change from one unit, or set of units, to others of the same kind.

Ex.—One pound is 20 shillings.
£3 4s. is 3 × 20s. + 4s. = 64s.
17 pence is 12 pence + 5 pence = 1s. 5d.

Reduction may be either from a larger unit (or **denomination**) to a smaller, or from a smaller to a larger. These cases are taken separately.

I. Reduction from a Larger to a Smaller Unit.

58. Ex. 1.—*Reduce* $6\frac{3}{4}$d. *to farthings.*

As 1 penny = 4 farthings, ∴ 6 pence = 6 × 4 farthings = 24 farthings;
∴ $6\frac{3}{4}$d. = 24 + 3f. = 27 farthings. *Ans.*

Ex. 2.—*Reduce* 14s. 6d. *to pence.*

As 1 shilling = 12 pence, ∴ 14 shillings = 12 × 14 pence = 168 pence;
∴ 14s. 6d. = 168 + 6 pence = 174 pence. *Ans.*

Ex. 3.—*Reduce* £13 *to shillings.*

£13 = 13 × 20s. = 260 shillings. *Ans.*

From these examples it is clear that, in reducing from a larger to a smaller unit, the process adopted is one of multiplication, with addition when necessary, as in Examples 1, 2.

N.B.—Always put down what your answer represents. Thus, 27 farthings, NEVER 27 only.

59. Turning back to § 49, it is seen that

(1) £1 = 2 half-sovereigns = 4 crowns = 8 half-crowns
= 5 double florins = 10 florins.
(2) 1 half-crown = 5 sixpences.
(3) 1 shilling = 2 sixpences = 4 threepences.*

Reduction may take place, therefore, into any one of these intermediate terms:—

Ex. 4.—*How many half-crowns in £6?*

£1 = 8 half-crowns ∴ £6 = 6 × 8 half-crowns = 48 half-crowns.

Ex. 5.—*Reduce £3 to florins.*

£1 = 10 florins; ∴ £3 = 30 florins

60. We now apply the method to more complicated examples.

Ex. 6.—*Reduce* £14 6s. 10½d. *to farthings.*

£	s.	d.
14	6	10½
20		
280		
6		
286	shillings	
12		
3432		
10		
3442	pence	
4		
13768		
2		
13770	farthings	

The actual working is as follows:—

£	s.	d.		£	s.	d.
14	6	10½	*or*	14	6	10½
20				286 s.		
286 s.				3442 d.		
12				13770 f.		
3442 d.						
4						
13770 f.						

Each step is an example of combined multiplication and addition.

After a time you may safely omit the multipliers, but should never omit the name of the unit.

Ex. 7.—*Reduce* £120 18s. 6d. *to sixpences.*

£	s.	d.
120	18	6
20		
2418 s.		
2		
4837 six-pences.		

Explanation.—One shilling is 2 sixpences; therefore, at the last stage, we multiply by 2 and add 1. Be careful not to add *6* instead of 1. Practise omitting the multipliers.

* Such results should be not so much learned as practised with the actual coins until familiar. Toy coins may be obtained for the purpose.

When the sum of money to be reduced just falls short of an exact number of shillings, or an exact number of pounds, we may work as in the following examples:—

Ex. 8.—*Reduce* 4s. 11¾d. *to farthings.*

This is just ¼d. less than 5s.; therefore we reduce 5s. to farthings, and subtract 1. *Ans.* 239.

Ex. 9.—*How many sixpences in* £11 18s. 6d. *?*

This is 1s. 6d., *i.e.* 3 sixpences, less than £12. Reduce £12 to sixpences, and subtract 3. *Ans.* 477.

Verification.—(1) The converse process (see II. below) may be employed.

(2) The reduction may be effected in several ways, *e.g.* in Example 6 £14 may be reduced to farthings, then 6s. also, then 10d., then ½d., and the results added.

EXAMPLES XX.

Reduce to farthings—

1. 1s. 6d., 2s. 6d., 3s., 6s. 8d., 7s. 11¾d.

2. 5s. 7½d., 3s. 3¾d., 4s. 11¼d., 14s. 1¼d., 17s. 2d., 8s. 11¾d.

3. £1 3s. 1d., £2 16s. 4½d., £8 9s. 2d., £14 6s. 1¾d.

4. £80 0s. 4d., £40 10s. 7¼d., £116 7s. 0¾d., £301 1s. 4½d.

5. £19 19s. 11¾d., £47 0s. 11½d., £861 17s. 6¼d., £159 17s. 11¾d.

6. 10 half-crowns, 5 guineas, 85 florins, 17 half-guineas.

Reduce to pence—

7. £1 2s. 6d., £3 3s., £5 13s. 11d., £8 14s. 1d.

8. £68 7s. 4d., £60 0s. 6d., £120 1s., £25 19s. 11d.

9. £29 19s. 11d., £320 16s. 8d., £719 19s. 10d., £1000.

10. 27 florins, 86 half-guineas, 52 double florins, 271 guineas, 87 half-sovereigns.

How many halfpence in?—

11. £3, £14 10s., 8s. 6d., 3 half-crowns.

12. 217 guineas, £73 4s. 10½d., £1 19s. 10½d.

13. £18 4s. 1d., 420 half-guineas, 283 shillings, 721 sixpences.

14. Reduce to sixpences—£10, £10 10s., 420 half-crowns, 273 guineas, £19 19s. 6d.

15. How many shillings in £100, 100 guineas, £215 15s., £820 10s., £1720 19s., £99 19s.?

16. Reduce (1) to half-crowns, (2) to threepences—£20, £19 17s. 6d., £41 2s. 6d., £850 10s., £17 12s. 6d.

Reduce—

17. £111 11s. 11¼d. to farthings.

18. £18 4s. 6d. to sixpences.

19. £117 5s. to crowns.

20. £812 17s. 11d. to pence.

21. £941 16s. to double florins.

22. £82 11s. 3d. to threepences.

23. £10 to halfpence.

24. £142 18s. 2d. to pence.

25. £123 4s. 8d. to fourpences.

26. How many boys could each receive sixpence out of £17 12s. 6d.? How many could receive half-a-crown?

27. Reduce the following amounts to pence, and add all the results together:—£3, 3 half-crowns, 3 guineas, 3 crowns, 3 half-sovereigns, 3 florins, 3 sixpences, 3 shillings.

28. What number of halfpenny stamps could be bought for £7 10s.

29. How many penny articles can be bought for £14 15s. 3d.?

30. If one article cost half-a-crown, how many such can be purchased for £50 10s.?

II. Reduction from a Smaller to a Larger Unit.

61. If you had to find the value of a large heap of farthings, the quickest way to do this would be to arrange them in piles of 4 each, and then to group these piles in dozens. Each pile would represent a penny, each group of piles a shilling. Thus, 103 farthings would give 25 piles and 3 farthings over. The 25 piles would give 2 groups of a dozen piles each and 1 pile over. Thus we have altogether 2 groups, 1 pile, 3 farthings, *i.e.* 2s. 1¾d. But the number of piles is the number of times that 4 is contained in, *i.e.* will divide into, the number of farthings; therefore our result is obtained by dividing first by 4 to get the number of pence-piles, then by 12 to get the number of shilling-groups.

Ex. 1.—*Reduce* 8682 *farthings to* £ s. d.

4) 8682
12) 2170 pence and **2** farthings over, *i.e.* one halfpenny.
20) 180 shillings and **10** pence over.
9 pounds and **0** shillings over. *Ans.* £9 0s. 10½d.

Ex. 2.—*Reduce* 867 *threepences to* £ s. d.

4) 867
20) 216 shillings and 3 threepences, *i.e.* **9**d. over.
10 pounds and **16**s. over. *Ans.* £10 16s. 9d.

62. If the units named in the question have no direct

relation with one another (*e.g.* pounds and guineas, florins and half-crowns), we must seek for some unit which is contained an exact number of times in each, and work through this unit. Thus, to reduce pounds to guineas, we should reduce pounds to shillings, and then shillings to guineas. Be careful to choose the largest unit which is contained exactly in both. Thus, in reducing florins to half-crowns, we must work through sixpences, and *not* through pence. These questions, however, are better done by the methods of FRACTIONS. (See Chap. xix.).

Ex. 3.—*How many guineas are there in* £840 *?*

Work thus—

$$\begin{array}{r} £840 \\ 20 \\ \hline 16800 \end{array} \text{ shillings.} \qquad 21\left\{\begin{array}{l} 3\,)\,\underline{16800} \text{ shillings} \\ 7\,)\,\underline{5600} \\ \quad 800 \text{ guineas.} \end{array}\right. \quad \textit{Ans.}$$

Ex. 4.—*Reduce* 83 *half-crowns to florins.*

$$\begin{array}{r} 83 \text{ half-crowns} \\ \underline{5} \\ 4\,)\,\underline{415} \text{ sixpences} \\ 103 \text{ florins and 3 sixpences over.} \end{array}$$

Ans. 103 florins and 1s. 6d. over.

Explanation.—There are 5 sixpences in a half-crown; therefore 83 half-crowns contain 83×5 sixpences $= 415$. There are 4 sixpences in a florin; therefore 415 sixpences contain $415 \div 4$ florins $= 103$ florins and 3 sixpences over.

Ex. 5.—*Reduce* 327 *guineas to pounds and shillings.*

To reduce guineas to pounds and shillings, you must remember that one guinea is £1 + 1s., and, therefore, that 327 guineas is £327 + 327s., *i.e.* £327 + £16 7s., *i.e.* £343 7s.

EXAMPLES XXI

Reduce to pounds, shillings, &c.—

1. 7491 farthings, 8723 pence, 10,000 farthings, 8961 pence.

2. 7236 halfpence, 847 threepences, 49632 farthings.

3. 896 half-crowns, 1000 florins, 9000 sixpences.

4. 92634 farthings, 7413 half-crowns, 49136 pence.

5. 7681 florins, 941723 shillings, 2491763 farthings.

6. 7244 threepences, 961 farthings, 241 pence, 1941 crowns.

7. How many guineas in (anything over express in shillings, pence, &c.) £210, 14700 shillings, 826 sixpences, £300, 24713 pence, 738064 farthings, 499968 farthings?

8. Reduce 7823 guineas to pounds and shillings.

9. What number of half-crowns are contained in 720 florins, in 845 sixpences, in 6705 guineas, in 96480 farthings?

10. Reduce 731 guineas to pounds and shillings.

11. What number of florins are contained in 320 half-crowns, in 847 sixpences, in 1111 shillings?

Reduce (express what is over in shillings, pence, &c.)—

12. 7059 half-crowns to shillings.

13. 4071 pence to florins.

14. 71999 threepences to £. s. d.

15. 48061 farthings to £. s. d.

16. 4713 guineas to £. s.

17. What will be the cost of 7247 penny stamps?

18. If 100,000 people contribute half-a-crown each to a fund, what will be the total collected?

19. Find the cost of 7043 articles at 1d. each.

20. What will be the cost at $\frac{1}{2}$d. each of 7 gross of pencils? (1 gross = 12 dozen.)

CHAPTER IX.

REDUCTION: WEIGHTS AND MEASURES.

63. The principles involved are similar to those used in reduction of money. We illustrate by examples.

I. Avoirdupois Weight, Troy Weight, &c.

Ex. 1.—*Reduce* 3 *tons* 11 *cwts.* 15 *lbs.* 6 *oz. to ounces.*

tons	cwts.	qrs.	lbs.	oz.
3	11	0	15	6
20				
71 cwts.				
4				
284 qrs.				
28				
568				
2287				
7967 lbs.				
16				
7967				
47808				
127478 oz.				

or

tons	cwts.	qrs.	lbs.	oz.
3	11	0	15	6
71 cwts.				
284 qrs.				
1136 } lbs.				
7967 } lbs.				
31868 } oz.				
127478 } oz.				

Explanation.—The multipliers 20, 4 are omitted. The multiplier 28 is replaced by 4×7. 284×4 gives 1136, and 1136×7, with 15 added, gives 7967. *Be very careful* not to add the 15 until the second of these factor-multiplications. The multiplier 16 is replaced by 4×4, and the 6 is added during the second of these multiplications.

Apothecaries' Weight.—Great care will be needed in dealing with this table on account of the changes introduced in 1885 (see § 50). If a question is given involving ounces, drachms, scruples, and grains, the reference is probably to the older Apothecaries' Weight, in which 8 drachms made 1 ounce, so that the ounce contained 480 grains, and was the same as the ounce troy. This ounce is sometimes still used in prescriptions.

Troy Weight involves no such difficulty.

Ex. 2.—*Reduce* 782351 *grs. to pounds, ounces, &c.* (*troy*).

```
24 { 4 ) 782351 grs.
   { 6 ) 195587          3 remainder } Final remainder not 5, but 5 × 4 + 3
     20 )  32597 dwts.   5 remainder }   grs. = 23 grs.  (See p. 41.)
     12 )   1629 oz.    17 dwts.
            135 lbs.     9 oz.
```

Ans. 135 lbs. 9 oz. 17 dwts. 23 grs.

To reduce avoirdupois weights to troy weights, and *vice versa*, we must first reduce to grains, and remember that

1 *pound avoirdupois* = 7000 *grains*,
1 *pound troy* = 5760 *grains*.

II. Length, Area (not introducing poles or square poles).

Ex. 3.—*Reduce* 3 *mls.* 1 *furl.* 2 *yds.* 1 *ft.* 7 *in. to inches.*

```
 mls.    furl.  yds.  ft.  ins.
  3       1      2    1    7
  8
 25 furls.
220
 50
502
5502 yards
   3
16507 feet
  12
198091 inches
```

Ans.

or

```
 mls.  furl.  yds.  ft.  ins.
  3     1      2    1    7
 25 furls.
 500 }
5502 } yds.
16507 ft.
198091 ins   Ans.
```

Explanation.—The multipliers are omitted. The multiplier 220 is replaced by 20 × 11, and the 2 is added in on the second of these factor-multiplications. *Be very careful* not to forget the 0 here.

Ex. 4.—*Reduce* 12345678 *sq.in. to square feet, yards, &c.*

```
144 { 12 ) 12345678 sq.ins.
    { 12 )  1028806          6 remainder } Final remainder not 10 sq. in.,
      9 )    85733 sq.ft.,   10 remainder }   but 10 × 12 + 6 = 126 sq.in.
   4840 )     9525 sq.yds. 8 sq.ft.                [(See p. 41.)
              1 ac. 4685 sq.yds.
```

Ans. 1 ac. 4685 sq.yds. 8 sq.ft. 126 sq.in.

N.B.—If the number of square yards is very great, the division by 4840 is, of course, performed by long division. It might be worked as 40 × 11 × 11 by the method of p. 41; but there is little advantage in this. The division by 640 needed to bring acres to square miles would be worked as 80 × 8.

III. Capacity.

Ex. 5.—*How many quarters, bushels, &c., are there in* 654321 *pints? Verify your answer.*

2) 654321	pts.
4) 327160	qts. 1 pt.
2) 81790	galls. 0 qt.
4) 40895	pks. 0 galls.
8) 10223	bush. 3 pks.
1277	qrs. 7 bush.

Verification.

qrs.	bush.	pks.	galls.	qts.	pt.
1277	7	3	0	0	1

10223 bush.
40895 pks.
81790 galls.
327160 qts.
654321 pts.

IV. Time.—A reduction question in time is often very vaguely stated.

If "weeks, days, hours, &c.," are named, there is no difficulty.

If "months, weeks, days, &c.," are named, you are intended to take 4 weeks = 1 month.

If "years, weeks, days, &c.," are named, you are intended to take 52 weeks = 1 year.

If "years, days, hours, &c.," are named, you are intended to take 365 days = 1 year.

If "years, months, weeks, days, &c.," are named, the question is indefinite; but you are probably intended to take 1 year = 52 weeks, and 1 month = 4 weeks. The answer should, however, state this, *e.g.* 3 years (of 52 weeks), 2 months (of 4 weeks), 3 weeks, 2 days.

Ex. 6.—*How many seconds are there in* 100 *years?*

(1) If a "year" means 365 days, there are 36,500 days in 100 years, and, therefore, 36500 × 24 × 60 × 60 seconds = 3,153,600,000 seconds.

(2) If a "year" means $365\frac{1}{4}$ days, one day is added to the 365 every four years; therefore in 100 years 25 days are added; therefore there are 36525 days, *i.e.* 3,155,760,000 seconds.

(3) If the date is specified, great care is needed. In the century commencing January 1, 1801, there are 36,524 days, *not* 36,525, while in the century commencing January 1, 1901, there are 36,525 days. See § 55 (2) and also §§ 242, 243.

V. Length, Area (Complete).

[*N.B.—This section is to be deferred until Multiplication and Division of Fractions have been mastered.**]

Ex. 7.—*Reduce* 21 *pls. to yards.*

```
 21
 5½
-----
105
 10½
-----
115½
```

Since 1 pole = $5\frac{1}{2}$ yards, 21 poles = $21 \times 5\frac{1}{2}$ yards. The work is arranged as shown, the multiplication by 5 and that by $\frac{1}{2}$ being performed in two separate lines.

Ex. 8.—*Reduce* 21 *pls.* 1 *yd.* 2 *ft.* 1 *in. to inches.*

```
pls.  yd.  ft.  in.        pls.  yd.  ft.  in.
 21    1    2    1   or    21    1    2    1
 5½                         106 }
-----                       10½ }
106                        116½ yds.
 10½                       351½ ft.
-----                     4219  ins.
116½ yds.
  3
-----
351½ ft.
 12
-----
4219 inches
```

Explanation.—We multiply first by 5, adding in the 1 yard at this stage, then by $\frac{1}{2}$, and add these together. At the next stage the working is:—3 times $\frac{1}{2}$, $1\frac{1}{2}$; write $\frac{1}{2}$ and carry 1; 3 times 6, 18, *and* 1, 19, and *2* (2 feet given), 21; write 1, and carry 2, &c.

Ex. 9.—*Reduce* 762998 *ins. to miles, &c.*

```
12 ) 762998 ins.
 3 )  63583 ft. and 2 ins.
      21194 yds. and 1 ft.
          2
11 )  42388 half-yds.
40 )   3853 pls. and 5 half-yds = 2½ yds.
 8 )     96 furls. and 13 pls.
         12 mls. and 0 furl.
```

Explanation. — The only difficulty is to divide by $5\frac{1}{2}$. But, to divide by $5\frac{1}{2} = \frac{11}{2}$ is (see § 118) the same as to multiply by 2, and then divide by 11. To multiply by 2 turns yds. into half-yds. Therefore the remainder, on dividing by 11, is a remainder of so many half-yds. The first form of the answer is

* If a rule need be taught at all which an educated man who is delivered from the bondage of examinations will employ not more than half-a-dozen times in his life, it should certainly be deferred until the subject of fractions has been mastered. It is logically unsound to attempt to master or evade the $5\frac{1}{3}$, $30\frac{1}{4}$ difficulties until this stage has been reached. Hours will be wasted in the attempt, and the most probable result will be an unintelligent use of a rule of thumb.

12 mls. 0 furl. 13 pls. $2\frac{1}{2}$ yds. 1 ft. 2 in. But $\frac{1}{2}$ yd. is 1 ft. 6 ins. Therefore the answer is 12 mls. 0 furl. 13 pls. 2 yds. 1 ft. 2 ins. + 1 ft. 6 in.: *i.e.* is 12 mls. 0 furl. 13 pls. 2 yds. 2 ft. 8 in. *Ans.*

Ex. 10.—*To reduce* 3 *acs.* 2 *rs.* 27 *po.* 27 *yds.* 7 *ft.* 25 *ins. to inches.*

The mention of "acres" shows that the poles, yards, feet, and inches are square measure.

acs.	rs.	pls.	yds.	ft.	in.
3	2	27	27	7	25
4					
14 rs.					
40					
587 pls.					
$30\frac{1}{4}$					
17637					
$146\frac{3}{4}$					
$17783\frac{3}{4}$ yds.					
9					
$160060\frac{3}{4}$ ft.					
12					
1920729					
12					
23048748 ins.					
25					
23048773 inches.					

acs.	rs.	pls.	yds.	ft.	ins.
3	2	27	27	7	25
14 rs.					
587 pls.					
17637					
$146\frac{3}{4}$					
$17783\frac{3}{4}$ yds.					
$160060\frac{3}{4}$ ft.					
1920729 } in.					
23048773 } in.					

Explanation.—(Full work.) The working is as follows:—

For roods: 4 *threes*, 12, and **2** (given), 14. Thus, 14 roods.

For poles: We have to multiply 14 by 40 and add the **27** (given). Write **7**; then 4 fours, 16, and 2 (from the 27), 18. Write 8 and carry 1.

4 ones 4, and 1, 5. Thus, 587 pls.

For yards: The multiplication by 30 is similar to that just made by 40; the **27** (given yards) is added in here. The multiplication by $\frac{1}{4}$ is division by **4**. These results are added.

For feet: Multiply by 9, and at the same time add the **7** (given).

For inches: We must multiply by 144, *i.e.* by 12×12. The **25** (given) is added *at the second of these multiplications, not at the first.* It is easy to do this addition at the same time as the multiplication, as is done in the contracted work.

Ex. 11.—*To reduce* 368427 *sq. ins. to square ft., yards, &c.*

```
144 { 12 ) 368427
    { 12 )  30702        3 } Final remainder
        9 )   2558 sq ft. 6 } 6 × 12 + 3 = 75 sq. ins.
              284 sq. yds. 2 sq. ft.
                4
121 { 11 )   1136 qr. yds. (sq.)
    { 11 )    103        3 } Final remainder
                9 p.     4 } 4 × 11 + 3 = 47 qr. yds. = 11¾ yds.
```

Explanation.—Division by 144 is worked by dividing twice by 12. Division by $30\frac{1}{4}$ (*i.e.* by $\frac{121}{4}$) is the same as to multiply by 4, turning yds. into qr. yds., and then divide by 121 (*i.e.* by 11 × 11). The remainder is a remainder of qr. yds. Thus the answer is at first obtained as 9 pls. $11\frac{3}{4}$ yds. 2 ft. 75 ins. (sq.). But $\frac{3}{4}$ yd. is 6 ft. 108 ins., and therefore the answer is better written 9 pls. 11 yds. 2 ft. 75 ins. + 6 ft. 108 ins. : *i.e.* 9 pls. 12 yds. 0 ft. 39 ins. *Ans.*

EXAMPLES XXII.

Avoirdupois.

Reduce to pounds—

1. 4 tons 15 cwts. 3 qrs. **2.** 227 cwts. **3.** 1 ton 2 cwts. 3 qrs. 4 lbs.

How many ounces in ?—

4. 2 tons 7 cwts. 1 qr. 7 lbs. 8 oz. **5.** 7 tons 14 cwts. 3 qrs.
6. 2 tons 14 cwts. 13 lbs. **7.** 15 lbs. 15 oz.
8. 1 ton 1 cwt. 1 lb.
9. How many drams in 52 tons 6 cwts. 2 qrs. 17 lbs. 15 drs.?
10. How many grains (troy) in 57 lbs. (avoir.) ?

Reduce—

11. 15 tons to pounds. **12.** 3 tons 11 cwts. 15 lbs. to ounces.
13. 4 qrs. to drams. **14.** 34 lbs. 15 oz. to ounces.
15. 3 tons 2 cwts. 7 oz. to ounces.

Reduce to tons, hundredweights, quarters, &c.—

16. 5781 lbs. **17** 9637 lbs. **18.** 42874648 drs.
19. 735934 oz. **20.** 81584 drs. **21.** 65173 oz.
22. 5279 oz.

Troy.

Reduce to grains—

23. 3 lbs. 7 oz. 14 dwts 17 grs. **24.** 3 lbs. 5 oz. 2 dwts. 3 grs.
25. 29 lbs. 3 oz. 13 dwts. 21 grs. **26.** 3 lbs. 7 oz. 5 dwts. 9 grs.
27. 5 oz. 17 dwts. **28.** 3 lbs. 6 oz. 7 dwts. 17 grs.

Reduce to pounds, ounces, &c.—

29. 19857 grs. **30.** 15897 grs. **31.** 437625 grs.
32. 6325 grs. **33.** 7251 oz. **34.** 7612 dwts.
35. 94623 grs.
36. How many pounds (avoir.) in 144 lbs. 7 oz. 8 dwts. 8 grs.?

Length (omitting poles).

Reduce—

37. 3 mls. 3 furls. to feet. **38.** 17 mls. 3 yds. to inches.
39. 500 yds. to inches. **40.** 22 furls. to yards.
41. 7 mls. 6 furls. to yards. **42.** 18 mls. to yards.
43. 3 mls. 4 furls. to inches. **44.** 4 miles 7 furls. 11 yds. to feet.
45. 5 mls. 2 furls. 3 yds. to feet.

Reduce to miles, furlongs (omit poles), yards, &c.—

46. 263369 ins. **47.** 28504 ft. 6 ins. **48.** 86130 ins.
49. 100,000 yds. **50.** 374621 ft. **51.** 19,000,000 ins.
52. 433206 ft.

Area (omitting square poles).

Reduce—

53. 10 acs. to square yards. **54.** 30 acs. 2 rs. to square feet.
55. 50 sq.yds. to square inches. **56.** 217 acs. 3 rs. to square yards.
57. 21 acs. to square feet.
58. 30 acs. 3 rs. 506 sq.yds. 3 sq.ft. 112 sq.ins. to square inches.

Reduce to acres, roods (omit square poles), square yards, &c.—

59. 84304202 sq.ins. **60.** 9680 sq.yds. **61.** 47631 sq.ft.
62. 4963 sq.ins. **63.** 90,000,000 sq.ins. **64.** 72463 sq.yds.

Capacity.

Reduce—

65. 3 qrs. 1 bush. 2 pks. 3 qts. to pints.
66. 13 qrs. 2 bush. 2 pks. 1 gall. to gallons.
67. 5 qrs. 6 bush. 6 galls. to gallons.
68. 1 qr. 2 bush. 2 pks. 3 qts. to pints.

Reduce to quarters, bushels, &c.—

69. 2559 pts. **70.** 1715 qts. **71.** 668 pts.
72. 253721 pts. **73.** 1741 pts. **74.** 46321 pts.

Time.

Reduce—

75. 13 hours 10 minutes to seconds.
76. 4 weeks 2 days 11 hours to minutes.
77. 9 weeks 1 day 7 hours 12 minutes 35 seconds to seconds.

Reduce—

78. 69 days 10 hours 39 seconds to seconds.

79. 7 weeks 3 days 7 hours 11 minutes to minutes.

80. 365 days to minutes.

81. How many minutes in January, in November, in February (1898)?

82. Find the total number of hours in the last six months of the year.

Reduce to years (*i.e.* 52 weeks), weeks, days, &c.—

83. 237249162 minutes.

84. 162360150 seconds.

85. 92317 hours.

86. 110543 seconds.

87. 42631 days.

88. 9241 weeks.

Length (General).

[Examples 89-110 should be omitted until § 118 has been reached.]

Reduce—

89. 8 mls. 4 furls. 38 pls. 3 ins. to inches.

90. 7 furls. 4 pls. 1 ft. 4 ins. to inches.

91. 5 leagues 2 mls. 6 furls. 7 pls. to feet.

92. 1 ml. 7 furls. 28 pls. 2 yds. to yards.

93. 3 mls. 2 furls. 4 pls. 5 yds. 2 ft. 5 ins. to inches.

Reduce to miles, furlongs, poles, yards, &c.—

94. 241985 ins.

95. 47715853 ins.

96. $3907\frac{1}{4}$ yds.

97. 12358 yds.

98. 847659014 ins.

99. 1014468 ft.

100. 167382 ins.

Area (General).

Reduce—

101. 5 acs. 3 rs. 4 pls. 7 yds. to square feet.

102. 23 acs. 1 r. 27 pls. 24 yds. 6ft. 103 ins. to square inches.

103. 3 acs. 1 r. 4 pls. 3 yds. to square feet.

104. 15 acs. 3 rs. $19\frac{3}{4}$ yds. to square feet.

105. 3 sq.pls. 4 sq.yds. 4 sq.ft. 84 sq.ins. to square inches.

Reduce to acres, roods, poles, yards, &c.—

106. 34760583 sq.ins.

107. 87964257 sq.ins.

108. 345241 sq.ft.

109. 10,000 sq.yds.

110. 8435798 sq.ins.

Volume.

Reduce—

111. 3 c.yds. 23 c.ft. 1119 c.ins. to cubic inches.

112. 14 c.yds. 7 c.ft. 470 c.ins. to cubic inches.

113. How many cubic yards in a cubic mile?

114. Reduce 361662 c.ins. to cubic yards, &c.

CHAPTER X.

THE FOUR RULES: MONEY.

64. In the following chapters (X. and XI.), which treat of the addition, subtraction, multiplication, and division of concrete quantities (§ 1), there is no really new principle involved; but since, in counting money, we count first in groups of fours (making farthings into pence-groups), then in twelves (making pence into shillings), then in twenties (making shillings into pounds), instead of always in tens, as in Chaps. ii. to v., we must modify the processes of these chapters accordingly. The addition, subtraction, &c., of concrete quantities is called *compound addition*, &c. The quantities must, of course, be expressed in units of the same kind.

Ex. 1.—*Add together* £18. 16s. 4d., £741. 3s. 6½d., £160. 14s. 8¾d., £427 8s. 6¼d.

	£	s.	d.		
	18	16	4		
	741	3	6	½	
	160	14	8	¾	
	427	8	6	¼	
	1346	41	24	6f	A
i.e.	1346	1	0	½	B
	2	2	1		C
i.e.	1348	3	1	½	D

Explanation.—This is exactly on the lines of § 11. The sums of money are carefully arranged—farthings under farthings, pence under pence, and so on. The beginner may make four columns, £ s. d. f., and write £741. 3s. 6½d., for instance, as £741. 3s. 6d. 2f.; we then add the columns separately. We now have the sum (line A), but it is not in its simplest form. For instance, there are 6 farthings: *i.e.*, 1 penny and 2 farthings. Put down the 2 farthings: *i.e.*, ½d. (line B) and add the 1 to the pence column (line C). So the 24 pence (line A) are 2 shillings (line C) and 0 pence (line B), and the 41 shillings are 2 pounds (line C) and 1 shilling (line B). The sum of B and C gives D, the final answer. In practice this is conducted as follows:—

Farthings: 1, 4, 6. But 6 farthings = $1\frac{1}{2}$d. Write $\frac{1}{2}$d., and carry 1 to pence column.

Pence: 1 (carried), 7, 15, 21, 25 = 2s. 1d. Write 1d. and carry 2 to shillings column.

Shillings: 2 (carried), 10, 14, 17, 23. Write 3 and carry 2 to the tens-of-shillings column; 2 (carried), 3, 4. But 4 tens-of-shillings = £2. Carry 2 to pounds column. Add the pounds as in ordinary addition.

Tests of Accuracy.—See § 13 (1), and § 71.

Hint.—The practice recommended in § 10 is a useful training in Compound Rules. A class should be drilled in such series as 6d., 1s., 1s. 6d., 2s., &c.

EXAMPLES XXIII.

Add together the following amounts:—

1. 2d., 1s. 3d., 2s., 7d., 10d., 1s. 1d.

2. 6d., 3s. 4d., 11d., 2s. 1d., 8d., 3s.

3. $3\frac{1}{2}$d., $4\frac{1}{4}$d., 10d., $3\frac{3}{4}$d., $9\frac{1}{2}$d., $7\frac{1}{4}$d.

4. $2\frac{1}{4}$d., $3\frac{1}{2}$d., $10\frac{3}{4}$d., 7d., 6d., $8\frac{1}{4}$d.

5. 1s. 7d., 4s. 1d., 11d., 3s. 6d., 4s. 6d., 7s.

6. 4s., 7s. 4d., 8d., 8s. 6d., 9s. 1d., 11d.

7. 4s. 8d., 7s. 3d., 10s. 5d., 14s. 7d., 16s. 10d., 18s. 1d.

8. 3s. 5d., 4s. 8d., 17s. 6d., 9s. 11d., 19s. 4d., 3s. 9d.

The following twenty examples (Nos. 9 to 28) may be read either from top to bottom or from left to right. The numbers in black type refer to the answers:—

9.		**10.**		**11.**		**12.**		**13.**		**14.**		
s.	d.	s.	d.	s.	d.	s.	d.	s.	d.	s.	d.	
3	6	2	1	4	7	3	2	8	9	4	10	**15.**
	7	1	9	6	1	4	8	9	7	8	1	**16.**
4	7	16	4	8	11	13	2	4	3	9	9	**17.**
17	9		11	15	3		3	13	7	16	7	**18.**
4	$2\frac{1}{4}$	7	$3\frac{1}{2}$	19	0	14	$7\frac{1}{4}$	11	0		$11\frac{3}{4}$	**19.**
8	$7\frac{3}{4}$	19	4		$9\frac{3}{4}$	17	$3\frac{1}{2}$	14	$5\frac{1}{2}$	10	$10\frac{1}{4}$	**20.**

21.			**22.**			**23.**			**24.**			
£	s.	d.	£	s.	d.	£	s.	d.	£	s.	d.	
13	4	7	11	2	1		19	7	5	3	6	**25.**
2	7	11	20	16	9	3	8	9		17	11	**26.**
	19	4	1	17	$3\frac{1}{4}$	17	13	$10\frac{1}{2}$	8	14	$9\frac{3}{4}$	**27.**
	17	$3\frac{1}{2}$		14	$8\frac{1}{2}$	2	9	6	1	9	$10\frac{1}{2}$	**28.**

29. A man paid away the following amounts:—£19 8s. 4d., £170 14s. 11d., £4 0s. 10d., £41 7s. 1d., £81 12s. 9d., £1 11s. 7d. How much did he pay altogether?

30. On Monday a man pays into his bank £18 17s. 10d., on Tuesday £81 10s. 7d., on Wednesday £101 0s. 4d., on Thursday £256 8s. 11d., on Friday £94 19s. 3d., on Saturday £84 17s. 6d. How much did he pay in altogether?

31. A man spends £150 18s. 4d. during each of four weeks. What does he spend in all?

32. Find the total of £201 19s. 6d. repeated four times.

Perform the following additions:—

33.

£	s.	d.
271	8	10
91	19	$4\frac{1}{2}$
87	17	$3\frac{3}{4}$
	19	0
		$7\frac{3}{4}$
195	3	6

34.

£	s.	d.
19	8	$7\frac{1}{4}$
133	13	11
127	6	0
781	3	9
	18	$11\frac{1}{2}$
1	0	$6\frac{1}{4}$

35.

£	s.	d.
981	17	11
39	4	$8\frac{3}{4}$
9	0	10
73	16	4
9	14	$0\frac{3}{4}$
	18	$4\frac{3}{4}$

36.

£	s.	d.
8146	0	$8\frac{1}{4}$
723	19	5
4073	4	$9\frac{3}{4}$
94	16	$0\frac{1}{2}$
327	8	$2\frac{1}{2}$
7609	0	11

37.

£	s.	d.
9460	17	3
20	4	$9\frac{3}{4}$
373	14	$0\frac{3}{4}$
2909	3	10
729	15	$7\frac{3}{4}$
1111	11	1

38.

£	s.	d.
2730	9	$8\frac{1}{2}$
4703	8	$9\frac{1}{2}$
256	0	$11\frac{1}{2}$
2909	17	$2\frac{1}{4}$
29	18	$5\frac{3}{4}$
470	3	1

39.

£	s.	d.
4097	0	0
246	18	$9\frac{3}{4}$
706	4	$9\frac{1}{4}$
2407	3	10
4	9	1
723	12	$2\frac{1}{4}$
2001	2	$1\frac{1}{2}$
990	19	9

40.

£	s.	d.
2407	19	8
246	3	$2\frac{1}{4}$
76	13	$2\frac{1}{4}$
9	3	$2\frac{1}{4}$
451	17	0
2415	9	$1\frac{1}{4}$
729	10	4
4306	9	$6\frac{1}{4}$

41.

£	s.	d.
9909	2	9
3303	0	$11\frac{1}{4}$
1101	16	$3\frac{1}{2}$
7	19	$3\frac{3}{4}$
17	2	$1\frac{3}{4}$
296	4	0
7020	13	$10\frac{1}{4}$
290	6	$8\frac{1}{4}$

42. If a man receives £130 8s. 6d. on Monday, and if on each succeeding day of the week up to and including Saturday he receives £2 13s. 6d. more than on the preceding day, find (1) how much he receives on the Saturday; (2) how much he receives during the whole week.

43. There are six articles, of which the first costs £17 14s. 9d., and each of the others costs a guinea and a half more than the preceding one. Find (1) the price of the sixth, and (2) the price of the whole.

44.

£	s.	d.
7463	18	11
216	4	7
809	13	4
7603	4	9
27	3	10
621	9	2
4716	8	5
1930	0	3
7964	2	7
8214	17	1
976	14	6
£328	15	8

45.

£	s.	d.
1749	18	11
240	3	9
709	15	6
216	3	1
9999	19	9
4334	17	10
2916	3	7
206	15	2
7	13	8
493	14	5
7931	0	0
475	0	6

46.

£	s.	d.
2401	14	5
3706	8	11
207	3	10
14	15	9
2704	13	1
4927	4	8
256	3	4
9	19	5
473	4	0
5245	2	6
249	9	7
23	14	2

47.

£	s.	d.
43456	7	8
19101	1	2
234	5	6
7890	14	11
23049	9	9
37216	6	6
15	17	10
491	18	9
12517	3	2
749	0	7
68686	8	6
4207	19	0

48.

£	s.	d.
10480	19	11
24801	9	1
496	10	4
3432	8	5
20725	14	3
64837	16	6
2418	3	4
942	12	10
34362	3	7
5214	15	0
12986	3	5
270	0	4

49.

£	s.	d.
205	14	5
17205	1	4
42502	15	11
798	3	2
25	15	3
23986	7	4
298	11	6
94621	3	7
62801	17	6
47	3	1
257	18	9
85908	14	4

50.

£	s.	d.
762491	7	4
37501	11	8
29063	15	7
407321	5	10
98634	2	11
528370	0	2
40623	17	6
249018	7	9
987654	3	2
10987	16	5
20015	14	3
41123	4	0
80791	19	9
274168	2	1
320073	18	11
32037	5	5
903198	19	8
29162	13	0

51.

£	s.	d.
396412	15	2
95003	17	3
26071	4	8
916362	5	4
82371	16	8
42016	17	5
281703	4	6
270316	18	4
25807	3	11
23508	11	3
900065	10	10
731982	11	11
250873	15	8
21625	17	1
258	0	6
40612	4	3
261430	7	4
280498	3	7

52.

£	s.	d.
498470	18	10
926314	3	2
4090	0	0
40376	2	1
333333	3	3
809765	15	9
27304	18	7
4761	14	5
247065	13	4
82519	19	11
290162	14	8
24703	8	11
941806	13	2
41823	8	5
2406	16	7
90137	12	5
204796	7	4
731303	3	9

Compound Subtraction (Money).

(The beginner is advised to read again §§ 14–20.)

65. In Compound Subtraction we find the difference between two concrete quantities which are expressed in different units of the same kind, or, in other words, *what quantity must be added to the smaller to make it equal to the larger.*

Ex. 1.—*Subtract* £14 7s. $2\frac{1}{2}$d. *from* £25 12s. $6\frac{3}{4}$d.

£	s.	d.
25	12	$6\frac{3}{4}$
14	7	$2\frac{1}{2}$
11	5	$4\frac{1}{4}$

Explanation.—We have to make the second line up to the first. This involves the four questions :—

(1) How many farthings added to $\frac{1}{2}$d. will make $\frac{3}{4}$d.? *Ans.* $\frac{1}{4}$d.

(2) How many pence added to 2d. will make 6d.? *Ans.* 4d.

(3) How many shillings added to 7s. will make 12s.? *Ans.* 5s.

(4) How many pounds added to £14 will make £25? *Ans.* £11.

Hence £11 5s. $4\frac{1}{4}$d. is wanted altogether to make the second line up to the first.

This result is called the remainder (because it is also what remains when the smaller sum is taken from the greater) or the difference (because it expresses how much one differs from the other). (See §§ 14, 15.) In the particular example just taken each of the four parts (£ s. d. f.) in the lower line is less than the corresponding part in the upper line. When this is not the case we have to employ the method of Equal Additions. (See § 18.)

Ex. 2.—*Subtract* £16 15s. $8\frac{1}{2}$d. *from* £23 9s. $6\frac{1}{4}$d.

£	s.	d.	f.
23	9	$6\frac{1}{4}$ *or*	1
16	15	$8\frac{1}{2}$ *or*	2
23	29	18	5
17	16	9	2
6	13	9	3

Explanation.—We add 4 farthings to top line and its equivalent 1 penny to lower line. We add 12 pence to top line and its equivalent 1s. to lower line. We add 20 shillings to top line and its equivalent £1 to lower line. The example now stands as in the second form, and the difference may be found at once, as in Ex. 1. The result is £6 13s. $9\frac{3}{4}$d.

The *mental work* in the example as it originally stood is as follows :—

Farthings : 2 and *three*, 5. Carry 1.
Pence : 1 (and 8) 9 and *nine*, 18. Carry 1. Omit words in brackets.
Shillings : 1 (and 15) 16 and *thirteen*, 29. Carry 1.
Pounds : 1 (and 16) 17 and *six*, 23.

As the stress of the voice falls on the words *three*, *nine*, *thirteen*, *six*, write down the corresponding figures in the answer.

66. *Test of Accuracy.* (See §§ 19, 20 and 71.)

Hints.—(1) Accustom yourself to subtract without always writing the smaller quantity below the larger. Thus, to take £86 14s. 8d. from £104 7s. 11d. *without rewriting.*

(2) To subtract 4s. 10d. from £10, think, mentally, of the £10 as £9 19s. 12d., and the work can be done at once.

Combined Addition and Subtraction.

67. It is of the greatest importance in "balancing accounts" to be able to do any number of additions and one subtraction in a single operation.

Ex. — *Add together* £81 15s. 6d., £17 0s. 3d., *and* 17s. 6d., *and subtract the sum from* £193 4s. 8d.

£	s.	d.
193	4	8
81	15	6
17	0	3
	17	6
93	11	5

Explanation.—The work is as follows:—

Pence: 6, 9, 15 = 1s. 3d., *and 5d.*, 1s. 8d. Write 5.

Shillings: 1, 18, 33 = £1 13s., *and 11s.*, £2 4s. Write 11.

Pounds: 2, 9, 10 *and 3*, 13, &c, as in ordinary addition.

You should compare this work carefully with that of § 20.

EXAMPLES XXIV.

Find the remainder in the following twelve examples:—

1.

£	s.	d.
99	18	4
23	14	11

2.

£	s.	d.
191	14	$7\frac{1}{2}$
89	15	$8\frac{3}{4}$

3.

£	s.	d.
256	13	4
17	19	3

4.

£	s.	d.
290	0	0
14	13	$0\frac{1}{2}$

5.

£	s.	d.
498	3	$4\frac{1}{4}$
23	19	$11\frac{1}{2}$

6.

£	s.	d.
500	0	0
2	19	$8\frac{3}{4}$

7.

£	s.	d.
2163	14	10
918	3	4

8.

£	s.	d.
7194	0	8
5037	2	$5\frac{1}{4}$

9.

£	s.	d.
9051	19	0
752	18	$4\frac{1}{2}$

10.

£	s.	d.
21716	5	$9\frac{1}{4}$
2913	4	8

11.

£	s.	d.
91732	14	$8\frac{3}{4}$
81732	15	9

12.

£	s.	d.
10000	0	0
9876	14	$8\frac{1}{4}$

Work Exx. 13–18 without arranging the quantities under one another.

13. From £168 11s. 4d. take £19 4s. 8d.

14. From £256 18s. $3\frac{1}{2}$d. take £128 9s. $4\frac{3}{4}$d.

15. From £781 19s. take £390 4s. $10\frac{1}{2}$d.

16. From £1000 take £850 19s.

17. From £4913 7s. 8d. take £2416 3s. 9d.

18. From £2417 8s. 5d. take £950 17s. 4d.

19. Find the difference between £19 4s. $8\frac{1}{4}$d. and £7 3s. 2d.

20. Find the difference between £28 14s. 6d. and £54 18s. $3\frac{1}{2}$d.

21. By how much is £19 3s. 4d. greater than £10 11s. 8d.?

22. By how much is £48 17s. 6d. less than £100 13s. 5d.?

23. The sum of two quantities is £864 3s. 5d. One of them is £432 11s. $8\frac{1}{2}$d. What is the other?

24. If the sum of two quantities is £9763 14s. 1d., and the smaller is £850 0s. 11d., what is the larger?

Fill up the missing lines in the following four addition sums without rearranging the lines:—

	£	s.	d.
25.	957	3	10
	....	.	.
	3821	16	2

	£	s.	d.
26.	537	14	8
	...	.	.
	1000	0	0

	£	s.	d.
27.	..	.	.
	471	15	$8\frac{1}{4}$
	500	15	9

	£	s.	d.
28.	...	.	.
	8765	3	4
	9432	17	1

29. The larger of two sums of money is £15 7s. 11d. What is the smaller when the two together amount to £20?

30. What is the difference between £99 19s. $11\frac{1}{2}$d. and £200?

31. The difference between two sums of money is £18 14s. 5d. Find the smaller, if the larger is £198 17s. 1d.

32. The smaller of two sums of money is £1801 4s. 7d. What is the larger, if the difference between them is £740 18s. 9d?

33. What must be added to £571 13s. 9d. to make up £1001 4s. 11d.?

34. What must I take from £1875 4s. 8d. so that I may have £1114 16s. 8d. left?

In Exx. 35 to 40 the second and following quantities are to be added together and subtracted from the first.

	£	s.	d.
35.	234	6	8
	17	15	9
	4	3	2
	156	14	10

	£	s.	d.
36.	571	18	$10\frac{1}{4}$
	200	3	9
	20	14	$0\frac{1}{2}$
	193	0	$10\frac{1}{4}$

	£	s.	d.
37.	1483	17	11
	941	8	$1\frac{1}{4}$
	432	2	6
	5	0	$9\frac{3}{4}$

38.

£	s.	d.
2168	17	5½
256	17	5
82	14	0
91	16	3
22	18	1

39.

£	s.	d.
9416	0	0
2703	1	11½
491	14	7
2608	3	1
28	17	10¼

40.

£	s.	d.
10000	0	0
3416	0	0
725	17	4½
91	19	3¼
8	3	11½

In Exx. 41 to 43 you are given an incomplete addition sum. One line is missing, and you are required to find it.

41.

£	s.	d.
947	3	2
17	16	4
	18	11
.....		
1000	0	0

42.

£	s.	d.
583	19	4½
27	6	5
3	17	4¼
.....		
985	19	3

43.

£	s.	d.
.....		
856	2	7¼
278	14	8
126	5	9½
1681	3	10¼

44. Subtract the sum of £18 11s. 4d., £23 17s. 11d., £81 6s. 8d. from £300.

45. What is the difference between £469 18s. 4d. and the sum of £17 4s. 1d., £18 4s. 10d., £19 17s. 6d.?

46. What is left out of £40 after paying bills of £3 17s. 6d., £10 18s. 1d., £4 8s. 5d., £19 19s.?

47. A man owes the following sums of money:—£4 8s. 7d., £11 8s. 3d., £17 0s. 6d., £18 15s. 11d.; whilst the following sums are due to him:—£6 8s. 11d., £17 3s. 1d., £23 8s. 4d., £1 19s. 4d. Is he in debt or otherwise, and by how much?

48. Find the value of £23 11s. 8d. − £17 16s. 2d. − £1 11s. 8d + £14 19s. 11d. − £7 6s. 8d.

49. Find a single sum of money which is equivalent to £93 4s. 9d. + £17 6s. 8d. − £23 11s. 5d. − £16 4s. 2d.

50. A man has in his pocket £5; he loses 10s., gives away 25s., pays two bills of £1 7s. 8d. and £1 11s. 4d. How much has he left?

51. A man buys a house for £750, spends £160 18s. 4d. in furnishing it, and then sells house and furniture together for £1000. What is his gain?

52. Add the sum of £27 3s. 4d. and £19 11s. 5½d. to their difference.

53. From the sum of £43 16s. 7d. and £15 8s. 4d. subtract their difference.

54. From what sum of money must I subtract £8915 15s. 9d. so as to leave £858 4s. 11d.?

55. From £1001 19s. 6d. I deduct a certain sum, and have £756 18s. 7d. left. How much do I deduct?

56. The receipts in a certain business for six days are £178 11s. 4d., £95 18s. 2d., £110 15s. 4d., £152 18s. 6d., £201 17s. 7d., £99 11s. 9d. The payments for the same time are £96 15s. 11d., £104 17s. 7d., £49 8s. 5d., £106 11s. 8d., £121 14s. 5d., £17 3s. 2d. By how much do the total receipts exceed the total payments?

Compound Multiplication (Money).

68. **Compound Multiplication** is the process of finding the sum of a given number of repetitions of the same concrete quantity, expressed in different units of the same kind.

Thus, multiplication is simply another form of addition. (See § 21.) The *quantity* which is to be repeated or multiplied is called the **multiplicand.** The *number* which expresses how often the multiplicand is repeated is the **multiplier.** Note that the multiplicand is a concrete quantity, the multiplier an abstract number. We cannot speak of £4 13s. × £4, nor of 4 ft. × 4 ft.; though the latter phrase is often used as an abbreviation.

Ex. 1.—*Multiply* £4 13s. $7\frac{3}{4}$d. *by* 3.

£	s.	d.	f.
4	13	7	3
			3
14	0	11	1

Explanation.—If the explanation of § 24 has been thoroughly understood, it will be clear that the working is as follows:—

Farthings: 3 times 3, 9. But 9 farthings = 2d. and $\frac{1}{4}$d.

Pence: 3 times 7, 21, and 2, 23. But 23 pence = 1s. and 11d.

Shillings: 3 times 13, 39, and 1, 40. But 40s. = £2 and 0s.

Pounds: 3 times 4, 12, and 2, 14.

Thus, we get £14 0s. $11\frac{1}{4}$d. as answer. There is no need, after a while, to write the farthings in a separate column.

69. Multiplication by Factors.—When the multiplier is larger than 12, and can be separated into factors less than 12, it is better (at any rate, at first) to proceed as in Ex. 2.

Ex. 2.—*Multiply* £26 4s. $10\frac{1}{4}$d. *by* 28

£	s.	d.
26	4	$10\frac{1}{4}$
		4
104	19	5
		7
734	15	11

Explanation.—Since 28 is 4 × 7, to multiply by 28 is the same as to multiply first by 4 and then to multiply this product by 7. (See Factor Law, § 22.)

Hint.—The same result would have been reached if we had multiplied first by 7 and then by 4. (See Commutative Law, § 22.) But the work is lessened if we make 4 the first multiplier, because the multiplier 4 gets rid of the farthings. If there are no farthings in the multiplicand, and the multiplier is 36, say, the best factors would be first 12 (which gets rid of the pence) and then 3. So in other cases.

Verification.—(1) Change the order of the factors, and obtain the same product. (2) Divide your result by the multiplier, and reproduce the multiplicand.

EXAMPLES XXV.

Multiply—

1. £1 2s. 6d., £3 2s. 5d., £4 7s. $0\frac{1}{2}$d., £18 4s. 6d. (1) by 3, (2) by 5, (3) by 7, (4) by 9. (16 answers.)

2. £2 4s. 7d., £3 17s. 1d., £4 5s. 1d., £16 18s. 4d. (1) by 4, (2) by 6, (3) by 8, (4) by 10.

3. £4 17s. 1d., £18 4s. $5\frac{1}{4}$d., £70 16s. $9\frac{1}{2}$d., £163 5s. 10d. (1) by 5, (2) by 9, (3) by 10, (4) by 12.

4. £6 18s. 4d., £17 5s. 11d., £81 4s. $9\frac{3}{4}$d., £201 17s. $8\frac{1}{2}$d. (1) by 6, (2) by 8, (3) by 9, (4) by 11.

5. £30 4s. $7\frac{1}{4}$d. by 16, 20, 24, 28.

6. £84 17s. $2\frac{1}{2}$d. by 15, 25, 30, 35.

7. £14 7s. $6\frac{3}{4}$d. by 56, 48, 63, 81.

8. £47 14s. 7d. by 10, 100, 500, 1000.

70. When the multiplier is larger than 12, and cannot easily be separated into simple factors, adopt one of the following methods (or the method which will be explained in Chap. xxii.) :—

Ex. 3.—*Multiply* £26 4s. $10\frac{3}{4}$d. *by* 467.

		£	s.	d.	
First Method—	(*a*)	26	4	$10\frac{3}{4}$	$= N$, say
				10	
	(*b*)	262	8	$11\frac{1}{2}$	$= N \times 10$
				10	
		2624	9	7	$= N \times 100$
				4	
		10497	18	4	$= N \times 400$
Multiply (*b*) by 6		1574	13	9	$= N \times 60$
Multiply (*a*) by 7		183	14	$3\frac{1}{4}$	$= N \times 7$
Add		12256	6	$4\frac{1}{4}$	$= N \times 467$

Explanation. — To multiply by 467 is the same as to multiply by 400, 60, and 7 separately and add the results. (Distributive Law, § 22.)

This multiplication can be arranged as shown.

Second Method— 467
3
4) 1401 farthings. This line gives the result of multiply-
$350\frac{1}{4}$ pence. [ing $\frac{3}{4}$d. by 467.
4670 This line gives the result of multiplying 10d.
12) 5020 pence. [by 467.
418 shillings, **4**d.
1868 This line gives the result of multiplying 4s.
20) 2286 [by 467.
114 pounds, **6**s.
934 } These lines give the result of multiplying £26
2802 } by 467.
£12256 6s. $4\frac{1}{4}$d. *Ans.*

Ex. 4.—*If* 1 *lb. of tea cost* 2s. 4d., *what will be the change out of a sovereign after buying* 7 *lbs.?*

Explanation.—7 lbs. of tea will cost 7 times as much as 1 lb.; therefore, to get the cost of the 7 lbs., we multiply the cost of 1 lb., 2s. 4d., by 7 (*not* by 7 lbs.) The result is 16s. 4d.; and the change is £1 − 16s. 4d.: *i.e.* 19s. 12d. − 16s. 4d. = 3s. 8d. *Ans.*

71. *Notes on Multiplication.*

The accuracy of the result may be tested by varying the method, or by division, or by practice (Chap. xxii.).

Casting out Nines.—The method of "casting out nines" may be easily applied to check the accuracy of multiplication, &c., of money. The rule is as follows:—

Cast out nines from the number of pounds, multiply the remainder by 2 *and add to the number of shillings; cast out nines, multiply the remainder by* 3 *and add to the number of pence; cast out nines, multiply the remainder by* 4 *and add to the number of farthings; cast out nines. Apply the methods of pp.* 28 *and* 40 *to the remainder thus obtained.*

Ex.—We will test *Ex.* 3, § 70:—

£26 4s. $10\frac{3}{4}$d. treated thus gives 4 } 8 × 4 = 32, which gives 5 as
467 ,, 8 } remainder.
£12256 6s. $4\frac{1}{4}$d. ,, ,, ,, 5. Thus the sum is probably right.

We leave the proof of this rule as an exercise for the student.

Many of the devices applicable to ordinary multiplication can be applied to compound multiplication—*e.g.* that of § 33. To multiply N by 495, we multiply first by 5, then by 10, then by 10, getting $500N$; then we subtract our first product ($5N$), getting $500N - 5N$: *i.e.* $495N$.

It is easy to perform combined multiplication and addition, as in Exx. xii. Thus the advanced student ought to be able to find in one

operation the answer to £113 7s. 6d. + 5 × £31 6s. 11d. + 7 × £13 9s. 7d. = £364 9s. 2d.; or £1000 − 5 × £42 7s. 10d. − 8 × £14 7s. 9d. = £672 18s. 10d.

The student who desires to practise this rule may easily make examples for himself by adding together the answers in Exx. xxv.

EXAMPLES XXVI.

1. Multiply £3 2s. 7d. by 13, 23, 80, 100, 99.

2. Multiply £18 5s. 7¼d. by 17, 171, 260, 351, 99.

3. Multiply £56 14s. 10½d. by 93, 402, 710, 685, 997.

4. Multiply £173 17s. 4d. by 29, 176, 1030, 2300, 900.

5. Multiply £91 6s. 8¾d. by 129, 234, 2734, 10000, 9999.

6. Multiply £87 4s. 10d. by 130, 151, 3201, 3021, 6222.

7. Find the cost of (1) 20 articles at 1s. 6d. each, (2) 14 pairs of gloves at 3s. 9d. a pair, (3) 16 volumes at 3s. 6d. a volume, (4) 18 chairs at 7s. 8d. each.

8. What is the price of 310 things at 17s. 6d. each, and of 350 dozen oysters at 3s. 6d. a dozen?

9. If 1 lb. of tea cost 1s. 8d., what will be the price of a dozen (*i.e.* 12) lbs.? Also of a score (*i.e.* 20) lbs.?

10. If a person receives 4s. 6d. a day, how much will he receive in an ordinary year of 365 days?

11. What will be the price of 15 tons of coal at £1 1s. 6d. for each ton?

12. Calculate the cost of 3 gross of marbles at 2d. a dozen (1 gross = 12 dozen).

13. If 25 men receive as wages £1 1s. 6d. each, and 15 boys receive 15s. each, what is the total amount paid?

14. How much is left out of £100 after paying for 112 lbs. of tea at 2s. 6d. a lb., 50 lbs. of coffee at 1s. 6d. a lb., and 100 lbs. of butter at 1s. 5d. a lb.?

15. One hundred pounds of tea are bought as follows:—20 lbs. at 1s. 2d. a lb., 40 lbs. at 1s. 6d. a lb., 30 lbs. at 2s. a lb., and the rest at 2s. 6d. a lb. Find the total cost.

16. A farmer sells 25 calves at £2 6s. each, 33 pigs at 16s. each, 80 sheep at £2 2s. each, and three horses at £15 15s. each. What does he receive for the whole?

17. A builder lets a row of 20 houses at 5s. a week for each house, and six other houses at an annual rent of £30 each. What will be the total rent for a year of 52 weeks?

18. A person has an annual income of £1000, and spends £760 18s. out of it. What does he save in 13 years?

Compound Division (Money).

72. The nature of division has already been fully explained in § 35. It will be seen that in compound division, as in simple division, two problems present themselves.

Case I. To divide a given concrete quantity into any number of equal parts, *or* to divide a concrete quantity by an abstract number.

This case is sometimes called that of **Partition.**

Case II. To find how often one concrete quantity is contained in another of the same kind, *or* to divide a concrete quantity by another concrete quantity of the same kind.

This case is sometimes called that of **Quotition.**

Case I. Partition.

73. In some cases the answer is obvious. Thus, if we are asked to divide £12 9s. 6d. by 3, the answer is clearly £4 3s. 2d.; for £12, divided into 3 equal parts, gives £4 in each part, and so on. In other cases the method will be seen from the following examples :—

Ex. 1.—*Divide* £9 3s. $10\frac{3}{4}$d. by 7.

7)	£9	3s.	$10\frac{3}{4}$d.
	£1	6s.	$3\frac{1}{4}$d.

Explanation.—£9 divided by 7 is £1, with £2 over. This £2 is 40s., which with 3s. in the dividend makes 43s. Divided by 7 this is 6s., with 1s. over. This 1s. is 12d., which with 10d. in the dividend is 22d. Divided by 7 this is 3d., with 1d. over. This 1d. is 4f., which with 3f. in the dividend is 7f. Divided by 7 this is 1f.

Ex. 2.—*Divide* £20 4s. $8\frac{1}{4}$d. by 11.

This is worked as in Ex. 1, but it will be found that there are 10 farthings over, *i.e.* $2\frac{1}{2}$d. We write: *Quotient*, £1 16s. $9\frac{1}{4}$d.; *remainder*, $2\frac{1}{2}$d.

EXAMPLES XXVII.

Divide (in Nos. 1 to 4 there is no remainder)—

1. £154 5s. by 4, 5, 6, 10, 12.

2. £165 18s. by 3, 6, 7, 8, 9.

3. £686 6s. $10\frac{1}{2}$d. by 5, 6, 9.

4. £642 9s. $4\frac{1}{2}$d. by 7, 10, 11.

In numbers 5 and 6 give the remainder in its simplest form; thus, if 7 farthings remain, write it as $1\frac{3}{4}$d.

Divide—

5. £820 4s. $6\frac{1}{2}$d. by 3, 4, 5, 7, 10.

6. £750 18s. $5\frac{1}{4}$d. by 2, 6, 8, 9, 12.

Ex. 3.—*Divide* £7361 18s. $4\frac{1}{2}$d. by 67 and also by 100.

```
      £   s.  d.
     109  17   7   Quotient.
67 ) 7361  18  4½                  or
     67
     661
     603
      58        Pounds over.
      20        Multiply by 20 and add 18 from
     1178       Shillings.                [dividend.
      67
      508
      469
       39       Shillings over.
       12       Multiply by 12 and add 4 from
      472       Pence.                    [dividend.
      469
        3       Pence over.
        4       Multiply by 4 and add 2 from
       14 f.    Remainder.                [dividend.
```

```
      £   s.  d.
     109  17   7
67 ) 7361  18  4½
     661
      58
    1178s.
     508
      39
     472d.
       3
      14f.   Remainder.
```

In division by 100, 1000, &c., the abbreviated method should always be adopted.

```
        £   s.  d.
       73  12  4½
100 ) 7361  18  4½
      1238s.
       460d.
       242f.
```

The division by 100 is effected by cutting off two figures on the right, the **dark** figures on the left giving the quotient.

From the above working you ought to be able to answer the following problems:—

(1) *What is the least number of farthings which must be subtracted from* £7361 18s. $4\frac{1}{2}$d. *so that the remainder shall be exactly divisible by* 67? Ans. 14.

(2) *What is the least number of farthings which must be added to* £7361 18s. $4\frac{1}{2}$d. *to make the sum exactly divisible by* 67? Ans. 53, since $14+53=67$.

Notes.—Beware of the common error of omitting the 0 in the £ in the quotient.

The answer may be tested by multiplying the quotient by the divisor and adding in the remainder.

A quotient is often required "correct to the nearest penny." In the above case the quotient is £109 17s. 7d., and no mention would be made of the remainder. But, if the quotient had been £109 17s. $7\frac{3}{4}$d., then, "correct to the nearest penny," it would be written £109 17s. **8**d., not £109 17s. 7d. If we had been asked to

divide £7361 18s. **10**½d. by 67, "correct to the nearest farthing," it is easy to see from the work above that the remainder would have been 38 farthings. Hence, to make the dividend a multiple of 67, 29 (=67−38) farthings must be added, or 38 farthings subtracted from it. The former is the smaller number, and therefore the answer "to the nearest farthing" is £109 17s. 7¼d., not £109 17s. 7d.

Division by 36, 48, 60, and such easily factorized numbers may be accomplished by factors, exactly as in § 41.

EXAMPLES XXVII. (*continued*).

Divide, giving the remainder, when there is one, in its simplest form—

7. £102 10s. by 24 and 25.

8. £90 6s. 8d. by 16 and 17.

9. £188 6s. 6d. by 41 and 36.

10. £805 3s. 6d. by 56 and 61.

11. £3125 18s. 3¾d. by 171 and 100.

12. £22810 19s. 9d. by 402 and 10.

13. £179082 13s. 4d. by 1030 and 1000.

14. £913364 11s. 8d. by 750 and 10000.

15. £8216 14s. 8d. by 351 and 700.

16. £10000 10s. 10d. by 400 and 705.

17. £816324 8s. 6d. by 10, 100, 1000.

18. £94732 18s. 4d. by 20, 200, 2000.

19. (1) Divide £856 2s. 4½d. by 71.
(2) What is the least number of farthings to be subtracted from £856 2s. 4½d. that the difference may be exactly divisible by 71?
(3) What is the least sum of money to be added to £856 2s. 4½d. that the sum may be exactly divisible by 71?

20. Divide £4860 by 23, giving the answer to the nearest *penny*.

21. Divide £756 12s. 7d. by 43, giving the answer to the nearest *farthing?*

22. If £47321 17s. 6d. be divided into 19 equal parts, what is the value of each part?

23. If £1861 13s. 4d. be divided equally amongst 100 people, how much will each get?

24. Divide £22 4s. 2d. by 21, giving the answer to the nearest *penny*.

25. A sum of £186 19s. 4d. is divided amongst 20 persons. What does each get? How much remains?

26. What is the sum of money (1) just above £1000, (2) just below £1000, which is exactly divisible by 47?

27. Obtain the quotient of £3305763 14s. 8¼d. by 4001.

28. Divide £3002 11s. 6¾d. by 207.

Case II. Quotition.

74. To determine how many times one concrete quantity is contained in another concrete quantity of the same kind, the best method is to reduce both quantities to the same denomination, and then divide one result by the other. Care must be taken to choose that denomination in which the given quantities are expressed by the smallest numbers. Another method is illustrated in Ex. 3 below.

Ex. 1.—*How many times is* £13 2s. 6d. *contained in* £1640 12s. 6d. ?

We reduce both sums to half-crowns. Since there are 8 half-crowns in £1, £13 2s. 6d. $= 8 \times 13 + 1 = 105$ half-crowns. Similarly £1640 12s. 6d. $= 13125$ half-crowns. The question now becomes: How many times is 105 contained in 13125, and dividing 105 into 13125 we obtain 125 as the quotient. *Ans.* 125.

If we had reduced both quantities to sixpences or to pence we should have obtained the same answer, but the divisor and dividend would both have been larger numbers.

Ex. 2.—*Divide* £481 3s. 3d. *by* £17 3s. $8\frac{1}{4}$d.

£ s. d.	£ s. d.	
17 3 $8\frac{1}{4}$	481 3 3	16497) 461916 **28**
343s.	9623s.	131976
4124d.	115479d.	0
16497f.	461916f.	

Explanation.—Both sums are here reduced to farthings, giving 16497 and 461916 respectively. But 16497 is contained 28 times in 461916. Hence the answer is 28.

Ex. 3.—*Into how many equal parts, each* £13 15s. 7d., *can you divide* £8490 11s. 2d. ? *What remains ?*

£ s. d.	£ s. d.	
13 15 7 (1)	8490 11 2	
10	*8267 10 0*	= £13 15s. 7d. × **600**
137 15 10 (10)	223 1 2	
10	*137 15 10*	= £13 15s. 7d. × **10**
1377 18 4 (100)	85 5 4	
	82 13 6	= £13 15s. 7d. × **6**
	Rem. 2 11 10	**616**

Ans. 616 parts, with £2 11s. 10d. remainder.

This process might have been applied to Exx. 1 and 2, or, on the contrary, Ex. 3 might have been done by reducing both sums to pence, as in Exx. 1 and 2. The figures in Ex. 3 which are printed in italics may be omitted by the advanced student, and the work is then extremely concise.

EXAMPLES XXVIII.

[In Exx. 1–10 there is no remainder.]

How many times is—

1. $4\frac{3}{4}$d. contained in 15s. 10d.? **2.** 7s. 6d. contained in £30?

3. 19s. 2d. contained in £33 10s. 10d.?

4. £1 6s. 8d. contained in £46 13s. 4d. (N.B.—£1 = 3 times 6s. 8d.)

5. £5 4s. $9\frac{1}{4}$d. contained in £104 15s. 5d.?

6. £10 8s. 6d. contained in £156 7s. 6d.?

7. £17 18s. $9\frac{1}{4}$d. contained in £304 19s. $1\frac{1}{4}$d.?

8. £156 14s. contained in £11439 2s.?

9. £201 17s. 6d. contained in £21398 15s.?

10. £375 19s. 1d. contained in £75566 15s. 9d.?

Find the quotient and remainder in its simplest form when—

11. £83 4s. 9d. is divided by £6 8s. 4d.

12. £874 18s. $2\frac{1}{4}$d. is divided by £63 11s. 5d.

13. £1000 is divided by £150.

14. £924 is divided by £15 6s. 8d.

15. How often must £17 16s. 8d. be repeated to make £107?

16. How many times can £40 0s. 10d. be subtracted from £800 17s.? How many pence remain?

17. Subtract £4 7s. 6d. from £18 19s. 2d. From the remainder again subtract £4 7s. 6d., and so on as often as possible. How often can the subtraction be performed? What is the final remainder? How many times is this final remainder contained in £4 7s. 6d.?

18. How often is £1 7s. contained in 27 guineas?

19. How many half-crowns are there in (1) 10 guineas, (2) £180 17s. 6d.?

20. To how many different persons could I pay £1 3s. 6d. out of a sum of £12 18s. 6d.?

21. £100 is divided equally amongst 27 people, each receiving as much as possible. What will each get, and what will remain over?

22. If a man spend £1 6s. 8d. a week, how many weeks will £500 last him?

23. What is the largest sum of money that can be deducted 69 times from £1000?

24. If the railway fare for a journey is £1 7s. 8d., how many tickets can be bought for £23 10s. 4d.?

25. Divide £9652 6s. by £100 10s. $10\frac{3}{4}$d.

26. How often can £74 5s. $1\frac{1}{4}$d. be subtracted from £14776 15s. $8\frac{3}{4}$d.?

75. We proceed to give a few problems in the use of money.

Ex. 1.—*The sum of* £3 11*s.* 6*d. is made up of half-crowns, florins, and shillings, an equal number of each. How many coins are there in all?*

Make a pile consisting of one of each of these coins. The value of the pile will be 2s. 6d. + 2s. + 1s. = 5s. 6d. We must now see how many of such piles must be taken to give a total of £3 11s. 6d.

Reducing 5s. 6d. to sixpences—this is shorter than reducing it to pence—we get 11 sixpences. Reducing £3 11s. 6d. to sixpences, we get 143 sixpences. 143 sixpences contains 11 sixpences as many times as the number 143 contains the number 11, that is 13 times. There are, therefore, 13 piles, or 39 coins. *Ans.* 39.

Ex. 2.—*A clock and three watches altogether cost* £5 17*s.*; *the clock alone costs* £3 3*s.* *Find the cost of each watch, supposing that two of them cost* 3*s.* *each more than the third.*

Cost of clock and three watches = £5 17s.
Cost of clock alone = £3 3s.
∴ By subtraction we get cost of the three watches = £2 14s.

If, now we further subtract 6s., the extra cost of the two dearer watches, we should get that the price of three watches of the cheaper kind is £2 8s.

∴ Dividing by 3, we get that the cost of the cheaper watch is 16s.

∴ The cost of each of the others is 19s. *Ans.* 19s., 19s., 16s.

Ex. 3.—*How many yards of cloth at* 5s. 6d. *a yard are equal in price to* 55 *lbs. of tea at* 2s. 6d. *a lb.?*

The cost of 55 lbs. of tea at 2s. 6d. a lb. is 55 half-crowns = £6 17s. 6d. 5s. 6d. is contained in £6 17s. 6d. as often as 11 sixpences is contained in 275 sixpences, that is 25 times.

We require therefore 25 yds. of cloth. *Ans.* 25.

Ex. 4.—A *and* B *together have* £1 11s. 3d., B *and* C *together* £1 13s. 6d., A *and* C *together* £1 10s. 7d. *Find what they each have.*

A and B together have £1 11s. 3d.
B and C together have £1 13s. 6d.
A and C together have £1 10s. 7d.

∴ By addition we get—
Twice A's money + twice B's money + twice C's money amounts to £4 15s. 4d., or dividing by 2, we get

A's money + B's money + C's money amounts to £2 7s. 8d.
But A's money + B's money amounts to £1 11s. 3d.
∴ By subtraction, C's money is 16s. 5d.

So by subtracting £1 13s. 6d. from £2 7s. 8d. we get that A's money is 14s. 2d.

And by subtracting £1 10s. 7d. from £2 7s. 8d. we get that B's money is 17s. 1d.

Ans. A has 14s. 2d.; B has 17s. 1d.; C has 16s. 5d.

Ex. 5.—A *and* B *between them have* £3 7s. 9d., A *having twice as much as* B. *Find what each of them has.*

Suppose B (the one who has the less amount) to have any convenient sum of money, say 1d. A would then have 2d.

Now see how many piles each containing 3d. we can get out of £3 7s. 9d., that is 271 threepences.

We thus have 271 piles, from *each* of which B is to have 1d., while A has 2d. Therefore B has 271d. or £1 2s. 7d.; and A has 271 × 2d. or £2 5s. 2d.

Ans. A has £2 5s. 2d.; B has £1 2s. 7d.

EXAMPLES XXIX.

(Miscellaneous Exercises on Money.)

1. Six chairs and a table together cost £6 13s., and the table itself costs £3 10s. Find the price of each chair.

2. How much is left out of £10 when the following have been paid for:— 20 lbs. of tea at 1s. 4d. a lb., 3 doz. lbs. of sugar at 2d. a lb., 14 lbs. of coffee at 1s. 6d. a lb., and 25 packets of cocoa at 6d. a packet?

3. Tom and Harry begin a game of marbles with 100 each; Tom wins 30. What is the value of the marbles which each now has, if 10 cost 1d.?

4. A bag of money contains an equal number of shillings, sixpences, and pennies. If the total sum in the bag be £1 11s. 8d., how many coins of each kind are there?

5. What will be the total rent for a year of 52 weeks for a row of 25 cottages at 3s. 6d. a week?

6. If a carriage and pair together cost £172 13s., and each horse costs £48 16s. 6d., find the cost of the carriage.

7. Find the sum of money nearest to £100 which can be divided equally, without remainder, amongst 27 persons.

8. What number of pence must be added to £25 14s. 3d. so that 13 persons may get £1 19s. 7d. each?

9. What will be the total wages for 5 weeks of 37 men receiving 18s. 6d. a week each?

10. A hospital collection consisted of 220 sovereigns, 401 half-sovereigns, 508 half-crowns, 716 shillings, 808 sixpences. This is divided amongst four hospitals, three of them receiving equal amounts, but the fourth receiving twice as much as each of the other three. What do they each receive?

11. How many lbs. of tea at 2s. 1d. a lb. must be given in exchange for 50 yds. of cloth at 4s. 6d. a yard?

12. How many miles will a man walk at 4 miles an hour who starts at 8 a.m. and finishes at 5 p.m., having rested for two intervals of half-an-hour each?

13. A farmer sells 30 sheep for £45 to a butcher, who sells them again at 6d. a lb., thus gaining £15. How many lbs. did each sheep weigh?

14. I spend £1 10s. in the purchase of equal numbers of penny and halfpenny stamps. How many did I buy altogether?

15. If a boy's education costs 19s. 6d. each quarter of a year, what will be the total cost for 8 years?

16. A lady pays the following bills:—£4 7s. 3d., £1 11s. 7d., £2 1s. 11d., 19s., £1 7s. What remains out of £12?

17. If my gas bill amounts to £17 1s., how many cubic feet have I consumed at the rate of 2s. 7d. per 1000?

18. A person buys 78 yards of linen, at 2s. 6½d. a yard, and 50 yards of calico. If the total cost is £11 6s. 4½d., what is the price per yard of calico?

19. The sum of £8 1s. 6d. is divided amongst an equal number of men and women, each man receiving 5s. and each woman 6d. less than each man. How many men were there?

20. What is the difference in cost of 253 things at £10 12s. 6d. each and 82 things at £31 17s. 4d. each?

21. A dinner is provided for 250 persons at 12s. 6d. a head, but an extra charge of 5s. each is made for 190 of them who take wine. Find the total cost?

22. Find the difference in farthings between £120 15s. 10d. and 12000 farthings.

23. An equal number of half-crowns, florins, shillings, sixpences, and pennies are taken to make up £8 4s. 3d. How many coins are used altogether?

24. A man pays £50 a year as rent; his household expenses are £8 a week; his taxes are £12 10s. a year; and his other expenses are £150 a year. What does he save out of an income of £700? (N.B.—52 weeks in a year.)

25. Add together £756 14s. 2d., £82 17s. 8d., £147 11s. 5d., £376 0s. 4d., £276 15s. 7d., and subtract the sum from £1934 19s. 1d.

26. How many tons of coke at 8s. 6d. a ton will cost the same as 153 tons of coal at £1 3s. 6d. a ton?

27. If 8s. 11d. be the cost of a railway journey, how many journeys can be taken for £31 4s. 2d.?

28. A pound of butter and one egg cost 1s. 8d.; the pound of butter and six eggs cost 2s. 6d. Find the cost of the pound of butter and one dozen eggs altogether.

29. Find the cost of 500 herrings at 2 for $1\frac{1}{2}$d.

30. How many payments of 15s. $6\frac{1}{2}$d. will amount to £89 7s. $3\frac{1}{2}$d.?

31. If a sheep is bought for £1 10s. 6d. and sold for £2 1s. 3d., what is the profit on 250 sheep?

32. What is the price of a ton (2240 lbs.) of cheese at 7d. a lb.?

33. A grocer buys 112 lbs. of tea for £11 4s., intending to sell again at 2s. 6d. a lb.; but, 40 lbs. being damaged, he sells this portion at 1s. 10d. a lb. What did the tea cost him a lb.? What profit did he make on the whole?

34. Find the difference between 827 half-crowns and 941 florins.

35. How often is the sum of half-a-crown, a florin, a sixpence, and a farthing contained in £13 1s. 1d.?

36. A and B together have £1 5s., B and C together have £1 3s. 6d., A and C together have £1 2s. 6d. Find what A, B, and C have separately.

37. A sum of £23 15s. is divided amongst an equal number of men, women, and children, each man receiving 5s., each woman 3s., and each child 1s. 6d. How many persons were there altogether?

38. What sum of money subtracted 17 times from £50 will leave a remainder of £8 16s. 11d.?

39. Find a sum of money which contains £17 14s. 10d. 29 times.

40. The sum of £53 12s. is divided between 25 men, 17 women, and 12 boys, each woman getting 18s. 6d., and each boy 10s. What does each man get?

41. If a man spends daily 15s. 7d. and saves £24 6s. 8d. a year, what is his income for a year of 365 days?

42. How many persons can receive £3 17s. $8\frac{1}{2}$d. each out of a sum of £221 9s. $4\frac{1}{2}$d.?

43. How many guineas are there in 22680 half-crowns?

44. By how much does the sum of 83 florins, 241 shillings, 73 sixpences, and 90 farthings exceed 697 sixpences?

45. If £915 17s. be divided equally amongst 82 persons, how much will each get? How much remains over?

46. Into how many portions of £2 14s. 6d. each can £129 15s. 8d. be divided? What number of pence remain over?

47. What sum of money nearest to £250 can be exactly divided into 71 equal parts. What is the value of each part?

48. How many pounds are there in a shoulder of mutton which costs 4s. 9d. at $9\frac{1}{2}$d. a lb.?

49. What is the cost of the postage for 5,000 circulars at $\frac{1}{2}$d. each?

50. A man takes a penny and a halfpenny newspaper six days in the week. What does it cost him for a year of 52 weeks?

51. A clock strikes once at one o'clock, twice at two o'clock, and so on. How often does it strike in 7 days?

52. What is the cost of one million penny stamps?

53. After paying bills of £14 7s. 6d., £10 5s. 8½d., £4 11s. 7d., £13 7s. 6½d., what is left out of £90?

54. Find the total of a penny, a halfpenny, and a farthing, each repeated a thousand times.

55. How many oranges can be bought for 7s. 6d. at ¾d. each?

56. A and B have 15s. between them, A having twice as much as B. How much has each of them?

57. Divide £13 7s. 8d. into two portions, so that for every penny that one contains the other may contain threepence.

58. If cloth rises in price 2d. a yard, what will be the extra payment for 830 yards?

59. A man is employed to work for 36 days at 2s. 6d. a day, but one day in every three he works only half-time. Find his total wages.

60. Divide £101 15s 6d. into an equal number of sovereigns, half-sovereigns, half-crowns, and florins. How many coins of each kind are there?

61. What must be subtracted from £8216 15s. 8d. so that the remainder may contain £7 6s. 8d. an exact number of times?

62. Divide £394 12s. 6d. by 15.

63. A sum of £10 19s. 9¼d. has to be divided equally among 137 people. How much does each receive?

64. If the total revenue of a country whose population is 34,908,560 is £107,780,179, find the average amount of revenue raised per head of the population.

65. Multiply £741 10s. 4½d. by 306.

66. Divide £6263 7s. by 167.

67. A sum of £50 7s. 5d. is to be equally divided among the boys of a certain school, and it is found that each boy gets half-a-sovereign, half-a-crown, and sevenpence. How many boys were there?

68. How many pounds of tea at 3s. 2½d. a pound must be given in exchange for 92 yards of silk at 8s. 0¼d. a yard?

69. The sum of £113 9s. 10¼d. is made up of £5 notes, guineas, half-sovereigns, half-crowns, and farthings, of each an equal number. What is this number?

70. How many payments of 14s. 11¾d. amount to £89 2s. 6¼d.?

71. What must be added to £7543 19s. $7\frac{3}{4}$d. to make £9000 17s. $8\frac{1}{2}$d.?

72. What is the total of £7046 18s. 4d. repeated 236 times?

73. A collecting-box is found to contain 22 farthings, 75 halfpence, 132 pence, 97 threepences, 69 sixpences, 29 shillings, 11 florins, and 3 half-crowns. What is the total amount?

74. A father leaves £1191 13s. 4d. to be divided equally between his seven sons and six daughters. How much will each receive? Also how much would each have received if the share of each son had been double of the share of each daughter?

75. How many times is £237 6s. 8d. contained in £9256?

76. A clergy list contains the names of 18980 beneficed clergymen. What will it cost to send a copy of a certain circular to each of these, the cost of printing the circulars being 4d. for 10 and the postage of each being $\frac{1}{2}$d.?

77. Find the total cost of 36 pocket-handkerchiefs at 1s. 10d. each, 29 collars at $9\frac{1}{2}$d. each, and 13 neckties at 1s. $11\frac{3}{4}$d. each.

78. Find the number of pence in £8 19s. 5d. repeated 59 times.

79. How often is £1337 11s. 3d. contained in £76241 1s. 3d.?

80. How many times can £12 12s. 7d. be subtracted from £34161 17s. 11d.?

81. If a franc be worth $9\frac{1}{2}$d., how many are equivalent to £86 1s. $10\frac{1}{2}$d.?

82. If 7 yards of cloth cost £1 19s. $4\frac{1}{2}$d., how many yards can be bought for £4 15s. $7\frac{1}{2}$d.? (First find what one yard costs.)

83. Two persons between them have £13 17s. 6d., and one has 17s. 6d. more than the other. Find how much each has.

84. A man buys 420 articles at $\frac{1}{2}$d. each, and 200 others at a different price, spending altogether £1 10s. What was the price of each of the second kind?

85. Divide £2 1s. 8d. amongst three persons, A, B, and C, so that for every penny A has B may have threepence and C sixpence.

CHAPTER XI.

THE FOUR RULES: WEIGHTS AND MEASURES.

Compound Addition.

76. The principles are exactly the same as those used in dealing with money. A slight difficulty arises in long measure and square measure if poles are introduced, and you should study the worked examples in these measures very carefully.*

Ex. 1.—*Add together* 3 *tons* 10 *cwts.* 2 *qrs.* 5 *lbs.*; 17 *tons* 14 *cwts.* 20 *lbs.*; 5 *tons* 3 *qrs.* 11 *lbs.*; *and* 2 *qrs.* 27 *lbs.*

	tons	cwts.	qrs.	lbs.
	3	10	2	5
	17	14	0	20
	5	0	3	11
			2	27
Ans.	26	6	1	7

Explanation.—Arrange the quantities underneath one another, tons beneath tons, &c. Add the lbs. The total is 63, but 28 lbs. is 1 qr.; therefore 63 lbs. is 2 qrs. 7 lbs. Write the 7 and carry the 2 to the qrs. column. The total of this, with this 2 added is 9: *i.e.* 2 cwts. 1 qr. Carry 2 to the cwts. column. The total is 26: *i.e.* 1 ton 6 cwts. Then, lastly, the total of the tons column is 26.

* *Ex.* 2.—*Add together* 5 *acs.* 3 *rs.* 31 *sq.pls.* 22 *sq.yds.*, 18 *acs.* 1 *r.* 24 *sq.pls.* 18 *sq.yds.*, 17 *acs.* 37 *sq.pls.* 17 *sq.yds.*, 8 *acs.* 2 *rs.* 25 *sq.pls.* 25 *sq.yds.*, 4 *acs.* 1 *r.* 28 *sq.yds.*

acs.	rs.	sq.pls.	sq.yds.
5	3	31	22
18	1	24	18
17	0	37	17
8	2	25	25
4	1	0	28
54	2	0	$19\frac{1}{4}$

Explanation.—Adding the sq.yds., we get 110. Now, $30\frac{1}{4}$ sq.yds. = 1 sq.pl., and 110 clearly contains $30\frac{1}{4}$ three times. Taking away $3 \times 30\frac{1}{4}$, *i.e.* $90\frac{3}{4}$, from 110, we have $19\frac{1}{4}$ sq.yds. left, and 3 to carry to the sq.pls. The rest of the work presents no difficulty.

Note.—The beginner who is using this rule before he understands fractions may reason thus: 110 sq.yds. is 440 quarter-sq.yds; but there are 121 quarter-sq.yds. in a sq.pl., and $440 = 3 \times 121 + 77$. Therefore we get 3 pls. and 77 quarter-sq.yds. over, *i.e.* $19\frac{1}{4}$ sq.yds.

* But a wise teacher will postpone all dealings with the $5\frac{1}{2}$ and $30\frac{1}{4}$ until fractions have been mastered. (See foot-note to § 63, V.) The following examples in this chapter involve this difficulty: *Addition*, 17, 22. *Subtraction*, 11, 12, 14; *Multiplication*, 6, 7, 8, 11; *Division*, 5, 8, 11, *Miscellaneous*, 18, 26, 42, 56. They are marked *.

EXAMPLES XXX.

Perform the following additions, reducing all answers, unless otherwise directed, to as high a denomination as possible:—

Avoirdupois.

1.

tons.	cwts.	qrs.	lbs.	oz.
1	0	1	13	0
	1	2	12	0
			2	4
1	5	2	0	0

2.

tons.	cwts.	qrs.	lbs.
3	17	3	17
	14	0	5
2	3	4	10
1	7	6	3

3.

lbs.	oz.	drs.
24	15	15
3	11	4
	10	1
12	5	11
16	14	10

4.

cwts.	qrs.	lbs.
16	3	27
4	0	5
13	2	0
	1	18
3	2	10

5.

tons.	cwts.	qrs.
418	13	1
796	4	3
14	1	0
750	0	2
56	18	3

6.

cwts.	qrs.	lbs.
14	3	27
27	2	16
73	0	10
4	1	5
18	2	4

7.

tons	cwts.	qrs.	lbs.	oz.
24	19	3	20	15
	3	0	25	14
3	3	2	6	9
47	16	1	11	4
16	15	1	21	10
2	7	3	20	1
1	1	0	5	3

8.

cwts.	qrs.	lbs.	oz.	drs.
19	3	27	15	15
2	0	1	1	3
14	1	2	3	4
	2	16	11	12
5	0	9	10	11
16	1	14	8	10
10	2	15	9	11

Troy.

9.

dwts.	grs.
13	21
7	14
	11
19	23
18	6
3	2

10.

oz.	dwts.
10	13
2	16
9	19
6	5
17	8

11.

oz.	dwts.	grs.
11	17	23
3	19	9
9	4	17
6	18	4
15	2	10

12.

oz.	dwts.	grs.
17	4	20
13	17	4
2	11	9
9	3	18
8	2	3
26	8	6

13.

lbs.	oz.	dwts.	grs.
21	3	4	17
2	9	19	23
7	11	2	7
14	3	0	1
9	10	17	14
10	4	5	11

14.

lbs.	oz.	dwts.	grs.
2	10	17	17
3	2	3	13
10	3	2	0
5	4	1	1
9	11	16	6
7	1	5	2

Length.

(In Nos. 15 and 16 leave the answer in yards.)

15.

yds.	ft.	ins.
4	2	9
17	1	0
15	0	11
7	1	8
3	2	7

16.

yds.	ft.	ins.
11	2	8
23	0	10
4	0	10
7	1	11
18	2	3

17.*

furls.	pls.	yds.
7	31	4
2	7	3
14	18	5
11	29	0
5	5	2

18.

mls.	furls.	yds.
1000	7	79
99	3	104
703	0	21
496	2	5
73	6	16

19.

mls.	yds.
476	850
27	494
130	150
89	29
41	306

Square Measure.

(In Exx. 20–21 leave the answer in square yards.)

20.

sq.yds.	sq.ft.	sq.ins.
29	3	70
4	0	29
33	0	94
16	8	11
7	5	21

21.

sq.yds.	sq.ft.	sq.ins.
27	0	9
16	2	3
21	7	93
5	1	104
9	5	27

22.*

acs.	rs.	sq.pls.	sq.yds.
217	0	29	$19\frac{1}{4}$
93	1	17	20
161	2	0	7
104	2	14	3
27	0	2	11

23.

acs.	rs.	sq.pls.
17	2	20
6	1	13
173	0	39
29	2	0
53	3	5

Cubic Measure.

24.

c.yds.	c.ft.	c.ins.
19	26	1000
2	1	0
4	20	29
16	9	168
83	14	890

Capacity.

(In Exx. 25–26 leave the answers in gallons. In 28 reduce to weeks.)

25.

galls.	qts.	pts.
4	3	1
7	0	0
20	2	0
17	1	0
31	0	1

26.

galls.	qts.
27	3
31	1
10	0
17	2
4	1

27.

bushs.	pks.	galls.	qts.
7	1	1	3
4	0	1	1
6	3	0	2
1	0	0	3
10	2	1	0

Time.

28.

days	hours	minutes	seconds
25	3	59	21
13	20	23	42
27	11	1	50
111	14	31	4
98	5	17	10

29.

years	days	hours
10	213	17
4	111	10
15	7	0
21	104	1
20	96	13

Subtraction.

77. The following example will illustrate the method in its most difficult case:—

Ex.—*Subtract* 3 *pls.* 4 *yds.* 1 *ft.* 10 *in. from* 6 *pls.* 2 *yds.* 2 *ft.* 5 *ins.*

pls.	yds.	ft.	ins.	
6	2	2	5	
3	4	1	10	
2	$3\frac{1}{2}$	0	7	(A)
		1	6	
2	3	2	1	(B)

Explanation.—Work thus:—

Inches: 10 and **7**, 17 (*i.e.* 5 + 12).

Feet: 1, 2, and **0**, 2.

Yards: 4 and $\mathbf{3\frac{1}{2}}$, $7\frac{1}{2}$ (*i.e.* $2 + 5\frac{1}{2}$).

Poles: 1, 4, and **2**, 6.

The answer is thus given in the Form (A). The half-yard is now written as 1 ft. 6 ins., and added in, giving the final form (B).

EXAMPLES XXXI.

Subtract—

Avoirdupois.

1. 14 cwts. 3 qrs. 27 lbs. from 1 ton.

2. 13 lbs. 15 oz. 10 drs. from 1 qr. 10 lbs. 10 oz.

3. 25 tons 19 cwts. 42 lbs. from 30 tons 1 qr. 4 lbs.

4. 14 cwts. 17 lbs. 3 oz. from 19 cwts. 3 qrs. 14 oz. 3 drs.

Troy.

5. 13 dwts. from 2 oz. 14 grs.

6. 2 lbs. 10 oz. 4 grs. from 4 lbs. 13 dwts. 2 grs.

7. 1 oz. 1 dwt. 1 gr. from 1 lb.

8. 14 oz. 14 dwts. 14 grs. from 3 lbs. 3 oz. 10 dwts.

Length.

9. 4 yds. 2 ft. 7 ins. from 5 yds. 1 in.

10. 2 mls. 500 yds. from 5 mls. 190 yds.

11.* 3 furls. 27 pls. 5 yds. from 1 mile.

12.* 10 pls. 3 yds. 2 ft. 6 ins. from 1 furl. 2 pls. 1 ft. 7 ins.

Square Measure.

13. 23 sq.yds. 7 sq.ft. 102 sq.ins. from 31 sq.yds. 2 sq.ft. 30 sq.ins

14.* 30 acs. 1 r. 30 sq.pls. 17 sq yds. from 40 acs. 12 sq.pls

Cubic Measure.

15. 26 c.ft. 143 c.ins. from 1 c.yd.

16. 14 c.yds. 17 c.ft. 852 c.ins. from 20 c.yds. 76 c.ins.

Capacity.

17. 3 galls. 3 qts. 1 pt. from 5 galls.

18. 6 bushs. 3 pks. 1 pt. from 13 bushs. 1 gall.

Time.

19. 50 minutes 50 seconds from 1 day.

20. 30 days 19 hours 11 minutes from 1 year.

Multiplication.

78. Ex.—*Multiply* 3 *days* 12 *hours* 40 *minutes* 30 *seconds by* 123.

days	hours	minutes	seconds	
3	12	40	30	$= N$
			10	
35	6	45	0	
			10	
352	19	30	0	$= N \times$ **100**
70	13	30	0	$= N \times$ **20**
10	14	1	30	$= N \times$ **3**
433	23	1	30	$= N \times$ **123**

Or thus (*cf.* p. 82) :—

```
        123
         30
60 ) 3690  seconds
        61  30 seconds remainder
      4920  i.e. 40 × 123
60 ) 4981  minutes
        83  1 minute
      1476  i.e. 12 × 123
 6 ) 1559
 4 )  259  5 }
        64  3 } Rem. 3 × 6 + 5 = 23
       369  i.e. 3 × 123
       433
```

The second method is recommended if the multiplier is large.

EXAMPLES XXXII.

Multiply—

1. 3 qrs. 4 oz. by 134. **2.** 3 tons 1 cwt. 3 qrs. 17 lbs. by 37.

3. 7 cwts. 2 qrs. 13 lbs. by 47. **4.** 3 qrs. 10 lbs. 8 oz. by 112.

5. 8 oz. 12 dwts. 14 grs. by 17.

6.* 53 mls. 5 furls. 23 pls. 4 yds. by 19.

7.* 3 yds. 2 ft. 4 ins. by 37. **8.*** 3 furls. 16 pls. 4 yds. by 33.

9. 5 mls. 6 furls. 115 yds. by 44.

10. 5 sq.yds. 10 sq.ft. 27 sq.ins. by 16. (Ans. in sq.yds.)

11.* 11 acs. 30 sq.pls. 21 sq.yds. by 25.

12. 26 c.ft. 1727 c.ins. by 1728. **13.** 4 pks. 3 qts. 1 pt. by 47.

14. 1 qr. 3 bushs. 1 pk. 1 gall. by 87.

15. 4 days 22 hours 33 minutes 49 seconds by 401.

16. 2 days 11 hours 12 seconds by 25.

Division.

79. Ex. 1.—*Divide* 273 *acs.* 3 *rs.* 27 *sq.pls. by* 53.

```
     acs.    rs.  sq.pls.  acs.  rs.  sq.pls.
53 ) 273     3     27     ( 5    0    26
     265
      8
      4
     35rs.          Ans. 5 acs. 0 rs. 26 sq.pls., with 49 sq.pls. or
     40           1 r. 9 sq.pls. remainder.
   1427sq.pls.
   106
    367
    318
     49     Stop here.
```

Ex. 2.—*How often does* 2 *tons* 5 *cwts.* 7 *lbs. contain* 1 *qr.* 5 *lbs.* 9 *oz.?*

```
qr.  lbs.  oz.          tons  cwts.  qrs.  lbs.
 1    5     9            . 2    5     0     7
28                         20
33 lbs.                     45 cwts.
16                           4
33                          180 qrs.
207                          28
537 oz.                     360
                           1447
                           5047 lbs.
                             16
                          5047
                         30282
                         80752 oz.
```

Explanation.—Here we reduce both quantities to the lowest unit named, that is, ounces. We have now to find how often 537 is contained in 80752. By ordinary division we find that 537 goes into 80752 150 times, with a remainder 202. Hence the answer is 150 times, with the remainder 202 oz., *i.e.* 12 lbs. 10 oz.

EXAMPLES XXXIII.

Divide, giving remainder, if there is one—

1. 134 tons 14 cwts. 3 qrs. 9 lbs. by 39.

2. 417 tons 14 cwts. 1 qr. 9 lbs. by 45.

3. 7 lbs. 11 oz. 5 drs. by 116. **4.** 2 lbs. 3 oz. 15 dwts. 14 grs. by 27.

5.* 1 ml. 2 furls. 35 pls. 4 yds. 2 ft. 11 in. by 17.

6. 317 days 13 hours 31 minutes 57 seconds by 53.

7. 1000 galls. by 11. **8.*** 257 acs. 3 rs. 27 pls. $21\frac{1}{4}$ sq.yds. by 16.

9. 3411 c.yds. 2 c.ft. 1000 c.ins. by 309.

10. How often does 135 tons 4 cwts. 3 qrs. 14 lbs. contain 3 tons 4 cwts. 1 qr. 17 lbs.?

11.* How many times must 7 mls. 6 furls. 22 pls. 4 yds. be repeated to obtain 211 mls. 1 furl. 13 pls. 3 yds. 1 ft. 6 in.?

12. If 13 hours 32 minutes 15 seconds be multiplied by a certain number, the result is 2 weeks 1 day 19 hours 3 minutes. What is the number?

13. If 60 acs. 3 rs. be divided into plots of 1 ac. 2 rs. 3 pls each, how many such plots will there be?

14. How many times is 10 oz. 10 dwts. contained in 17 lbs. 6 oz.?

80. We now give examples of miscellaneous problems in weights and measures.

Ex. 1.—*How many revolutions will a carriage-wheel* 6 *ft.* 2 *ins. in circumference make in a journey of* 3 *mls.* 970 *yds.* 2 *ft.* 10 *ins.?*

When the wheel has made one revolution the carriage has moved through a distance of 6 ft. 2 ins. Therefore the required number of revolutions is the number of times that 6 ft. 2 ins. is contained in 3 mls., &c. Reducing both to inches, we have:—

```
 ft. ins.                    mls.  yds.   ft.  ins.
  6   2                        3    970    2    10
 12                         1760
 74 ins.                    6250 yds.
                               3
                           18752 ft
                              12
                     74 ) 225034 ins. ( 3041
                          222
                            303
                            296
                              74
                              74
                                 Ans. 3041.
```

Ex. 2.—*The total weight of* 5 *trucks and their contents is* 31 *tons* 13 *cwts.* 3 *qrs. If each truck weighs* 1 *ton* 2 *cwts.* 3 *qrs. find the weight in each truck, supposing that they all contain equal weights.*

	tons	cwts.	qrs.	
5 trucks and contents weigh	31	13	3	
∴ 1 truck and its contents weigh	6	6	3	
But 1 truck alone weighs	1	2	3	
∴ The contents of each truck weigh	5	4	0	*Ans.*

Ex. 3.—*Two bags together contain* 185 *lbs.* 4 *oz.* (*avoir.*), *one of them containing* 16 *lbs.* 8 *oz. more than the other. How much is there in each bag?*

If, from the total of the two bags, 185 lbs. 4 oz., we deduct the extra 16 lbs. 8 oz., we obtain 168 lbs. 12 oz. This would be the quantity in two bags, the contents of each of which were the same as those of the lighter bag. Therefore, dividing by 2, we get—

Contents of the lighter bag are 84 lbs. 6 oz.
∴ Contents of the heavier bag are 100 lbs. 14 oz. } *Ans.*

Ex. 4.—*If a train travels* 30 *miles in* 1 *hour, how many feet will it travel in* 1 *second, supposing its rate to be uniform?*

In one hour it travels 30 × 5280 ft., *i.e.* 158400 ft., and, as there are 60 × 60 or 3600 seconds in 1 hour, it will in 1 second travel 158400 ÷ 3600 ft., *i.e.* 44 ft. *Ans.* 44 ft. in 1 second.

EXAMPLES XXXIV. (Miscellaneous).

(For the meaning of the asterisk see footnote to § 76.)

1. If 1 lb. of sugar cost 2½d., what is the cost at the same rate of (1) a cwt., (2) a ton?

2. How many grains of gold are there in a cup weighing 6 oz. 13 dwts.?

3. How many days are there altogether in the four years beginning 1895? Also in the four years ending 1900?

4. Find the total weight of 15 trucks with their contents, each truck weighing 1 ton 3 cwts., and containing 3 tons 2 qrs. 17 lbs.?

5. What is the total weight of 42 silver spoons, if each weighs 2 oz. 3 dwts. 8 grs.?

6. How often does 359 mls. 1 furl. 34 pls. contain 8 mls. 4 furls. 17 pls.?

7. If light travels at the rate of 186000 miles a second, and the sun is 93930000 miles distant, how long will it take for the sun's light to reach us?

8. How many seconds were there in the month of February, 1899?

9. A gallon of water weighs 10 lbs., and 1000 oz. is the weight of 1 c.ft. of water. How many cubic feet of water will be required to fill a tank which holds 375 gallons?

10. (1) Add together 20 cwts. 2 qrs. 14 lbs. 5 oz., 7 cwts. 22 lbs. 11 oz., 2 qrs. 4 oz., and 4 cwts. 1 qr. 4 oz. (2) If the whole be divided into parcels each containing 1 lb. 8 oz., how many such parcels would there be?

11. How often is (1) £22 6s. 3d., (2) £120 6s. 3d. contained in £6135 18s. 9d.?

12. How many minutes from 8.50 a.m. Monday to 7.25 p.m. the following Friday?

13. A box of eggs weighs 29 lbs. 2 oz., and the box itself weighs 6 lbs. 10 oz. If each egg weighs 2 oz., how many does the box contain?

14. Into how many plots of 16 sq.pls. each can 5 acs. be divided?

15. A workman is fined 3d. for every time he is 10 minutes late. If his fines for a year amount to a guinea and a half, how many hours' work did he lose?

16. Divide 219 days 2 hours 35 minutes 11 seconds by 4 days 3 hours 13 minutes 7 seconds.

17. What is the total weight of 1001 packets each weighing 3 lbs. 3 oz. 3 drs.?

18.* How many revolutions will a carriage wheel 3 yds. 2 ft. 6 ins. in circumference make in a journey of 7 mls. 3 furls. 34 pls. 4 yds. 1 ft.?

19. A man pays as income-tax $6\frac{1}{2}$d. on each £ of his income, which is £750. What has he left?

20. A boy was born June 8, 1880. Including this day, how many days will he have lived at the end of November 11, 1899?

21. A man pays £14 6s. a year for the rent of his house. How much is that each week?

22. The diameter of a half-penny is 1 in. How many such coins placed close together in a straight line will reach to half a mile?

23. A clock strikes once at each hour and at each quarter-hour. How often does it strike in 5 years, including two leap-years?

24. A hogshead (54 gallons) of beer is used to fill pint bottles. Allowing 2 gallons for waste, how many bottles are filled?

25. What is the cost of 3 qrs. 10 lbs. at 3d. an ounce?

26.* A builder buys four plots of ground, comprising respectively 20 acs. 1 r. 30 pls. $12\frac{1}{2}$ sq.yds., 16 acs. 3 rs. $5\frac{1}{4}$ sq.yds., 10 acs. 15 pls. 17 sq.yds., 5 acs. 33 pls. $25\frac{3}{4}$ sq.yds., and divides the whole into 120 equal parts. Find the size of each part.

27. How many rings each weighing 4 dwts. 15 grs. can be made from a mass of gold which weighs 1 lb. 4 oz. 13 dwts.?

28. A person takes 99 steps in a minute, each 30 ins. long. How many miles and yards does he walk in an hour?

29. A circular race-course has eight laps to a mile. How many yards does the winner of a three-mile race run if he had 300 yds. start? How often does he come to the winning post?

30. A man buys 500 yds. of cloth at 3s. $6\frac{1}{2}$d. a yard, and sells it at a profit of $1\frac{3}{4}$d. a yard. What did he sell the whole for?

31. If the large and small wheels of a railway engine are respectively 16 ft. 11 ins. and 7 ft. 3 ins. in circumference, how many more revolutions will the latter make than the former in a journey of 203 miles?

32. How many days are there from May 7th, 1890, to July 4th, 1891, including one only of these two days?

33. How many rings each weighing 4 dwts. 18 grs. can a goldsmith make from a mixture of 1 lb. 10 oz. 1 dwt. 18 grs. of pure gold with 7 oz. 7 dwts. 6 grs. of alloy?

34. A guardsman in a march of 4 mls. 6 furls. takes 10032 steps. Find the length of his steps.

35. What sum of money will it take to pay the following bill?—14 lbs. of pork at $6\frac{3}{4}$d. per pound, 10 lbs. of beef at $10\frac{1}{2}$d. per pound, 18 kidneys at $1\frac{3}{4}$d. each, 9 lbs. of sausages at $8\frac{1}{2}$d. per pound.

36. An estate of 1416 acs. 2 rs. 16 pls. was divided into allotments each 4 acs. 3 rs. 27 pls. in area. How many allotments were there?

37. Subtract the total number of days in those months which contain 30 days from the total number in those which contain 31 days.

38. Each ton of ore obtained from a gold mine yields on the average 2 oz. 1 dwt. 15 grs. of fine gold. How much fine gold will be obtained from 293 tons?

39. Multiply the difference between 12 cwts. 3 qrs. 17 lbs. 10 oz. and 5 cwts. 2 qrs. 23 lbs. 11 oz. by 528.

40. How many yards of calico are there in 4 cwts. if 1 yd. weighs 8 oz.?

41. What will be the extra cost of 10 tons of coal if the price is raised 1d. per hundredweight?

42.* Divide 40301 acs. 38 sq.pls. 9 sq.yds. into 251 equal portions.

43. A grocer buys a ton of currants for £40, and sells it again at 6d. per pound. What is his profit?

44. What is left out of an income of £1000 after paying income-tax at 8d. for each £1?

45. A truck of coals contains 6 tons 6 cwts. How many trucks will be required for 189 tons?

46. How many grains of gold in a cup weighing 11 oz. 3 dwts.?

47. A consignment of 53 boxes of fruit weighs 6 qrs. 7 lbs. 9 oz. What is the weight of each box if they are all of equal weight?

48. Divide 108 bushs. 2 pks. 1 qt. into portions each containing 4 bushs. 2 pks. 1 gall. 3 qts. How many such portions will there be?

49. Divide the sum of 12 cwts. 3 qrs. 17 lbs. 10 oz. and 5 cwts. 2 qrs. 23 lbs. 11 oz. by 324, giving the number of ounces which remain.

50. A load of coals weighing 7 tons 7 cwts. 2 qrs. 16 lbs. is to be removed in 6 carts. If each cart contains the same quantity of coals, what will the weight of that quantity be?

51. A man's housekeeping expenses amount on an average to £1 15s. 3½d. a day. What did they amount to for the year 1890?

52. If a sovereign weighs 5 dwts. 8 grs., what is the value of 16 lbs. 3 oz. 4 dwts. of the same metal?

53. A truck is loaded with 100 sacks; the sack itself in each case weighs 6 lbs. 11 oz., and each sack contains 85lbs. of grain. What is the weight of the whole load in tons, hundredweights, &c.?

54. What is the saving of cost in the consumption of 150,000 c.ft. of gas owing to a reduction in price of 3d. per thousand?

55. The circumference of a carriage wheel is 2 yds. 2 ft. 2 ins. How many times will it revolve during a journey of 27 mls. 578 yds. 2 ft. 10 ins.?

56.* Into how many plots of ground each containing 154 sq.yds. could 2 acs. 3 rs. 8 pls. be divided?

57. How often can 1 ton 6 cwts. 3 qrs. 21 lbs. be subtracted from 113 tons 2 cwts. 3 qrs.?

58. At midday on November 20th, 1899, a person had lived 10480 days. When was he born?

59. The fore wheel of a carriage is 8 ft. in circumference, and the hind wheel is 11 ft. How many revolutions will one make more than the other while the carriage travels 3 miles?

60. If 3 cwts. 2 qrs. of sugar are made up into bags containing respectively 1 lb. 8 oz., 2 lbs. 8 oz., and 3 lbs., an equal number of each, how many bags are there altogether?

61. 10 lbs. of tea, at 2s. 6d. a pound, are mixed with 6 lbs. at 1s. 10d. a pound, what is the average price of 1 lb. of the mixture?

62. What is the total cost for 52 weeks of the following papers?—the *Graphic* and *Spectator*, each 6d. a week, and the *Times* (six days a week) at 3d. a day.

63. What is the cost of 100,000 halfpenny stamps?

64. What will be the cost of 3 tons 7 cwts. 1 qr. 13 lbs. at 1d. an ounce?

65. An innkeeper mixes 10 galls. of spirits at 11s. a gallon, 8 at 9s., and 6 at 9s. 2d. At what price per gallon must he sell the mixture so as to gain £1 19s. on the whole?

66. Find the equivalent in lbs. Avoirdupois of 29 lbs. 2 oz. Troy.

67. A greengrocer bought 500 oranges for 25s.; he sold 300 at 10d. a dozen and the rest at 8 for 6d. How much did he gain?

68. A boy was set to divide a certain number of tons, hundredweights, &c., by 26, but, copying the sum wrongly, he obtained as his answer 1 cwt. 1 qr. 18 lbs., which was 13 lbs. less than the correct answer. What was the number of tons, hundredweights, &c., set him?

69. The total weight of six trucks and their contents is 36 tons 6 cwts. If the trucks weigh 1 ton 5 cwts. 1 qr. each and contain equal loads, find the weight of each load.

70. Along a course of 350 yards posts are placed at intervals of 2 ft. 6 ins. If there be one at the beginning and one at the end, how many posts are there in all?

71. A certain number of articles were bought at 7s. 6d. each. A reduction in the price of 3d. each would have effected a saving of 15s. How many were bought and what was paid for them?

72. How many lengths of 1 furl. 185 yds. must be taken to make 3 mls. 7 furls. 65 yds.?

73. On dividing 5 cwts. 3 qrs. 22 lbs. into equal portions of 7 lbs. 2 oz. each, it is found that 3 lbs. 6 oz. remain over. How many portions are there?

74. How many times will a bicyclist in riding 10 mls. go round a course of 352 yds.?

75. Express 120 lbs. Avoirdupois in pounds and ounces Troy.

76. Two bags contain together 25 lbs. 12 oz. (Avoirdupois), the contents of one of them weighing 12 oz. more than those of the other. How much do they each contain?

77. Two bags containing the same number of pounds cost respectively £2 3s. 9d. and £2 0s. 10d., the price of the first being 6s. 3d. a pound. How many pounds were in each bag, and what was the price of the second per pound?

78. If a train travels 60 miles in 1 hour, how many feet does it travel in 1 second at the same rate?

79. What number of hundredweights and pounds must be multiplied by 93 to give the same as 4 cwts. 17 lbs. multiplied by 91?

80. November 16, 1899, was a Thursday. What day of the week was March 4 in the same year?

SECTION III.

FACTORS AND MULTIPLES.

CHAPTER XII.

FACTORS AND PRIME NUMBERS.

81. **Factor.**—The factors of a number are the numbers which are *exactly* divisible into it, *i.e.* divisible without remainder.

Thus the *factors* of 12 are 1, 2, 3, 4, 6, and 12.

Common Factor.—When two or more numbers have the same number as factor it is called a Common Factor of these numbers.

Thus 3 is a *common factor* of 12 and 27.

Measure is another name for Factor.

The name is less frequently used and is gradually disappearing.

Multiple of a number is a number which contains it an exact number of times more than once.

Thus 12 is a *multiple* of 2, for it contains it 6 times. So also 12 is a *multiple* of 3, 4, and 6. It is not usual to speak of 12 as a multiple of 12, but, apart from this exception, any number may be described as a multiple of any of its factors.

Consecutive Numbers are numbers which differ from one another by 1.

Thus 12, 13, 14 are *consecutive numbers.* So are 300, 299, and 298.

Even Numbers are those numbers which are multiples of 2.

Thus 2, 4, 6, 8, 10, ... are *even* numbers. *Note that* 0 *is an even number.* 2 and 4, 4 and 6, &c., are sometimes called *consecutive even numbers.*

Odd Numbers are those numbers which are not multiples of 2.

Thus 1, 3, 5, 7, 9, ... are *odd* numbers. 3 and 5, 5 and 7, &c., are *consecutive odd numbers.*

Prime Numbers or **primes** are those numbers which have no factors except themselves and 1.

Thus 1, 2, 3, 5, 7, 11, 13, ... are *prime numbers*. See Appendix i.

Composite Numbers are numbers which are not prime.

Thus 12 is a composite number.

Prime Factors are those factors of a number which are primes.

Thus 1, 2, and 3 are the *prime factors* of $12 = 2 \times 2 \times 3$; 4 and 6 are not.

Numbers are **prime to one another** when they contain no common factor except 1.

Thus 15 and 32 are *prime to one another*. Note carefully that numbers prime to one another are NOT generally prime numbers. Thus $15 = 5 \times 3$ and $32 = 2 \times 2 \times 2 \times 2 \times 2$. They contain no common factor, but they are not prime numbers.

Two numbers are **complementary** when their sum is a number consisting only of nines. Each is then the **complement** of the other.

Thus 25 and 74 are complementary, for $25 + 74 = 99$.

The *complement* of a number is sometimes defined as the smallest number necessary to make it up to a power of 10. Thus the complement of 25 would then be 75. We adhere to the first definition.

DIVISIBILITY BY 2, 4, 8, ... OR BY 5, 25, 125, ...

82. In certain cases it is possible to tell in a moment whether a number is or is not divisible by a certain number. We shall examine these cases.

I. **Divisibility by 2 or 5.**

Any number = a certain number of tens + its units figure.

Each ten can be split into $\left\{\begin{matrix}5 \text{ twos} \\ 2 \text{ fives}\end{matrix}\right\}$; therefore, if the units figure (last figure) is either 0 or divisible by $\left\{\begin{matrix}2 \\ 5\end{matrix}\right\}$, the whole number can be split into $\left\{\begin{matrix}\text{twos} \\ \text{fives}\end{matrix}\right\}$, *i.e.* is divisible by $\left\{\begin{matrix}2 \\ 5\end{matrix}\right\}$.

Hence: *A number is divisible by* 2 [*or by* 5] *if the* LAST *figure either is* 0 *or is itself divisible by* 2 [*or by* 5].

II. **Divisibility by 4 or 25.**

Any number of more than two figures = a certain number of hundreds + number formed by its last two figures.

Each hundred can be split into $\left\{\begin{matrix}25 \text{ fours} \\ 4 \text{ twenty-fives}\end{matrix}\right\}$; therefore,

if the number formed by the last two figures is either 00 or is divisible by $\left\{\begin{matrix}4\\25\end{matrix}\right\}$, the whole number can be split into $\left\{\begin{matrix}\text{fours}\\\text{twenty-fives}\end{matrix}\right\}$, *i.e.* is divisible by $\left\{\begin{matrix}4\\25\end{matrix}\right\}$.

Hence: *A number is divisible by* 4 [*or by* 25] *if the number formed by the* LAST TWO *figures either is* 00 *or is divisible by* 4 [*or by* 25].

III. **Divisibility by 8 or 125.**

8 and 125 both divide 1000.

Hence: *A number is divisible by* 8 [*or by* 125] *if the number formed by the* LAST THREE *figures either is* 000 *or is divisible by* 8 [*or by* 125].

Ex.—7**2**, 7**4**, 8**8**, 9**6** are divisible by 2 because **2**, **4**, **8**, and **6** are.

7**5**, 8**0**, 9**5** are divisible by 5 because the last figure is 5 or 0.

1**75**, 2**25**, 3**50**, 4**00** are divisible by 25 because the last two figures are so divisible.

6**125**, 7**250**, 8**375**, 9**625** are divisible by 125 because the last three figures are so divisible.

6**128**, 7**256**, 8**376**, 9**624** are divisible by 8 because the last three figures are so divisible.

Divisibility by 3 or 9.

83. *A number is divisible by* 3 *or by* 9 *if the sum of the digits is so divisible.*

This theorem will be proved generally later [see § 284, (i.)], An illustration will show the reason of it. Take any number whatever, no matter how many figures, say 35784—

$$\begin{array}{rrrrrrrrr}
35784 = & 30000 & = & 3\times 10000 & = & 3\times(9999+1) & = & 3\times 9999 & +3\\
+ & 5000 & & +5\times\ 1000 & & +5\times\ (999+1) & & +5\times\ 999 & +5\\
+ & 700 & & +7\times\ 100 & & +7\times\ (99+1) & & +7\times\ 99 & +7\\
+ & 80 & & +8\times\ 10 & & +8\times\ (9+1) & & +8\times\ 9 & +8\\
+ & 4 & & +4 & & +4 & & & +4\\
\hline
 & & & & & & & = \text{a multiple of } 9 & \\
 & & & & & & & +3+5+7+8+4 &
\end{array}$$

But 9 is divisible both by 3 and by 9.

Therefore 35784 is divisible by 3 or by 9 if $3+5+7+8+4$ is, *i.e.* if the sum of the digits is.

It should be clear also from the above illustration that the *remainder*, on dividing a number by 3 or by 9, is the same as that on dividing the sum of the digits by 3 or by 9.

Divisibility by 11.

84. *Add together alternate digits beginning with the units digit; then add alternate digits beginning with the tens digit. If these two totals are equal, or differ by* 11, *or a multiple of* 11, *the number is divisible by* 11.

Here again the proof will be deferred [see § 284, (ii.)], and an illustration given. Consider again 35784—

35784 =	4 =	4 =	4 =	4
+ 80	+ 8 × 10	+ 8 × (11 − 1)	+ 8 × 11 − 8	
+ 700	+ 7 × 100	+ 7 × (99 + 1)	+ 7 × 99 + 7	
+ 5000	+ 5 × 1000	+ 5 × (1001 − 1)	+ 5 × 1001 − 5	
+ 30000	+ 3 × 10000	+ 3 × (9999 + 1)	+ 3 × 9999 + 3	
			= a multiple of 11 + (4 + 7 + 3) − (8 + 5) = a multiple of 11 if 4 + 7 + 3 differs from 8 + 5 by a multiple of 11 (which, of course, it does not).	

Note.—In the last column but one in the above and similar work, we have to deal with the numbers 11, 99, 1001, 9999, 100001, 999999, .. It is clear that 99, 9999, 999999, ... and all numbers which involve an *even* number of 9's are divisible by 11. The series 1001, 100001,... may be treated thus—

$$1001 = 990 + 11,$$
$$100001 = 99990 + 11,$$

and it is clear that they also are divisible by 11.

Rules of Divisibility.

85. We may now give the very important *Rules of Divisibility*, which must be thoroughly mastered

A number is divisible by

2 if the last figure is even,

3 if the sum of the digits is so divisible,

4 if the number formed by the last two figures is so divisible,

5 if the last figure is 0 or 5,

6 if it is divisible by 2, and also by 3 (see § 41)

7 (no simple Rule) (see Exx. xl. 30, and also § 87)

8 if the number formed by the last three figures is so divisible,

9 if the sum of the digits is so divisible,

10 if the last figure is 0,

11 if the sum of one set of alternate digits differs from the sum of the other set by 0 or a multiple of 11,

12 if it is divisible by 4 and also by 3.

Caution, *Divisibility by* 12.—Do not test by 6 and 2. Any number divisible by 6 is *necessarily* divisible by 2 also. The factors selected for the test must be prime to each other.

EXAMPLES XXXV.

1. Of the following numbers write down those divisible (1) by 2, (2) by 3, (3) by 4, (4) by 5, (5) by 6, (6) by 8, (7) by 9, (8) by 10, (9) by 11, (10) by 12:—321, 125, 28, 280, 3364, 33648, 333648, 333, 330, 3333, 7296, 74965, 312, 123, 261, 216, 612, 216612, 1234, 4235.

2. State the rules for divisibility by (1) 15, (2) 18, (3) 22, (4) 30, and (5) 45. (See *Caution* in § 85.)

3. State the rules for divisibility by (1) 16, (2) 20, (3) 24, (4) 36, and (5) 40. (See *Caution* in § 85).

4. Of the following numbers write down those divisible by (1) 15, (2) 16, (3) 18, (4) 20, (5) 22, (6) 24, (7) 30, (8) 36, (9) 40, (10) 45:—240, 242, 360, 9360, 374, 473374, 374242, 8360, 660, 6160.

5. *Write down* the remainders after dividing the following numbers (1) by 3, (2) by 9:—137, 3721, 14685, 45987, 571234.

6. *Write down* the remainder after dividing the following numbers (1) by 3, (2) by 9:—371, 3207, 14576, 48569, 516314.

Resolution into Factors.

86. We can now resolve a number into its prime factors, in many cases, without difficulty, *e.g.*, 174240. We proceed thus:—

$$\begin{array}{r|l} 10 & 174240 \\ \hline 4 & 17424 \\ \hline 4 & 4356 \\ \hline 9 & 1089 \\ \hline & 121 \end{array}$$

Explanation.—It is clearly divisible by **10**; divide it.

24 is divisible by **4**, therefore the number is; divide it.

56 is divisible by **4**; therefore the number is; divide it.

Add the digits in 1089. They make 18; therefore the number is divisible by **9**. 121 is **11** × **11**.

Therefore Number $= 10 \times 4 \times 4 \times 9 \times 121$

$$= 2 \times 5 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 11 \times 11$$

$$= 2^5 \times 3^2 \times 5 \times 11^2.$$

Note.—It is better to take advantage of the obvious fact that 24 is divisible by 4 than to test whether 424 is divisible by 8.

If, after using all our Divisibility Rules and dividing by the factors found, we have a number left which we do not recognize as a prime, we have no method but to test it by the various primes, 7, 13, 17, 19, ... (see App. II.), until we discover its factors, if it has any.

Ex.—

4) 154836	**Explanation.**—**4** is detected at once.
9) 38709	The sum of the digits is 27; divide by **9**.
11) 4301	The eleven test applies; divide by **11**.
17) 391	We are left with 391, and must try it successively by 7, 13, **17**; divide by **17**.
23	**23** is recognized as prime.

Therefore Number $= 4 \times 9 \times 11 \times 17 \times 23 = 2^2 \times 3^2 \times 11 \times 17 \times 23$.

The factors should be finally arranged in **ascending order,** *i.e.*, with the smallest first.

87. *Limitation of Tests.*—Suppose that, after removing all the factors which the Divisibility Rules showed, we had the number 401 left. We should test it without success by 7, 13, 17, 19.

The quotients would be, respectively, 57+, 30+, 23+, 21+. We should then, *without testing further*, know that the number was prime. For the next prime to 19 is 23, and it will clearly produce a quotient less than 23, and 23 therefore cannot divide 401; for, if it did, the quotient, the number less than 23, would itself have satisfied our previous tests.

So with 701 we should test up to 23, but no farther, for the next prime is 29, and $29 \times 29 = 841$; therefore, $29 \times$ some number *less than* $29 = 701$.

You will learn later that the tests need not go further than the "Square Root" of the number. The Square Root of 701 is 26·47...; therefore no number greater than 26 need be tried.

Grouping of the Number.—The grouping of the number will often suggest a factor. For instance, 8989 is clearly divisible by 89. So 12827 is divisible by 127, though this is not quite so easy to see. So also $127127 = 127 \times 1001$, $79079 = 79 \times 1001$, and the factors of 1001 are 7, 11, and 13.

88. *Factorization* is generally understood to mean splitting into prime factors. It is only possible to do this in one way. But it is possible to split into factors which are not prime in more ways than one. For instance, $12 = 6 \times 2$, or 4×3.

Number of Factors.—We can easily state the total number of factors which a number has. For instance, we have proved above

$$174240 = 2^5 \times 3^2 \times 5 \times 11^2;$$

therefore 1*, 2, 2^2, 2^3, 2^4, 2^5, will all divide 174240.

So 1, 3, 3^2, ,, ,, ,, ,

1, 5, ,, ,, ,, ,

1, 11, 11^2, ,, ,, ,, ,

and any number made by multiplying one term from each of these four lines, *e.g.*, $2^3 \times 3 \times 1 \times 11$, will divide 174240.

Now there are 6 terms in the first line, and 3 in the second.

Therefore 6×3 pairs may be made from the first two lines [$1 . 1$, $1 . 3$, $1 . 3^2$, $2 . 1$, $2 . 3$, $2 . 3^2$, &c.].

Therefore $6 \times 3 \times 2$ products may be made from the first three lines.

Therefore $6 \times 3 \times 2 \times 3$ products may be made from the first four lines, *i.e.*, 108.

Hence, in all, there are 108 factors. These include 1, and the number itself.

The number of ways of splitting a number up into pairs of factors is easily seen to be one half the number of single factors, unless the number is a perfect square.

Prime Numbers.

89. No process has yet been discovered for determining whether a number, 1000003 for instance, is, or is not, a prime number, except the very tedious one of successive division by the primes up to the limit fixed by the last article. In the case suggested we should have to divide by the primes 7, 11, 13,...997, and, as there are (see App. II.) 164 such, we should have 164 division sums to make before we reached the conclusion that the number was prime.†

With larger numbers the difficulty is immensely increased. For instance, with 100,000,000,000,003, supposing this to be a prime, we should have 664579 division sums before us.†

It would seem as if the great mathematician Fermat had discovered some process. "In answer to Mersenne, who asked if he could tell without much trouble whether the number 100895598169 was a prime, Fermat wrote that it was the product of 898423 and 112303, both of which were primes."—W. W. R. Ball.

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* Note the insertion of this 1 in each line, and explain it to yourself after you have read the proof.

† But in most cases the number of factors to be tested can be greatly diminished by methods explained in the *Theory of Numbers*. Thus it may be proved that the two numbers in the text can only have factors such that when divided by 12 the remainders are either 1 or 7. Therefore 17, for instance, cannot divide either.

Such being the difficulty of the process of factorisation, tables, called *Factor Tables*, have been constructed, a small portion of one of which is printed in the Appendix. These tables give the smallest prime factor of all the numbers contained in them. They run only, however, up to 9,000,000. Beyond this number no general process is available, and there is probably no living man who could tell, without months of work, what are the factors of 23,999,798,000,203, a number which is not a prime number. Mr. R. Satchivell, of Liskeard, wrote me on August 3rd, 1904, that he had factorized the number after five weeks' work :—

$$3999971 \times 5999993.$$

90. *Curiosities of Prime Numbers.*

The highest number at present known to be prime is 2305843009213693951. The number $5.2^{75} + 1$, namely

$$18889465931478580854784 1$$

(Dr. Morehead), is almost certainly also a prime, while M. Lucas believes himself to have proved that $2^{127} - 1$, namely

$$170141183460469231731687303715884105727$$

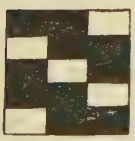
is also a prime. The following are other specimens of high numbers known to be prime :—85280581, 234750601, 2413941289, 77843839397, 370373 there is no prime number. So also from 4652353 to 4652507, and from 8421251 to 8421403. There are altogether 78496 prime numbers less than 1000000, 148932 less than 2000000, 216817 less than 3000000, ..., and probably 5761460 less than 100000000.

91. *Construction of Factor Tables.*—The method pointed out by Eratosthenes (*c.* 235 B.C.) is practically the one used to-day. Essentially the process is as follows.

The odd numbers are written down thus :—

1	11	21	31	41	51	61	71	81	91	101	111
3	13	23	33	43	53	63	73	83	93	103	113
5	15	25	35	45	55	65	75	85	95	105	115
7	17	27	37	47	57	67	77	87	97	107	117
9	19	29	39	49	59	69	79	89	99	109	119

Every third beginning with 9 is then crossed off as divisible by 3. This is done by means of a perforated cardboard with holes as indicated, which, being passed along the numbers, indicates at once the number to be crossed off. Such an arrangement is called a *Sieve*. The figure indicates the 3 Sieve. A similar arrangement would be made for 7, 13, &c. When a number shows itself in one of the holes of a sieve the number of that sieve (3, 7, 13, ... as the case may be) is entered in the Factor Tables against the number thus appearing. Numbers which appear in none of the sieves are prime. (For further information see Glaisher's *Factor Tables for the Fourth Million.*)



92. *Number of Primes.—The number of primes is infinite, i.e.*, however large a prime number may be, it is always possible to find one larger.

For let P be the greatest prime number yet discovered. Multiply together 2, 3, 5, 7, 11, 13,..., up to P and add 1. The number so formed leaves 1 as remainder when it is divided by 2, 3, 5, ... and any prime up to P, and therefore it is prime to all these numbers, and therefore it must either itself be prime or be divisible by some prime greater than P. In either case it has been proved that there is a prime greater than P.

Algebraical Factorization.

93. The advanced student can often bring algebraical results to bear upon the factorization of numbers otherwise intractable. Only a few examples can be noted here:—

Ex.—$63999991 = 64000000 - 9 = (8000)^2 - 3^2 = (8000-3)(8000+3)$
$= 7997 \,.\, 8003$
$= 11 \,.\, 53 \,.\, 151 \,.\, 727$, all of which are prime.

$1000001 = 10^6 + 1 = (10^2)^3 + 1$ and is divisible by $10^2 + 1$, *i.e.* by 101. The prime factors are 101, 9901.

$1000000001 = 10^9 + 1$ and is divisible by 1001. The prime factors are 7, 11, 13, 19, 52579.

$810901 = 30^4 + 30^2 + 1 = (30^2 + 30 + 1)(30^2 - 30 + 1)$. The prime factors are 7^2, 13, 19, 67.

$40001 = 4 \,.\, 10^4 + 1$. Now $4x^4 + 1 = (2x^2)^2 + 1$
$= (2x^2 - 2x + 1)(2x^2 + 2x + 1)$. $\therefore$ 40001 is divisible by $2 \,.\, 10^2 + 2 \,.\, 10 + 1 = 221$. The prime factors are 13, 17, 181.

Good practice in such methods is given by $2^n \pm 1$ and $3^n \pm 1$, a few of the results of which are given among the Harder Problems (Nos. 49–55). In these cases what is known as Aurifeuille's Theorem is of great use, namely,

$$2^{4n+2} + 1 = (2^{2n+1} - 2^{n+1} + 1)(2^{2n+1} + 2^{n+1} + 1).$$

There is a similar theorem for powers of 3, namely,

$$3^{6m+3} + 1 = (3^{2m+1} + 1)(3^{2m+1} - 3^{m+1} + 1)(3^{2m+1} + 3^{m+1} + 1).$$

Ex.—

$2^{26} + 1 = (2^{13} - 2^7 + 1)(2^{13} + 2^7 + 1) = 5 \,.\, 53 \,.\, 157 \,.\, 1613$, all primes.

$2^{30} + 1 = (2^{15} - 2^8 + 1)(2^{15} + 2^8 + 1) = 5^2 \,.\, 13 \,.\, 41 \,.\, 61 \,.\, 1321$, all primes.

$2^{34} + 1 = (2^{17} - 2^9 + 1)(2^{17} + 2^9 + 1) = 5 \,.\, 137 \,.\, 953 \,.\, 26317$, all primes.

The factorization of such expressions depends on Aurifeuille's Theorem and on the recognition of the fact that $2^{30} + 1$ can be written $(2^{10})^3 + 1$, or $(2^6)^5 + 1$, and is divisible both by $2^{10} + 1$ and by $2^6 + 1$.

EXAMPLES XXXVI.

[Examples 21–58 should be left to a second reading.]

Find the prime factors of:—

1. 125, 132, 144. **2.** 120, 135, 154. **3.** 240, 287, 343.
4. 216, 429, 550. **5.** 625, 621, 891. **6.** 375, 627, 729.
7. 1001, 2662, 1323. **8.** 1029, 3402, 3993. **9.** 4081, 5313, 6993.
10. 4941, 8262, 7623.
11. 32802. **12.** 34182. **13.** 36498. **14.** 19551 **15.** 132288.
16. 443520. **17.** 12447. **18.** 13473. **19.** 10101. **20.** 18291.

What is the highest *prime* number which need be tested for divisibility into?—

21. 9601. **22.** 9901. **23.** 14621. **24.** 15307. **25.** 486103.
26. 576001. **27.** 1056007. **28.** 1110103. **29.** 2244553.
30. 2555551.

How many factors, excluding unity and the number itself, will divide into?—

31. 125. **32.** 144. **33.** 6993. **34.** 39930. **35.** 32802.
36. 34182. **37.** 10530. **38.** 15390. **39.** 1009. **40.** 1061.

Find all the prime numbers lying between—

41. 720 and 750. **42.** 820 and 850. **43.** 1000 and 1020.
44. 1050 and 1070.

Using Appendix, find all the prime numbers lying between—

45.* 10000 and 10030. **46.*** 10030 and 10050.

47.* What is the prime number next greater than 100000?

48.* What is the prime number next less than 100000?

Factorize, by the methods of Art 93:—

49. 4^7+1. **50.** 3^9+1. **51.** 27^5+1. **52.** 7^6-1.
53. 8^6+1. **54.** $2^{18}-1$. **55.** 80999975. **56.** 2561601
57. 640001. **58.** 999271.

CHAPTER XIII.

HIGHEST COMMON FACTOR (H.C.F.).

94. The **Highest Common Factor** (H.C.F.) of two or more numbers is the greatest number which divides each of them exactly.

Thus 6 is a *common factor* of 72 and 120, so is 12, but 24 is the *highest common factor.*

The terms **Greatest Common Measure** (G.C.M.) **Highest Common Divisor** (H.C.D.), **Greatest Common Factor** (G.C.F.) are also used.

To find the H.C.F. two methods are used.

Method I.—When the numbers can be factorized (*i.e.* expressed in prime factors), their H.C.F. can be written down at a glance :—

Ex. 1.—*Find the H.C.F. of* 105, 135, 180.

$$\left.\begin{array}{l} 105 = \mathbf{3} \times \mathbf{5} \times 7 \\ 135 = 3 \times 3 \times \mathbf{3} \times \mathbf{5} \\ 180 = 2 \times 2 \times 3 \times \mathbf{3} \times \mathbf{5} \end{array}\right\} \text{ therefore H.C.F.} = \mathbf{3} \times \mathbf{5} = 15.$$

We may more conveniently work thus:

$105 = 3.5.7, \quad 135 = 3^3.5, \quad 180 = 2^2.3^2.5 \quad \therefore$ H.C.F. $= 3 \times 5 = 15.$

It is often not necessary, however, to *factorize more than one* of the numbers. If we are asked to find the H.C.F. of 6 and 10 mentally, we argue that $6 = 2 \times 3$; 2 divides into 10; 3 will not; therefore 2 is the H.C.F.

Ex. 2.—*Find the H.C.F. of* 126, 198, 270.

$126 = 2 \times 3 \times 3 \times 7$. We test for 2, and see that it divides all, then for 3, then for 3×3, lastly for 7. The 7 will not divide; the others will. therefore the H.C.F. is $2 \times 3 \times 3 = 18$.

EXAMPLES XXXVII.—H.C.F. METHOD I.

Find the H.C.F. of

1. 128, 324. **2.** 544, 425. **3.** 510, 561. **4.** 840, 945. **5.** 224, 336.
6. 429, 715. **7.** 900, 3474. **8.** 2121, 1313. **9.** 348, 1024.
10. 1225, 625. **11.** 264, 2321. **12.** 1904, 2720.
13. 115, 161, 207. **14.** 52, 78, 416. **15.** 24, 32, 48, 74.
16. 13, 104, 416, 780, 5200.

What prime factors are common to

17. 924, 1694. **18.** 1260, 5292. **19.** 8560, 9360. **20.** 1266, 4680?

Laws of Factors.

95. The following principles must be carefully grasped. They have sometimes been implicitly assumed in the previous work.

I. *A factor of a number is a factor of any multiple of that number.*

In general language, if f divides N it divides mN. For, if f divides N, N can be split up into groups each of which contains f, therefore mN can be split up into m times as many groups, *i.e.*, mN is divisible by f.

As an illustration: if a number of boys can be arranged in *fours*, then twice or seven times that number of boys can also be arranged in fours.

II. *A common factor of two numbers is a factor of their sum.*

That is, if f divides both M and N it divides $M+N$. For M can be arranged in groups of f each, and N can also be so arranged, and when these groups are put together we have $M+N$ arranged in f-groups.

As an illustration: if there are two squads of boys both in *fours*, then when they march together they can still be arranged in fours

III. *A common factor of two numbers is a factor of their difference.*

That is, if f divides both M and N it divides $M-N$. For M can be arranged in f-groups and from these groups a certain number making N can be taken away, and the remainder $(M-N)$ will still be in f-groups.

IV. *A common factor of two numbers is a factor of the sum or difference of any multiple of the one number and any multiple of the other number.**

For by I. the common factor of the numbers is a factor of their multiples also, and therefore, by II. and III., of the sum or difference of these multiples.

V. *A common factor of two numbers is a factor of the remainder obtained when the larger number is divided by the smaller.*

For this remainder is obtained as the difference between the larger number and a multiple of the smaller, or the smaller itself, and this difference [see I. and III.] contains the common factor.

VI. *The H.C.F. of two numbers is the H.C.F. of the smaller and of the remainder obtained when the larger number is divided by the smaller.*

Proof.

```
SMALLER ) LARGER ( QUOTIENT
          MULT. OF SMALLER
          ----------------
          REMAINDER.
```

Since the Larger Number = Remainder + a Multiple of the Smaller Number, therefore *all common factors* of Smaller and Remainder divide Larger, therefore H.C.F. of Smaller and Remainder divides the Larger. This must be the *Highest* C.F. of Smaller and Larger, for, if there were any higher common factor, by V. this would divide the Remainder also, *i.e.*, it would form a higher common factor of Smaller and Remainder.

H.C.F. Method II.

96. *Method I.* can only be used if the numbers are easy to factorise. *Method II.* depends upon Law VI., that the H.C.F. of two numbers is also the H.C.F. of the smaller and of the remainder obtained by dividing the larger by the smaller. Thus the finding of the H.C.F. of the numbers is made to depend upon finding the H.C.F. of two smaller numbers, and the process is then repeated until the numbers are small enough to allow the H.C.F. to appear, or until one number by dividing exactly into the other shows itself to be the H.C.F. required.

The process may be written in various forms. In the example given the first is the clearest, and the one which

* In IV., V., VI., it must be borne in mind that any number whatever may be regarded as a factor of *zero*, *i.e.* as dividing exactly into *zero*.

should be adopted at first. It may then, after a time, be replaced by the second. The third is the Italian method, in which the partial products are not written. Its only disadvantage, as before mentioned, is that if a mistake is committed that mistake is not so easy to detect.

Ex.—*Find the H.C.F. of* 851 *and* 943.

```
851 ) 943 ( 1
      851
       92 ) 851 ( 9
            828
             23 ) 92 ( 4
                  92
```

```
9 ) 851 | 943 ( 1
      828 | 851
       23 |  92 ( 4
           |  92
```

```
9 ) 851 | 943 ( 1
     23 |  92 ( 4
```

Ans. 23.

Explanation.—The second arrangement is simply a compression of the first.

The first stage in it is as shown on the right.

```
851 | 943 ( 1
    | 851
    |  92
```

The 92 is then divided into the 851 on the left without rewriting either, and the quotient is shown to the left of the 851.

When this division is completed, the new remainder 23 is usually just opposite the number 92 into which it is next to be divided.

The third arrangement is a further compression in which only the remainders are written.

Finally, 23 divides exactly into 92, *i.e.* is the H.C.F. of 23 and 92, and therefore the H.C.F. of 851 and 943.

The meaning of the process will be made clearer by exhibiting the work (first arrangement) as follows:—

```
37 times 23 ) 41 times 23 ( 1
              37 times 23
               4 times 23 ) 37 times 23 ( 9
                            36 times 23
                             1 times 23 ) 4 times 23 ( 4
                                          4 times 23
```

H.C.F. Miscellaneous Notes.

97. (1) If at any stage in Method II. the prime factors of the remainder can be seen it is generally best to revert to Method I. and test these factors separately.

Thus, in the Example given, 92 is the first remainder. Its factors are 4 and 23. 4 clearly will *not* divide the given numbers; therefore 23 must be the H.C.F. *if there is one.* But notice that 92 would have appeared in the same place if the numbers had been 853 and 945; and neither 4 nor 23 divides these numbers, which have, in fact, no common factor.

(2) At any stage in Method II. a factor which obviously does not belong to the H.C.F. may be rejected.

Thus, in the Example given, we might have rejected the 4 from the 92, and the work would then have been as shown on the right.

This device will often save much labour.

```
37 ) 851 | 943 ( 1
     851 | 851
         |
         | 92÷4
         | 23
```

(3) All factors which are clearly common should be removed before commencing Method II., and subsequently, of course, multiplied by the C.F. obtained by Method II.

Thus, if we had 4255 and 4715 to deal with, we should divide both by 5, obtaining 851 and 943. The H.C.F. of these is 23; therefore the H.C.F. of our numbers is $5 \times 23 = 115$.

(4) To find *H.C.F. of three or more numbers*, if Method I. will not apply, use Method II. to find the H.C.F. of the two smallest. This H.C.F. will probably be capable of factorization. If it is, we can revert to Method I. If not, apply Method II. to the H.C.F. and the next smallest number, and so on.

Thus, to *find the H.C.F. of* 722000, 913520, 15447. By *Note* 2 we can reject the 0's and 2's in the first two factors, and therefore the numbers become 361 (rejecting factor 2000, which does not divide the third number), 11419 (rejecting factor 80), and 15447. Applying Method I. to 361 and 11419, we obtain 19 as the H.C.F., and since this divides into 15447 it is the required H.C.F.

(5) To find the G.C.M.† of two concrete numbers, the numbers must both be reduced to the same unit.

Ex.—*Find the G.C.M. of* 7s. 8d., £1. 1s. 1d., and 1s. 11d.

We reduce all these sums to pence, obtaining 92, 253, and 23. 23 is the H.C.F. of 23, 92, and 253; therefore 23 pence, *i.e.* 1s. 11d., is the G.C.M. of the sums given.

23 pence cannot be spoken of as a *factor* of 7s. 8d. The word *factor* is used only of abstract numbers.

---

† H.C.F. is scarcely appropriate in this connection. The teacher can use this Example to explain the meaning of the word "measure." Thus, the G.C.M. of 3 ft. 6 in. and 1 ft. 3 in. is 3 in., *i.e.* a measure 3 inches long is the greatest measure which can be used to measure both these lengths.

EXAMPLES XXXVIII.

Find the H.C.F. of

1. 754, 1131. **2.** 770, 2431. **3.** 1379, 2401.

4. 2314, 3721 **5.** 928, 1073. **6.** 1785, 1683.

7. 1836, 1938. **8.** 1957, 9373. **9.** 2037, 6596.

10. 5292, 4473. **11.** 5473, 8463. **12.** 6239, 7684.

13. 8694, 9198. **14.** 3775, 10000. **15.** 11385, 16819.

16. 11529, 12411. **17.** 124321, 197451. **18.** 168543, 314675.

19. 1019527, 1231845. **20.** 2030107, 2263680.

21. 1311, 1610, 1978. **22.** 1236, 1632, 1728.

23. 2233, 2639, 4147. **24.** 1729, 3900, 5850.

25. 11781, 13923, 17017. **26.** 10670, 27742, 12804.

27. Find the greatest number by which 24797 and 10549 can both be divided exactly.

28. Find the greatest number by which both 4488 and 7293 can be divided exactly.

29. What is the highest number which both 2091 and 4551 will contain an exact number of times?

30. Find how often the H.C.F. of 150271 and 163647 is contained in each of them.

31. Find the greatest weight which can be contained exactly in 3 tons 5 lbs. and in 20 tons 3 cwt. 2 qrs.

32. Find the greatest weight which can be contained exactly in 4 tons 15 cwt. 3 qrs. and in 454 cwts.

33. £11. 12*s.* 9*d.* is divided into equal portions, and then £11. 1*s.* 8*d.* is divided into portions of just the same value. If the total number of portions is as small as possible, what is it, and how great is each portion?

34. £3. 17*s.* 7*d.* is divided into equal portions, and then £22. 3*s.* 4*d.* is divided into portions of just the same value. In how many ways can this be done without using half-pennies?

35.* Find the H.C.F. of 1047553 and 35184363700225.

36.* Find the H.C.F. of 1152921503533105153 and 1407374548009.

CHAPTER XIV.

LEAST COMMON MULTIPLE (L.C.M.).

98. The **Least Common Multiple** (**L.C.M.**) of two or more numbers is the smallest number which is exactly divisible by each of the numbers.

For example, 6 and 8 are both factors of 48, 48 is a *Common Multiple*, but it is not the *Least Common Multiple*, for both numbers are also factors of 24, which is the L.C.M. It is sometimes called the **Lowest Common Multiple.**

Method I. When the numbers whose L.C.M. is required can be written in prime factors the L.C.M. can always be written at a glance.

Ex.—*Find the L.C.M. of* 30, 32, 36, 40, 48.

$30 = 2 \times 3 \times 5$
$32 = 2^5$
$36 = 2^2 \times 3^2$
$40 = 2^3 \quad \times 5$
$48 = 2^4 \times 3$

Explanation.—The numbers contain the factors 2, 3, 5 and these only, therefore every multiple contains these factors and the least multiple has these only. Now 2 is repeated 5 times in 32, therefore it must be repeated 5 times in any multiple of 32, similarly 3 must be repeated twice, since it is so in 36, and 5 once.

$\therefore$ L.C.M. $= 2^5 \times 3^2 \times 5 = 1440$.

By a rearrangement of this work the process of factorization and that of finding the L.C.M. may be combined. The work is then written thus:—

2)	30,	32,	36,	40,	48
2)	15,	16,	18,	20,	24
2)	15,	8,	9,	10,	12
2)	15,	4,	9,	5,	6
3)	15,	2,	9,	5,	3
5)	5,	2,	3,	5,	1
	1,	2,	3,	1,	1

Explanation.—2 is contained in each of the numbers. Divide by it.

2 is again contained in some of the numbers. Divide those which contain 2 and leave unaltered those which do not.

Repeat the operation. Repeat it again.

Divide by 3 in the same way, rewriting unchanged numbers which are not divisible.

Divide by 5, rewriting a number where necessary.

The quotients are now all prime to one another.

The divisors on the left, together with the final quotients, form all possible divisors; therefore the product of all these, namely, $2 \times 2 \times 2 \times 2 \times 3 \times 5 \times 2 \times 3$, contains all the original numbers, *i.e.* it is a common multiple of them. But it is clear on studying the process that no number which contains fewer 2's, or fewer 3's, or fewer 5's, can contain all the numbers. Therefore this product is the L.C.M. *Ans.* 1440.

Hints.—(1.) At any stage of the process we may cancel any one of the quotients which will divide exactly into any of the others.

Thus in the above question we should cancel 5 at the fourth line, for it divides exactly into 15; and 3 at the fifth line. Any multiple of 15 would contain the 5 and the 3, which need not therefore be considered

(2.) At first you are advised to divide by prime numbers *only*, but after you are thoroughly familiar with the method you may divide by composite numbers, *provided that you cancel from any number which a composite number will not divide any factors which it has in common with the composite number.*

For instance we might commence the question treated by dividing by 6. When we come to 32, which is not divisible by 6, we divide it by 2, the factor which it has in common with 6. The work then proceeds as shown. *This process needs great care.*

$$\begin{array}{r|rrrrr} 6 & 30, & 32, & 36, & 40, & 48 \\ 4 & \cancel{5}, & 16, & 6, & 20, & \cancel{8} \\ & & 4, & 3, & 5, & \end{array}$$

Ans.
$6 \times 4 \times 4 \times 3 \times 5 = 1440$

(3.) The secret of success in the mental working of such questions is to deal with two numbers only at a time. The process is as follows:—

We factorize 30, and write down $2 \times 3 \times 5 \times \ldots$

But 32 is to divide into the L.C.M. and 32 has 5 2's; therefore we correct our answer into $2^5 \times 3 \times 5 \times \ldots$

But $36 = 2^2 \times 3^2$ divides into it; therefore we need 2 3's; therefore we correct into $2^5 \times 3^2 \times 5 \times \ldots$

We now find that 40 and 48 will divide into the answer as it stands; therefore $2^5 \times 3^2 \times 5$ is the L.C.M.

99. *Method II.* (1) To determine the L.C.M. of two numbers whose factors cannot be seen at a glance.

Let A, B be the numbers. Determine their H.C.F. Call it H. Then $A =$ some factor $\times H = a \times H$, $B = b \times H$, and a and b have no

common factors, for, if they had, H would not be the H.C.F. of A and B. Clearly L.C.M. of A and $B = a \times b \times H$.

Clearly also L.C.M. × H.C.F. $= a \times b \times H \times H = a \times H \times b \times H$
$= A \times B$

Hence: *Product of L.C.M. and H.C.F. of two numbers = product of the numbers.*

(2) For 3 numbers, A, B, C, find the L.C.M. of A and B, and then the L.C.M. of this result and C So on for more numbers than 3.

Ex.—*Find the L.C.M. of* 527, 1271, *and* 1891.

The H.C.F. of 527 and 1271 is 31. Divide each number by 31; $\therefore$ $527 = 1 \times 17$, $1271 = 31 \times 41$; $\therefore$ the L.C.M. of 527, 1271 is $17 \times 31 \times 41$. We test 1891 by these factors and easily find $1891 = 31 \times 61$. Hence the final L.C.M. is $17 \times 31 \times 41 \times 61 = 1318027$.

EXAMPLES XXXIX.

Find the L.C.M. of the following numbers:—

1. 4, 9, 12, 16. **2.** 16, 20, 24, 36. **3.** 16, 18, 24, 27.
4. 14, 20, 28, 30. **5.** 63, 12, 84, 14. **6.** 12, 18, 30, 48, 60.
7. 35, 18, 21, 12. **8.** 18, 25, 27, 45. **9.** 72, 168 54, 108.
10. 24, 27, 63, 84. **11.** 12, 32, 42, 63. **12.** 24, 30, 54, 16.
13. 15, 25, 18, 35. **14.** 18, 24, 39, 15. **15.** 11, 30, 42, 66, 77, 90.
16. 42, 64, 70, 80, 112. **17.** 33, 14, 110 35. **18.** 64, 72, 80, 84, 96.
19. 928, 1073. **20.** 1836, 1938. **21.** 777, 4329, 819.
22. 341, 308, 286, 217, 616. **23.** 1125, 2200, 2424.
24. 3875, 1160, 17500, 3784. **25.** 1131, 1885, 2175.
26. 1311, 2346, 3876. **27.** 3127, 3551, 3953.
28. 252, 322, 522, 667.

What is the smallest number which is exactly divisible by?—

29. 60, 15, 14, 35. **30.** 16, 18, 84, 63.

31. How often does the L.C.M. of the numbers 15, 23, 30, 16, 21 contain each of them?

32. What is the least number which can be exactly divided by 18, 30, 45, and 48?

33. Find the least number exactly containing 170, 204, and 225.

34. Find the least number which is divisible by all numbers up to 16 inclusive.

35.* Prove that the product of any three numbers and their H.C.F. is equal to the product of the L.C.M. of the numbers and the H.C.F.'s of the numbers taken in pairs.

100. The following examples illustrate the application of the principles of the last two chapters to problems:—

Ex. 1.—*What is the greatest sum of money which is exactly contained both in* £2. 2s. $1\frac{1}{4}$d. *and in* £6. 13s. $5\frac{3}{4}$d. ?

The first sum is 2021 farthings, the second 6407 farthings. The H.C F. of 2021 and 6407 is 43; therefore 43 farthings, *i.e.* $10\frac{3}{4}$d., is the greatest sum of money exactly contained in both.

Ex. 2.—*A courtyard measures* 56 *yds.* 0 *ft.* 5 *ins. by* 177 *yds.* 2 *ft.* 11 *ins., and is to be paved with equal square paving stones. What is the size of the edge of the largest stone which can be used?*

The dimensions reduce to 2021 inches and 6407 inches; therefore the number of inches in the edge of the paving stones must divide exactly into both 2021 and 6407. The H.C.F. of these two is 43; therefore 43 inches is the size of the largest stone which can be used.

Ex. 3.—*The front wheels of a cart are* 4 *ft.* 2 *ins. in circumference, and particular points on a front and on a back wheel are in contact with the ground at the same instant,* 2112 *times every* 35 *miles. Find the size of the back wheel, which is known to contain a whole number of inches in its circumference.*

2112 times in 35 miles means once in every 1050 inches; therefore every 1050 inches both wheels have revolved an exact number of times; therefore 1050 inches is the L.C.M. of their circumference. The problem now is: the L.C.M. of two numbers is 1050, and one of them is 50 (= 4 ft. 2 ins.), What is the other? $1050 = 21 \times 50$; therefore the other *must* contain 21 as a factor, and *may* contain any other factor of 50; therefore the other may be

$$21,\ 21\times 2,\ 21\times 5,\ 21\times 2\times 5,\ 21\times 5\times 5,\ 21\times 2\times 5\times 5,$$

i.e. 21, 42, 105, 210, 525, 1050;

therefore the circumference of the other wheel may be

1 ft. 9 ins., 3 ft. 6 ins., 8 ft. 9 ins., 17 ft. 6 ins., 43 ft. 9 ins., 87 ft. 6 ins.

A wheel 87 ft. 6 in. in circumference is not recommended for a cart.

Ex. 4.—*Find a number which on dividing* 34100 *leaves a remainder* 63, *and on dividing* 307100 *leaves a remainder* 93.

Since on dividing 34100 a remainder 63 is left, it follows that the number divides $34100-63$ exactly, *i.e.* that it is a factor of 34037. Similarly it is a factor of $307100-93$, *i.e.* of 307007. Hence it is a common factor of 34037 and 307007. Finding the H.C.F. of these two, we obtain 337, which is a prime number, and therefore the only number which satisfies the conditions.

Ex. 5.—*What is the least number which, when divided by* 32, *leaves* 22 *as remainder, when divided by* 42 *leaves* 32, *when divided by* 52 *leaves* 42 *?*

If 10 were added to this number, it would leave 22 + 10 as remainder after the division by 32, *i.e.* 32 would be contained once oftener in it, so also would 42 and 52. Hence, if 10 be added to the number, it becomes divisible by 32, 42, and 52. The L.C.M. of these numbers, *i.e.* the least number which is divisible by them all, is 8736 ; therefore the number required is 8726.

EXAMPLES XL.

(Miscellaneous Examples and Problems on Factors, H.C.F., and L.C.M.)

1. Write down all the prime factors contained in 2310.

2. What are the prime factors common to 12012 and 9072 ?

3. Find the H.C.F. and L.C.M. of 7648 and 13384.

4. Find the H.C.F. and L.C.M of 3432 and 3575.

5. How often does the L.C.M. of 1530, 61506, 10302 contain their H.C.F. ?

6. How often is the H.C.F. of 30600, 30753, and 30906 contained in the L.C.M. of the same three numbers ?

7. Decompose 7648, 13384, and 63096 into their prime factors ; and find the H.C.F. and L.C.M. of these three numbers.

8. Find the least number into which 616 and 385 will each divide without remainders.

9. Factorize 123456.

10. Factorize 135792.

11. Find three numbers such that the first multiplied by the second gives 527 ; the product of the second and third is 713, and that of the first and third is 391.

12. Find three numbers such that the product of the first and the second is 437, the product of the first and third is 779, and the sum of all three is 83.

13. The L.C.M. of two numbers is 244188 and their G.C.M. is 84. If one of the numbers is 1428, find the other.

14. The L.C.M. of two numbers is 28 times their G.C.M. The sum of the L.C.M. and the G.C.M. is 1740. One number is 240 : what is the other ?

15. Find the greatest number which shall divide 13956 and 14565 and leave a remainder of 7 in each case.

16. Find the greatest number such that, if 13850 and 17030 be divided by it, the remainder in each case is 17.

17. What is the greatest number which when used as the divisor of 68130 and 107275 will leave remainders 27 and 49 respectively ?

18. What numbers of four digits each can have 119 as their G.C.M. and 13923 as their L.C.M.?

19. Find a number which on dividing 3600, 5418, 7337, and 10064 leaves in every case a remainder 65.

20. Find the least number which when divided by 35 leaves a remainder 25, when divided by 45 leaves 35, when divided by 55 leaves 45.

21. Four bells begin tolling at the same time, and continue at intervals of 20, 25, 28, and 50 seconds respectively. Find the smallest integral number of minutes at the end of which they will all be tolling together again.

22. A man wishes to have his library arranged on shelves, each containing the same number of volumes. He tries successively to put 15, 16, and 17 volumes on each shelf, and each time he has 3 volumes remaining. If the number of volumes in his library be the least number of volumes for which this would be the case, show that he cannot arrange it in the way that he desires except upon shelves which each contain 3 volumes.

23. In going up a flight of stairs, if I go up 2 at a time there is 1 over, if 3 at a time there are 2, if 4 at a time there are 3, if 5 at a time 4, if 6 at a time 5, if 7 at a time 6. How many stairs are there?

24. Find the smallest quantity of a substance which can be divided into an integral number of parcels weighing respectively 25 lbs., 1 lb., 1 oz., and 17 lbs.

25. If the L.C.M. of 68, 170, and another number is 2380, and the H.C.F. of the three numbers is 17, what may the other number be?

26.* Four numbers have one factor in common, and there is no other factor which is common to any two of them. Their product is 65975910, and their L.C.M. 30030. What is their common factor?

27.* What is the least number which when divided by 48 leaves a remainder 40, when divided by 56 leaves 48, when divided by 64 leaves 56, and when divided into 41474 leaves 1234.

28.* Find as many prime factors as possible of

$$567943143205674320568567943 2.$$

29.* Find as many prime factors as possible of

$$1+1.2.3.4.5.6.7.8.9.10.11.12.$$

30.* Prove the following test for divisibility by 7, 11, or 13. Divide the number into groups of three figures from the right, treat these groups as the digits were treated in testing for divisibility by 11 (§ 29), adding the groups alternately, commencing from the right, and then subtracting the remaining groups. The number is divisible by 7, 11, or 13, if the number left by this process is so divisible.

SECTION IV.

VULGAR FRACTIONS.

CHAPTER XV.

Definitions.

101. In the tables of Long Measure which we gave on p. 51, the smallest *unit* (see § 47) which was named was the inch. If we take a foot-rule divided into inches, and measure the length of any object whatever, we shall probably find that it is not an exact number of inches. For instance, the page of this book is more than 6 ins. long, but not quite 7 in.—it is 6 ins. and a *broken inch.* Such a "broken inch" is called a "fraction" [Lat. *fractus*, "broken"] of an inch.

A **Fraction** is a part of a whole, more precisely defined as follows:—

If a unit be divided into any integral number of equal parts, and one or more of these parts be taken, a fraction of the unit is obtained.

Ex.—Thus 2s. 11d. is a fraction of a pound; for we have to "break into" a pound to pay it.

So 4 is a fraction of 8; for we may get 4 units from 8 units by "breaking" the 8 units into 2 sets of 4.

If you look at an ordinary foot-rule, you will see that the space between the inch-lines is divided (usually) into eight equal parts by shorter lines. These parts are called "eighths of an inch," because there are eight of them. Such a foot-rule will enable you to measure the length of this page much more accurately. You will find it 6 ins. and about six of these eighths. We express this as "six and six-eighths."

When the unit is divided into 2 equal parts the parts are called *halves* (sing. *half*).

When the unit is divided into 3 equal parts the parts are called *thirds*.

When the unit is divided into 4 equal parts the parts are called *fourths* or *quarters*.

When the unit is divided into 5 equal parts the parts are called *fifths*.

The notation for a half is $\frac{1}{2}$, for a quarter is $\frac{1}{4}$, for three-quarters $\frac{3}{4}$, &c.

We have, of course, used these signs before for one half-penny, one farthing, and three farthings; but we always wrote them $\frac{1}{2}$d., $\frac{1}{4}$d., $\frac{3}{4}$d., and meant one *halfpenny*, &c. Now we use them for a half, a quarter, &c.

We should read $\frac{31}{125}$ as "thirty-one one-hundred-and-twenty-fifths," or, often, "thirty-one one-twenty-fifths," or "thirty-one *over* one (hundred and) twenty-five."

It is sometimes written 31/125, and this notation is called the **Solidus Notation.**

The — and / have been mentioned previously as the symbols for division. This double use of the same symbol will be justified later. (See § 114, Ex. 3.)

102.—The above definitions enable us to interpret fractions of concrete quantities. We shall examine this more fully in Chap. xix., and, for the present, give only a simple example.

Ex.—*Interpret* $\frac{3}{8}$ *of* £1.

This means, and is read, "three-eighths of a pound," *i.e.* we are directed to divide £1 into eight equal parts, which is done by ordinary division, and take three of them, which is done by multiplication. *Ans.* 7s. 6d.

	£	s.	d.
8)	1	0	0
		2	6
			3
		7	6

EXAMPLES XLI.

1. Express in figures: Three-fifths, five-sevenths, seven-tenths, six-seventeenths, and seventeen-one-hundred-and twenty-fifths.

2. Express in figures: Five-sixths, three-sevenths, six-elevenths, seven-sixteenths, and sixteen-one-hundred and thirty-firsts.

3. Write in words: $\frac{4}{5}$, $\frac{7}{8}$, $\frac{9}{13}$, $\frac{3}{23}$, and $\frac{11}{111}$.

4. Write in words: $\frac{1}{6}$, $\frac{3}{8}$, $\frac{11}{12}$, $\frac{4}{25}$, and $\frac{10}{113}$.

5. Find the value of $\frac{1}{6}$, $\frac{2}{6}$, $\frac{3}{6}$, $\frac{4}{6}$, and $\frac{5}{6}$ of £1.

6. Find the value of $\frac{1}{10}$, $\frac{3}{10}$, $\frac{6}{10}$, $\frac{7}{10}$, and $\frac{9}{10}$ of £1.

103. We shall now explain the terms used in describing fractions.

The **Denominator** of a fraction is the number which shows into how many equal parts the unit is divided.

Thus in $\frac{3}{4}$ the 4 is the denominator, for it tells us that the unit is divided into 4 equal parts.

The **Numerator** of a fraction is the number which shows how many of these equal parts are taken to form the fraction.

Thus in $\frac{3}{4}$ the 3 is the numerator, for it tells us that of the 4 equal parts we take 3.

The *denominator* names (Lat. *nomen*, "name") the parts, and tells us whether they are fourths or fifths, &c. The *numerator* counts (Lat. *numerus*, "number") the parts, and tells us how many of them to take.

Fractions whose denominators are 10, 100, 1000, ... are called **Decimal Fractions.** They are represented, with their denominators omitted, by the special notation described in Chap. xvi.

All other fractions are called **Vulgar Fractions**. (Lat. *vulgus*, "common").

A **Proper Fraction** is one whose numerator is *less* than its denominator.

Thus $\frac{2}{3}$ is a proper fraction, for 2 (numerator) is less than 3 (denominator).

An **Improper Fraction** is one whose numerator is *not less* than its denominator.

Thus $\frac{4}{3}$ is an improper fraction, for 4 (numerator) is greater than 3 (denominator). A fraction was defined as a *part* of a whole; but, if we divide the unit into three parts, and *take four such parts*, we have clearly more than the unit we started with. Thus we divide a foot into twelfths and call them *inches*, but 13 such parts, *i.e.* $\frac{13}{12}$ ft. is 13 ins., *i.e.* 1 ft. 1 in., and is not a "part of a foot," but is more than a foot. Such quantities are not really "fractions" at all; hence their name of *improper fractions*.

A number consisting of units only is called a **whole number** or **integer** (Lat. *integer*, whole).

Thus 3, 4, 7, and 10 are whole numbers. Up to this point of the book whenever the word "number" has been used we have meant "whole number."

A **mixed number** is a quantity consisting of two parts, one an integer and the other a fraction. We write the two parts side by side.

Thus $3\frac{1}{2}$, $2\frac{1}{3}$ are mixed numbers. They are read as "three and a-half," "two and a-third." We now extend the meaning of the word "number," which is no longer "the result of counting," but the "result of measuring," and may therefore contain portions of a unit. Be careful not to confuse $3\frac{1}{2}$ with $3 \times \frac{1}{2}$.

104. (1) *An integer may always be expressed as a fraction (improper fraction) with any denominator we please.*

Thus sixpence is 24 farthings, *i.e.*, 6d. $= \frac{24}{4}$d., *i.e.* $6 = \frac{24}{4}$. So six feet is 72 inches, but an inch is the twelfth part of a foot; therefore 6 ft. $= \frac{72}{12}$ ft., *i.e.* $6 = \frac{72}{12}$.

Clearly, however many equal parts the unit is divided into, we can divide 6 units into 6 times that number of parts, 7 into 7 times, &c., *e.g.* $6 = \frac{60}{10} = \frac{54}{9} = \frac{48}{8} = \frac{42}{7}$, &c.; *i.e.* if the unit is divided into 10 equal parts, it takes 60 of these parts to make 6 units; if into 9, 54, &c. In general, if a, b be any numbers, $a = ab/b$.

(2) *A mixed number may always be expressed as an improper fraction.*

Ex. 1. $3\frac{1}{5} = 3 + \frac{1}{5} = \frac{15}{5} + \frac{1}{5} = \frac{15+1}{5} = \frac{16}{5}$.

Ex. 2. $4\frac{1}{7} = 4 + \frac{1}{7} = \frac{28}{7} + \frac{1}{7} = \frac{28+1}{7} = \frac{29}{7}$.

Note in these examples how 15 fifths and 1 fifth are written $(15+1)$ fifths, and thus only one denominator is needed.

(3) *An improper fraction may always be expressed as a mixed number or an integer.*

We have to work the process of (2) backwards. Take Ex. (1). We must first find how many 5's there are in 16, *i.e.* divide 16 by 5, and thus we can write it as $\frac{15+1}{5}$, then as $\frac{15}{5} + \frac{1}{5}$, then as $3\frac{1}{5}$.

Ex. 3. $\frac{17}{3} = \frac{15+2}{3} = \frac{15}{3} + \frac{2}{3} = 5 + \frac{2}{3} = 5\frac{2}{3}$,

Ex. 4. $\frac{23}{6} = \frac{18+5}{6} = \frac{18}{6} + \frac{5}{6} = 3 + \frac{5}{6} = 3\frac{5}{6}$.

EXAMPLES XLII.†

Express as improper fractions—

1. $5\frac{13}{19}$, $6\frac{17}{25}$, $7\frac{19}{27}$, $8\frac{21}{31}$, $2\frac{111}{113}$, $10\frac{17}{115}$.

2. $5\frac{11}{17}$, $6\frac{15}{23}$, $7\frac{17}{25}$, $8\frac{19}{29}$, $9\frac{109}{111}$, $10\frac{15}{117}$.

3. $15\frac{11}{19}$, $26\frac{13}{25}$, $37\frac{17}{27}$, $48\frac{19}{31}$, $59\frac{107}{113}$, $610\frac{15}{117}$.

4. $19\frac{11}{15}$, $25\frac{16}{23}$, $37\frac{15}{37}$, $41\frac{17}{38}$, $57\frac{109}{113}$, $617\frac{15}{109}$.

Express as mixed numbers—

5. $\frac{71}{11}$, $\frac{72}{13}$, $\frac{73}{17}$, $\frac{74}{25}$, $\frac{95}{27}$, $\frac{97}{32}$, $\frac{123}{35}$, $\frac{234}{41}$, $\frac{346}{57}$, $\frac{1234}{101}$.

6. $\frac{91}{11}$, $\frac{92}{13}$, $\frac{93}{17}$, $\frac{94}{25}$, $\frac{59}{27}$, $\frac{79}{32}$, $\frac{234}{35}$, $\frac{345}{41}$, $\frac{356}{57}$, $\frac{2341}{101}$.

105. *Illustration of Fractions.*—The meaning of fractions, and of the various laws which they obey, can be well brought out by several devices. We shall illustrate that of the *Divided Scale.*

The figure represents on a reduced scale a portion of an ordinary foot-rule, divided into inches, each inch again subdivided into 2, 4, and 8 parts. The rules of § 104 may be illustrated thus:—

Rule 1.—3 inches, *i.e.* the distance from the line 0 to the the line 3, is 6 half-inches (count them), *or* 12 quarter-inches, *or* 24 eighths, *i.e.* $3 = \frac{6}{2} = \frac{12}{4} = \frac{24}{8}$. So also $4 = \frac{8}{2} = \frac{16}{4} = \frac{32}{8}$, $5 = \frac{10}{2} = \frac{20}{4} = \frac{40}{8}$.

Rule 2.—$3\frac{1}{8}$ ins. is clearly 25 of the smallest divisions, *i.e.* 25 eighths; therefore $3\frac{1}{8} = \frac{25}{8}$.

Rule 3.—19 of the smaller divisions cover two complete inches, and reach to the third small division of the third inch, *i.e.* $\frac{19}{8} = 2\frac{3}{8}$.

Squared paper, which should always be in the hands of every beginner in mathematics, will provide even better illustrations. (See §§ 115, 117.) You should satisfy yourself by such of the truth of the following and similar statements:—$\frac{4}{8} = \frac{1}{2}$, $\frac{4}{36} = \frac{1}{9}$, $2\frac{1}{6} = \frac{13}{6}$, $\frac{1}{12} + \frac{1}{144} = \frac{13}{144}$, &c.

† Nos. 1, 2, 5, and 6 form excellent mental exercises. (See § 30.)

Reduction of Fractions.

106. The examples we have just been studying (§ 105) show that the same fraction may be expressed in a great many ways. We can prove by a figure, for instance, that

$$\tfrac{1}{4} = \tfrac{2}{8} = \tfrac{3}{12} = \tfrac{4}{16} = \tfrac{6}{24} = \tfrac{9}{36} = \tfrac{12}{48} = \tfrac{18}{72} = \tfrac{36}{144}.$$

Verify all these statements for yourself.

The laws underlying all these changes are

(1) *The value of a fraction is not altered by multiplying numerator and denominator by the same number.*

In general, if a, b, and c be whole numbers, $\dfrac{a}{b} = \dfrac{ac}{bc}$.

Illustration.

Consider the fraction $\frac{3}{4}$.

This means that the unit is divided into 4 equal parts, and we take 3 of them.

Subdivide each part into 3 equal portions.

The unit has now 4 parts, *i.e.* 4×3 portions.

The fraction has 3 parts, *i.e.* 3×3 portions.

Hence the fraction is $\dfrac{3 \times 3}{4 \times 3}$ of the unit.

$$\therefore \quad \tfrac{3}{4} = \frac{3 \times 3}{4 \times 3} = \tfrac{9}{12}.$$

Proof.

Consider the fraction a/b.

This means that the unit is divided into b equal parts, and we take a of them.

Subdivide each part into c equal portions.

The unit has now b parts, *i.e.* $b \times c$ portions.

The fraction has a parts, *i.e.* $a \times c$ portions.

Hence the fraction is $\dfrac{a \times c}{b \times c}$ of the unit.

$$\therefore \quad \frac{a}{b} = \frac{a \times c}{b \times c} = \frac{ac}{bc}.$$

(2) *The value of a fraction is not altered by dividing numerator and denominator by the same number.*

In general, if a, b be whole numbers divisible by another whole number c, $\dfrac{a}{b} = \dfrac{a \div c}{b \div c}$.

Proof.—If a be divisible by c, it can be arranged in groups containing c each. There are $a \div c$ of such groups. So b can be arranged in groups containing c each. There are $b \div c$ of such groups. We may

regard the unit as subdivided, not into b parts, but into $b \div c$ groups of parts, of which we take $a \div c$.

$$\therefore \quad \frac{a}{b} = \frac{a \div c}{b \div c}.$$

The division of both numerator and denominator by a common factor is called **Cancelling.**

When the numerator and denominator have no common factor the fraction is said to be in its **Lowest Terms.**

The process of changing the denominator of a fraction is called **Reduction.**

This is not really a new meaning of the word "reduction." When we say "reduce £3 to shillings" we mean "change £3 into twentieths of a £," *i.e.* "write 3 as $\frac{60}{20}$."

107. *Reduction to Lowest Terms.*—A fraction is reduced to lowest terms by dividing both numerator and denominator by all the factors common to both. Labour is often saved by finding the H.C.F. of numerator and denominator and dividing both by this. The fraction must then be in its lowest terms.

Ex. 1.—*Express* $\frac{7}{11}$ *with denominator* 55.

11 can be changed into 55 by multiplying by 5. The fraction is unchanged if both numerator and denominator are multiplied by 5; therefore $\frac{7}{11} = \frac{7 \times 5}{11 \times 5} = \frac{35}{55}$. *Ans.*

Ex. 2.—*Reduce* $\frac{32}{48}$ *to lowest terms.*

Work thus: $\frac{32}{48} = \frac{4}{6} = \frac{2}{3}$. The divisors of both numerator and denominator are 8 and then 2.

Ex. 3.—*Reduce* $\frac{8316}{7623}$ *to lowest terms.*

Work thus: $\frac{8316}{7623} = \frac{924}{847} = \frac{84}{77} = \frac{12}{11} = 1\frac{1}{11}$. The divisors are 9, 11, 7. These can be seen at once *if the Divisibility Rules* (see § 85) *are thoroughly known.*

Ex. 4.—*Reduce* $\frac{481}{629}$ *to lowest terms.*

Here no factors are shown by the Divisibility Rules. We find the H.C.F. It is 37. Dividing both numerator and denominator by 37, we obtain $\frac{13}{17}$, which is the answer.

Ex. 5.—*Reduce* $2\frac{2592}{3456}$. A comma is placed after the 2, to indicate that the other operations do not refer to it.

$2, \frac{2592}{3456} = \frac{324}{432} = \frac{81}{108} = \frac{9}{12} = \frac{3}{4}$. Divisors 8, 4, 9, 3. *Ans.* $2\frac{3}{4}$.

EXAMPLES XLIII.

[*Exx.* 1–4 *should be done mentally by advanced pupils.*]

Reduce to their lowest terms the following fractions, expressing as mixed numbers where possible :—

1. $\frac{16}{44}, \frac{20}{55}, \frac{26}{52}, \frac{29}{58}, \frac{64}{104}, \frac{65}{104}, \frac{68}{119}, \frac{70}{119}, \frac{84}{119}, \frac{42}{238}$.

2. $\frac{20}{48}, \frac{25}{65}, \frac{22}{66}, \frac{34}{68}, \frac{36}{66}, \frac{52}{117}, \frac{54}{117}, \frac{56}{119}, \frac{65}{117}, \frac{39}{234}$.

3. $\frac{56}{84}, \frac{56}{48}, \frac{65}{91}, \frac{64}{96}, \frac{60}{96}, \frac{60}{69}, \frac{70}{259}, \frac{72}{256}, \frac{75}{250}, \frac{72}{272}$.

4. $\frac{63}{91}, \frac{48}{84}, \frac{78}{91}, \frac{36}{96}, \frac{90}{66}, \frac{54}{69}, \frac{84}{259}, \frac{85}{255}, \frac{80}{248}, \frac{88}{132}$.

5. $\frac{324}{720}, \frac{324}{396}, \frac{720}{864}$.

6. $\frac{246}{342}, \frac{247}{323}, \frac{272}{425}$.

7. $\frac{330}{726}, \frac{630}{936}, \frac{324}{612}$.

8. $\frac{321}{749}, \frac{299}{529}, \frac{510}{561}$.

9. $\frac{1573}{1716}$. **10.** $\frac{2016}{3168}$. **11.** $\frac{4025}{4375}$. **12.** $\frac{825}{2709}$. **13.** $\frac{1786}{2451}$.

14. $\frac{3360}{3528}$. **15.** $\frac{3553}{5643}$. **16.** $\frac{1001}{1617}$. **17.** $\frac{882}{1386}$. **18.** $\frac{3920}{6720}$.

19. $\frac{2520}{8568}$. **20.** $\frac{7429}{8303}$. **21.** $\frac{1386}{3003}$. **22.** $\frac{1197}{5733}$. **23.** $\frac{1620}{5994}$.

24. $\frac{4641}{5187}$. **25.** $\frac{3663}{3993}$. **26.** $\frac{2379}{4498}$. **27.** $\frac{3857}{9823}$. **28.** $\frac{3383}{9333}$.

29. $\frac{14938}{23474}$. **30.** $\frac{25194}{88179}$. **31.** $\frac{52003}{66424}$. **32.** $\frac{10812}{22800}$.

33. $\frac{10265}{14371}$. **34.** $\frac{55247}{74841}$. **35.** $\frac{5184}{19008}$. **36.** $\frac{23535}{96780}$.

37. $\frac{93208}{13786}$. **38.** $\frac{37376}{27968}$. **39.** $\frac{46875}{265625}$. **40.** $\frac{342405}{661983}$.

41. $\frac{374192}{575680}$. **42.** $\frac{128352}{238368}$. **43.** $\frac{1089270}{1307124}$. **44.** $\frac{2396835}{3309915}$.

45. $\frac{2619344}{7483840}$. **46.** $\frac{2622048}{1411872}$.

108. *Comparison of Fractions.*—We can tell at once which of the fractions $\frac{8}{12}$, $\frac{9}{12}$, $\frac{10}{12}$ is the greatest and which the least, but we cannot see equally easily which of the fractions $\frac{3}{4}$, $\frac{2}{3}$, $\frac{5}{6}$ is the greatest and which the least. Clearly, to compare fractions with one another they should have the same denominator.

So also, if we were asked whether 251 pence or 21 shillings was the greater, we might not be able to answer at once. We should convert either the shillings into pence or the pence into shillings. We should be compelled, in fact, to reduce them to the same *denomination*. (See § 57.)

Any denominator will serve our purpose, provided that it is the same for all the fractions considered, but, clearly, the smaller the denominator the less trouble there will be to obtain it; and we generally work with what is called the *Least Common Denominator* (L.C.D). This will be the L.C.M. (see § 98) of the denominators.

Thus, with the fractions $\frac{3}{4}$, $\frac{2}{3}$, $\frac{5}{6}$ the denominators are 4, 3, 6; the L.C.M. of these is 12. Therefore we change the fractions into equivalent fractions with the denominator 12. They become $\frac{9}{12}$, $\frac{8}{12}$, $\frac{10}{12}$, and it is clear now that $\frac{8}{12}$, *i.e.* $\frac{2}{3}$, is the smallest of them all.

The same purpose would have been served by reducing all the fractions so as to have the same numerator, but this process of *Least Common Numerator* is not so useful afterwards. In the case given the L.C.N. is 30 and the fractions become $\frac{30}{40}$, $\frac{30}{45}$, $\frac{30}{36}$; and a little thought will show that the middle one is again the smallest.

Quantities are said to be arranged in **Ascending Order** of Magnitude when each is less than the one that follows it; in **Descending Order** of Magnitude when each is greater than the one that follows it.

Ex.—Arrange in ascending order $\frac{13}{28}$, $\frac{16}{35}$, and $\frac{19}{42}$.

The L.C.M. of the denominators is usually best found by the factor method (see § 98). The denominators are $2^2 . 7$, $5 . 7$, $2 . 3 . 7$, and the L.C.M. is $2^2 . 3 . 5 . 7 = 420$. The factors necessary to bring the denominators up to this are seen at once to be $3 . 5$, $2^2 . 3$, $2 . 5$, *i.e.* 15, 12, 10.

Hence $\frac{13}{28} = \frac{13 \times 15}{420} = \frac{195}{420}$, $\frac{16}{35} = \frac{16 \times 12}{420} = \frac{192}{420}$, $\frac{19}{42} = \frac{190}{420}$, and the fractions are, at present, in descending order, the ascending order being $\frac{19}{42}$, $\frac{16}{35}$, and $\frac{13}{28}$.

EXAMPLES XLIV.

1. Find fractions with denominators 168 which are equal to $\frac{7}{8}$, $\frac{11}{21}$, $\frac{13}{56}$, $\frac{13}{24}$, $\frac{19}{28}$.

2. Find fractions with denominators 252 which are equal to $\frac{6}{7}$, $\frac{11}{12}$, $\frac{13}{28}$, $\frac{13}{63}$, $\frac{19}{28}$.

Reduce to their L.C.D. (Least Common Denominator)—

3. $\frac{11}{12}$, $\frac{17}{90}$. **4.** $\frac{13}{14}$, $\frac{17}{35}$. **5.** $\frac{20}{63}$, $\frac{13}{84}$. **6.** $\frac{3}{22}$, $\frac{11}{36}$.

7. $\frac{7}{12}$, $\frac{8}{15}$, $\frac{9}{25}$. **8.** $\frac{7}{9}$, $\frac{11}{45}$, $\frac{17}{75}$. **9.** $\frac{7}{15}$, $\frac{10}{21}$, $\frac{16}{35}$. **10.** $\frac{5}{27}$, $\frac{11}{24}$, $\frac{4}{15}$.

11. $\frac{7}{16}$, $\frac{11}{18}$, $\frac{17}{24}$, $\frac{19}{36}$, $\frac{25}{48}$. **12.** $\frac{2}{3}$, $\frac{4}{9}$, $\frac{16}{27}$, $\frac{8}{81}$, $\frac{16}{243}$.

13. $\frac{4}{7}$, $\frac{3}{10}$, $\frac{5}{12}$, $\frac{17}{35}$, $\frac{4}{63}$, $\frac{15}{28}$. **14.** $\frac{11}{27}$, $\frac{17}{24}$, $\frac{5}{6}$, $\frac{7}{15}$, $\frac{2}{9}$, $\frac{35}{36}$.

Arrange in *descending* order of magnitude—

15. $\frac{3}{4}$, $\frac{7}{8}$, $\frac{15}{16}$, $\frac{31}{32}$. **16.** $\frac{5}{6}$, $\frac{5}{8}$, $\frac{2}{9}$, $\frac{13}{24}$.

17. $\frac{3}{5}$, $\frac{7}{10}$, $\frac{6}{25}$, $\frac{11}{30}$. **18.** $\frac{5}{7}$, $\frac{11}{12}$, $\frac{13}{15}$, $\frac{25}{27}$.

19. Which is the greatest of the fractions $\frac{15}{17}$, $\frac{23}{26}$, $\frac{68}{77}$?

20. Which is the least of the fractions $\frac{32}{91}$, $\frac{42}{119}$, $\frac{53}{143}$?

Arrange in *ascending* order of magnitude—

21.* $\frac{11}{13}$, $\frac{22}{25}$, $\frac{33}{37}$, $\frac{44}{49}$. **22.*** $\frac{2}{17}$, $\frac{4}{29}$, $\frac{6}{43}$, $\frac{12}{85}$.

23.* $\frac{3}{13}$, $\frac{6}{25}$, $\frac{9}{41}$, $\frac{12}{53}$, $\frac{18}{73}$. **24.*** $\frac{3}{10}$, $\frac{6}{19}$, $\frac{9}{28}$, $\frac{12}{37}$, $\frac{18}{59}$.

Addition of Fractions.

109. Only quantities of the same kind can be added together. It is impossible to add £3 to 5 ft. But even when quantities are of the same kind it is not always possible to add them. You cannot add 3s. to 4d. without changing the shillings into pence. It is true you may write it 3s. 4d., but this means only 3 shillings *and* 4 pence, and does not combine them. If you write it as 36d. + 4d., you can, however, combine the numbers and write 40d. In other words, *quantities can be added only when they are of the same kind* and also *of the same denomination* (*or name*). We infer from this that:—

Fractions with the same denominator may be added at once by adding their numerators.

3-fifths and 4-fifths together make 7-fifths.

Fractions with different denominators can only be added after they have been reduced to the same denominator.

Mixed numbers are added by adding the integers first and the fractions afterwards.

Just as when we have to add 3s. 5d. to 6s. 9d. (which are mixed concrete numbers), we add first the 9 and 5 and then the 6 and 3. The result of the process will often be an integer + an improper fraction, just as the result of adding 3s. 5d. to 6s. 9d. is 9s. 14d. It is necessary to convert the improper fraction to a mixed number, just as we convert the 14d. into 1s. 2d., and add the integer part to the integer already obtained, just as we write 9s. + 1s. 2d., *i.e.* 10s. 2d.

Hints :—

(1) Mixed numbers must *never* be converted into improper fractions. To do this would greatly increase the labour.

(2) Improper fractions must *always* be first converted into mixed numbers.

(3) Before commencing the addition usually reduce any fractions not already in their lowest terms to their lowest terms.

(4) The common denominator need only be written once, and all the numerators should be written in the line above it.

(5) The common denominator should usually be left in factors until the last step.

(6) Even when the fractions to be added are in their lowest terms, the answer is often not in its lowest terms. When this is the case reduce it, *e.g.* $\frac{1}{2}+\frac{1}{6}=\frac{3+1}{6}=\frac{4}{6}$. This is not in its lowest terms, and must be written $\frac{2}{3}$.

(7) When several fractions have to be added it is often advisable to add two or more together, leaving the others to be added afterwards. Only experience can decide; but this should usually be tried when several denominators have a simple L.C.M. which is not shared by the rest. (See Ex. xlv., 35.)

Ex. 1.—*Add* $\frac{5}{17}+\frac{2}{51}+\frac{3}{34}$.

The denominators are 17, 3.17, 2.17. The L.C.D. is 2.3.17. This is found *mentally*. The written work is

$$\frac{30+4+9}{2.3.17}=\frac{43}{2.3.17}=\frac{43}{102}.\ \textit{Ans.}$$

We see at a glance that 17 divides 2.3 times into 2.3.17, and write down 2.3.5, *i.e.* 30, as the numerator of the first fraction. So with the rest.

Ex. 2.—*Find the sum of* $2\frac{6}{8}$, $1\frac{1}{6}$, $3\frac{1}{10}$, $4\frac{4}{15}$.

The first of these fractions is not in its lowest terms. We reduce $\frac{6}{8}$ to $\frac{3}{4}$ before beginning. We then add the integers 2, 1, 3, 4—total, 10. We then deal with the proper fractions $\frac{3}{4}$, $\frac{1}{6}$, $\frac{1}{10}$, $\frac{4}{15}$.

The denominators are 4, 6, 10, 15,
i.e. 2^2, 2.3, 2.5, 3.5.

We see at once that the L.C.D. is $2^2.3.5$. The work is written as follows :—

$$2\tfrac{6}{8}+1\tfrac{1}{6}+3\tfrac{1}{10}+4\tfrac{4}{15}=2\tfrac{3}{4}+1\tfrac{1}{6}+3\tfrac{1}{10}+4\tfrac{4}{15}$$
$$=10\frac{45+10+6+16}{2^2.3.5}=10\tfrac{77}{60}\left[=10+1\tfrac{17}{60}\right]=11\tfrac{17}{60}.$$

The step enclosed in [ ] can usually be omitted. Moreover, in rewriting the question, fractions not in their lowest terms can usually be reduced *mentally* and written in their lowest terms at once.

Ex. 3.—*Add* $2\frac{56}{84}$, $3\frac{1}{9}$, $1\frac{13}{16}$, $\frac{217}{25}$.

$$\text{Sum} = 2\tfrac{2}{3} + 3\tfrac{1}{9} + 1\tfrac{13}{16} + 8\tfrac{17}{25}\ (^1)$$

$$= 14\,\frac{2400 + 400 + 2925 + 2448}{3600}$$

$$= 14\tfrac{8173}{3600} = 16\tfrac{973}{3600}\ (^4)\ \textit{Ans.}$$

$$\begin{array}{r} 4\ )\ 3600\ (^2) \\ 4\ )\ \ 900 \\ \hline 225 \\ 675 \\ \hline 2925 \end{array} \qquad \begin{array}{r} 5\ )\ 3600\ (^3) \\ 5\ )\ \ 720 \\ \hline 144 \\ 1008 \\ \hline 2448 \end{array}$$

Notes:—

(1) The fractions are reduced to their lowest terms on rewriting. (Hint 3.) The improper fraction is reduced to a mixed number. (Hint 2.) This can easily be done mentally in this case.

(2) The necessary working for reducing $\frac{13}{16}$ to the denominator 3600 is shown here. Such working should be exhibited *neatly* on one side of the main work. We have to divide 3600 by 16, and multiply the quotient by 13. The advanced student can divide 3600 by 16 in one operation. In multiplying by 13 do not waste time by writing the product of the 225 by the 1 of 13.

(3) This is similar working for the reduction of $\frac{17}{25}$. The advanced student can easily divide by 25 mentally. (See § 42.)

(4) If the rule of § 30 is thoroughly known, no intermediate step is needed here. A glance should be given to see that the answer is in its lowest terms.

EXAMPLES XLV.

Add together—

1. $4\frac{9}{14}$, $7\frac{24}{35}$. **2.** $5\frac{13}{27}$, $6\frac{11}{18}$. **3.** $7\frac{13}{28}$, $6\frac{29}{42}$. **4.** $11\frac{14}{15}$, $7\frac{13}{25}$.

5. $18\frac{16}{45}$, $1\frac{37}{54}$. **6.** $2\frac{6}{7}$, $4\frac{11}{35}$. **7.** $9\frac{13}{16}$, $11\frac{27}{40}$. **8.** $5\frac{9}{16}$, $6\frac{13}{24}$.

9. $5\frac{8}{9}$, $7\frac{140}{150}$. **10.** $5\frac{14}{15}$, $3\frac{70}{180}$. **11.** $4\frac{5}{12}$, $5\frac{11}{18}$. **12.** $7\frac{19}{25}$, $8\frac{14}{15}$.

13. $12\frac{22}{48}$, $11\frac{75}{108}$. **14.** $3\frac{91}{112}$, $\frac{31}{24}$. **15.** $6\frac{57}{81}$, $\frac{221}{45}$. **16.** $\frac{185}{21}$, $\frac{53}{14}$.

17. $2\frac{1}{4}$, $3\frac{1}{5}$, $4\frac{1}{6}$. **18.** $3\frac{1}{2}$, $4\frac{2}{5}$, $5\frac{1}{6}$. **19.** $4\frac{5}{6}$, $7\frac{2}{3}$, $11\frac{1}{2}$. **20.** $\frac{3}{7}$, $\frac{5}{8}$, $\frac{17}{56}$.

21. $2\frac{3}{4}$, $4\frac{5}{17}$, $\frac{13}{34}$. **22.** $5\frac{2}{3}$, $3\frac{5}{14}$, $4\frac{10}{21}$. **23.** $7\frac{8}{21}$, $10\frac{7}{12}$, $1\frac{11}{14}$.

24. $5\frac{2}{5}$, $\frac{17}{56}$, $4\frac{5}{14}$. **25.** $3\frac{5}{16}$, $2\frac{1}{7}$, $\frac{11}{48}$, $3\frac{2}{3}$. **26.** $4\frac{5}{13}$, $1\frac{1}{4}$, $\frac{3}{28}$, $10\frac{17}{65}$.

27. $\frac{4}{15}$, $3\frac{5}{16}$, $\frac{11}{48}$, $4\frac{7}{9}$. **28.** $2\frac{5}{14}$, $\frac{11}{12}$, $3\frac{8}{21}$, $\frac{7}{8}$. **29.** $\frac{11}{91}$, $4\frac{25}{52}$, $\frac{9}{35}$, $5\frac{27}{28}$.

30. $\frac{5}{18}$, $\frac{7}{45}$, 4, $7\frac{7}{30}$. **31.** $\frac{7}{12}$, $3\frac{4}{15}$, $\frac{11}{30}$, $\frac{25}{9}$. **32.** $\frac{3}{5}$, $6\frac{7}{65}$, $4\frac{17}{130}$, $2\frac{9}{20}$.

33. $\frac{9}{77}$, $2\frac{7}{33}$, $\frac{22}{63}$, $7\frac{10}{11}$. **34.** $\frac{97}{45}$, $\frac{13}{165}$, $\frac{17}{60}$, $\frac{104}{18}$. **35.** $\frac{99}{46}$, $\frac{38}{45}$, $\frac{57}{230}$, $9\frac{10}{18}$.

36. $\frac{4}{5}$, $\frac{5}{7}$, $\frac{3}{11}$, $\frac{17}{385}$. **37.** $2\frac{1}{5}$, $3\frac{1}{15}$, $1\frac{1}{18}$, $\frac{8}{45}$. **38.** $4\frac{5}{42}$, $6\frac{23}{91}$, $11\frac{51}{52}$, $1\frac{23}{156}$.

39. $3\frac{3}{8}$, $7\frac{8}{9}$, $4\frac{5}{6}$, $\frac{1}{30}$. **40.** $9\frac{1}{10}$, $3\frac{1}{11}$, $3\frac{17}{22}$, $1\frac{4}{55}$. **41.** $6\frac{11}{20}$, $5\frac{23}{25}$, $\frac{16}{45}$, $2\frac{13}{270}$.

42. $2\frac{7}{78}$, $\frac{8}{21}$, $5\frac{11}{65}$, $1\frac{5}{28}$. **43.** $3\frac{10}{33}$, $7\frac{13}{56}$, $1\frac{23}{132}$, $\frac{17}{126}$.

44. $13\frac{31}{65}$, $11\frac{17}{78}$, $9\frac{21}{26}$, $17\frac{29}{30}$.

45. $15\frac{11}{14}$, $9\frac{13}{84}$, $11\frac{15}{28}$, $17\frac{19}{72}$.

46. $\frac{51}{13}$, $\frac{51}{20}$, $\frac{21}{28}$, $\frac{23}{5}$.

47. $\frac{67}{12}$, $\frac{12}{35}$, $\frac{25}{48}$, $\frac{12}{5}$.

48. $7\frac{5}{14}$, $\frac{13}{36}$, $\frac{17}{11}$, $\frac{1}{77}$.

49. $11\frac{7}{24}$, $15\frac{24}{35}$, $19\frac{17}{105}$, $23\frac{25}{168}$.

50. $16\frac{4}{7}$, $\frac{215}{56}$, $\frac{11}{48}$, $\frac{17}{35}$.

51. $3\frac{5}{9}$, $\frac{11}{16}$, $2\frac{14}{27}$, $\frac{11}{72}$.

52. $5\frac{8}{17}$, $\frac{17}{20}$, $4\frac{11}{85}$, $\frac{8}{51}$.

53. $4\frac{5}{9}$, $\frac{14}{15}$, $3\frac{40}{81}$, $\frac{11}{30}$.

54. $2\frac{6}{7}$, $\frac{34}{35}$, $1\frac{1}{125}$, $\frac{24}{25}$.

55. $6\frac{91}{198}$, $5\frac{63}{286}$, $19\frac{23}{156}$, $27\frac{22}{39}$.

56. $9\frac{4}{13}$, $14\frac{23}{132}$, $3\frac{17}{24}$, $7\frac{9}{143}$.

57. $\frac{13}{42}$, $\frac{5}{72}$, $\frac{17}{63}$, $\frac{7}{24}$, $\frac{9}{56}$.

58. $\frac{5}{4}$, $\frac{29}{12}$, $\frac{4}{15}$, $\frac{21}{75}$, $\frac{11}{30}$.

59. $3\frac{4}{5}$, $9\frac{5}{8}$, $7\frac{3}{4}$, $41\frac{111}{125}$, $1\frac{2}{3}$.

60. $\frac{17}{34}$, $\frac{18}{27}$, $\frac{13}{78}$, $\frac{63}{105}$, $\frac{68}{255}$.

Subtraction of Fractions.

110. Just as was the case with addition, a fraction can only be subtracted from another fraction when it has the same "name," *i.e.* "denomination," *i.e.* "denominator." When this is the case the difference of two fractions can be found at once by taking the difference of the numerators and placing it over the same denominator.

Thus	7 *pence*	-3 *pence*	$= 4$ *pence.*
So	7 *tenths*	-3 *tenths*	$= 4$ *tenths,*
i.e.	$\frac{7}{10}$	$-\frac{3}{10}$	$= \frac{4}{10}$.

If the fractions have not the same denominator, they must first be brought to a common denominator.

The difference should be expressed in its lowest terms.

E.g. in the case given above $\frac{4}{10}$ should be written $\frac{2}{5}$.

In the case of mixed numbers the integral part and the fractional part must be treated separately. A difficulty may arise here when the fractional part of the smaller mixed number is greater than the fractional part of the larger mixed number. This difficulty will be considered in the next article.

Improper fractions should usually be reduced to mixed numbers.

Ex. 1.—*Subtract* $\frac{19}{36}$ *from* $\frac{7}{12}$.

Work thus: $\frac{7}{12} - \frac{19}{36} = \frac{21-19}{36} = \frac{2}{36} = \frac{1}{18}$. *Ans.*

Ex. 2.—*What is the difference between* $1\frac{3}{5}$ *and* $2\frac{5}{7}$?

Work thus: $2\frac{5}{7} - 1\frac{3}{5} = 1\frac{25-21}{35} = 1\frac{4}{35}$. *Ans.*

EXAMPLES XLVI.

Find the difference between the following fractions—

1. $\frac{11}{15}, \frac{8}{15}$. **2.** $\frac{13}{20}, \frac{7}{20}$. **3.** $\frac{4}{5}, \frac{1}{3}$. **4.** $\frac{7}{10}, \frac{7}{20}$.

5. $3\frac{3}{4}, 1\frac{1}{4}$. **6.** $3\frac{3}{4}, 2\frac{5}{8}$. **7.** $1\frac{3}{4}, \frac{4}{25}$. **8.** $1\frac{2}{3}, \frac{5}{24}$.

9. $1\frac{4}{7}, \frac{8}{25}$. **10.** $1\frac{4}{21}, \frac{1}{35}$. **11.** $\frac{27}{9}, 1\frac{13}{18}$. **12.** $2\frac{9}{10}, 1\frac{13}{15}$.

13. $\frac{17}{8}, \frac{17}{16}$. **14.** $\frac{21}{5}, \frac{31}{15}$. **15.** $5\frac{35}{36}, 2\frac{17}{64}$. **16.** $6\frac{31}{90}, 2\frac{19}{120}$.

[*Before reading this article beginners are advised to read again* § 18 *and Ex.* 2, § 65.]

111. When we had to subtract 18 from 42, *i.e.* 1 *ten* and 8 *units* from 4 *tens* and 2 *units*, we found it necessary to increase one quantity by a *ten* and the other quantity by 10 units and to subtract 2 *tens* and 8 *units* from 4 tens and 12 units. The same difficulty arose in compound subtraction, and was met in the same way. So also, when we have to subtract $1\frac{1}{2}$ from $2\frac{1}{4}$, we may increase the quantities by unity in one case, by 4 quarters in the other case, and subtract 2 *units* and a *half* from 2 *units* and 5 *quarters*, and, since a *half* is 2 *quarters*, the result will be 3 *quarters*, *i.e.* $\frac{3}{4}$.

Ex. 3.—*Subtract* $\frac{9}{10}$ *from* $1\frac{7}{15}$.

Here we have to add 1 to the subtrahend and $\frac{15}{15}$ to the minuend, *i.e.* we have to subtract $1\frac{9}{10}$ from $1\frac{22}{15}$. The integers and fractions are now treated separately. The work is written

$$1\tfrac{7}{15}-\tfrac{9}{10} = 1\tfrac{22}{15}-1\tfrac{9}{10} = \frac{44-27}{30} = \tfrac{17}{30}.$$

EXAMPLES XLVII.

Find the difference of the following pairs of numbers—

1. $5\frac{4}{27}, 2\frac{17}{18}$. **2.** $19\frac{3}{32}, 12\frac{1}{4}$. **3.** $6\frac{3}{16}, 5\frac{7}{20}$. **4.** $11\frac{5}{18}, 10\frac{7}{12}$.

5. $14\frac{15}{16}, 9\frac{23}{24}$. **6.** $8\frac{11}{14}, 3\frac{45}{49}$. **7.** $6\frac{13}{21}, 17\frac{1}{9}$. **8.** $2\frac{34}{35}, 6\frac{20}{21}$.

9. $\frac{67}{32}, \frac{11}{6}$. **10.** $\frac{61}{30}, \frac{49}{45}$. **11.** $19\frac{9}{35}, 9\frac{5}{14}$. **12.** $11\frac{5}{22}, 7\frac{18}{33}$.

13. $10\frac{4}{15}, 5\frac{17}{45}$. **14.** $10\frac{31}{84}, 7\frac{15}{28}$. **15.** $25\frac{17}{30}, 18\frac{19}{25}$. **16.** $11\frac{11}{78}, 7\frac{10}{13}$.

17. $9\frac{7}{132}, 5\frac{11}{12}$. **18.** $6\frac{7}{68}, 3\frac{5}{8}$. **19.** $19\frac{19}{54}, 14\frac{15}{28}$. **20.** $13\frac{9}{280}, 12\frac{6}{35}$.

21. $1\frac{19}{39}, 2\frac{7}{130}$. **22.** $8\frac{23}{35}, 11\frac{8}{15}$. **23.** $2\frac{9}{40}, 5\frac{3}{16}$. **24.** $7\frac{11}{15}, 9\frac{11}{27}$.

25. $12\frac{11}{42}$, $9\frac{13}{18}$. **26.** $11\frac{19}{45}$, $7\frac{22}{35}$. **27.** $17\frac{7}{165}$, $13\frac{15}{22}$. **28.** $26\frac{8}{63}$, $15\frac{19}{27}$.
29. $55\frac{16}{63}$, $47\frac{23}{27}$. **30.** $42\frac{16}{63}$, $28\frac{32}{35}$. **31.** $17\frac{37}{65}$, $13\frac{38}{39}$. **32.** $16\frac{15}{28}$, $11\frac{53}{98}$.
33. $10\frac{9}{14}$, $3\frac{13}{16}$. **34.** $120\frac{7}{60}$, $9\frac{43}{220}$. **35.** $17\frac{11}{153}$, $15\frac{19}{34}$. **36.** $63\frac{5}{76}$, $34\frac{46}{57}$.
37. $132\frac{5}{18}$, $127\frac{7}{15}$. **38.** $19\frac{1}{28}$, $10\frac{31}{48}$. **39.** $32\frac{17}{65}$, $28\frac{29}{50}$.
40. $15\frac{27}{143}$, $13\frac{11}{52}$. **41.** $25\frac{8}{29}$, $21\frac{23}{24}$. **42.** $42\frac{121}{343}$, $39\frac{45}{56}$.
43. $121\frac{17}{54}$, $97\frac{29}{63}$. **44.** $56\frac{7}{180}$, $47\frac{11}{78}$. **45.** $265\frac{8}{19}$, $242\frac{9}{17}$.
46. $20\frac{62}{63}$, $15\frac{161}{162}$. **47.** $95\frac{70}{143}$, $74\frac{90}{91}$. **48.** $41\frac{19}{84}$, $37\frac{41}{48}$.
49. $123\frac{17}{114}$, $117\frac{81}{95}$. **50.** $133\frac{23}{108}$, $123\frac{31}{64}$.

Combined Addition and Subtraction.

112. The following is the method of dealing with expressions involving several fractions to be added to or subtracted from one another :—

Reduce all fractions to their lowest terms. Combine any which have now the same denominator.

Reduce all improper fractions to mixed numbers.

Look through the proper fractions now left, to see whether the fractions to be added together are or are not less than the fractions to be subtracted from them. If less, add one or more units in fractions to that fraction among those which are to be added with the smallest denominator, and an equal number of units to the integers which are to be subtracted.

Bring the fractions to a common denominator, and arrange them so that the terms to be added come first, then those to be subtracted.

Add and subtract the numerators as may be necessary.

Bring the answer to its lowest terms.

Hints :—

(1) Time is often saved by grouping the fractions in pairs and simplifying each pair separately. Nothing but experience can guide you here, but this method should be tried when any pair of fractions has a simple L.C.D. which does not contain many of the factors of the other denominators. See Ex. 2 below, and Hint 7, § 109.

(2) When one of the proper fractions is near unity it is sometimes advisable to make it up to unity and subtract what is added for this purpose. Thus $13\frac{14}{15}$ might be dealt with as $14-\frac{1}{15}$. The advanced student should employ this method freely, *e.g.* in Exx. xlvii.

(3) In the evaluation of the numerator the method of § 20 should be employed where possible.

Ex. 1.—*Find the value of* $13\frac{2}{15} + 5\frac{1}{12} - 5\frac{5}{27} - 6\frac{3}{45}$.

Work thus :—

$$\begin{aligned}
\text{Value} &= 13\tfrac{2}{15} + 5\tfrac{1}{12} - 5\tfrac{5}{27} - 6\tfrac{1}{15} \\
&= 7\tfrac{1}{15} + \tfrac{1}{12} - \tfrac{5}{27} \\
&= 7\,\frac{1}{3\times5} + \frac{13}{3\times4} - 1\,\frac{5}{3\times9} \\
&= 6\,\frac{36+585-100}{3\times4\times5\times9} \qquad \begin{array}{r} 45 \\ \underline{135} \\ 585 \end{array} \\
&= 6\,\frac{621-100}{20\times27} \\
&= 6\tfrac{521}{540}. \quad \textit{Ans.}
\end{aligned}$$

Explanation.—The last of the fractions is not in its lowest terms. We begin by reducing it. We see that it can now be directly subtracted from the first. This subtraction is made, and at the same time the two 5's are dropped, because they balance each other. This gives us the second line. We now inspect the proper fractions, estimate roughly, mentally, whether their sum is greater or less than the sum of the fractions to be taken from them, and see that w must add a unit. We add this unit to the $\frac{1}{12}$ (this being the one with the smallest denominator of the fractions which are to be added), making it into $\frac{13}{12}$, and at the same time add a unit to the fractions which are to be subtracted. We split the denominators into factors, and thus obtain the third line. We next deal with the integers left, $7-1=6$, and write this 6 at the beginning of the next line. The L.C.D. can be seen by inspection to be $3\times4\times5\times9$, and the numerators are found at once mentally, except for the middle fraction, where the multiplication of 5×9, *i.e.* 45, by 13 is necessary. The work for this is shown. The numerators 36 and 585 are now added (mentally if possible), and the 100 subtracted, obtaining finally $6\frac{521}{540}$. The only factors of the denominator have been seen to be 3, 4, 5, and 9. 521 is not divisible by 3 ; for $5+2+1=8$ is not. Also clearly it is not divisible by 4 or 5 ; therefore the fraction is in its lowest terms.

Ex. 2.—*From the sum of* $5\frac{1}{13}$ and $2\frac{4}{117}$ *take the sum of* $3\frac{4}{51}$, $1\frac{16}{85}$.

To take the sum of two numbers away from a third is clearly the same as to take the numbers away separately ; therefore this question is the same as "*Find the value of* $5\frac{1}{13}+2\frac{4}{117}-3\frac{4}{51}-1\frac{16}{85}$," and might be dealt with in just the same way s in Ex. 1, reducing to the common denominator $5\times9\times13\times17$; but the numbers are large. We note that the L.C.D. of the first pair is 9×13, of the second pair $3\times5\times17$, and, as these L.C.D.'s have not much in common, we resolve to take the pairs separately.

Work—

$$5\tfrac{1}{13}+2\tfrac{4}{117} = 5\tfrac{1}{13}+2\,\frac{4}{9\times13} = 7\,\frac{9+4}{9\times13} = 7\,\frac{13}{9\times13} = 7\tfrac{1}{9},$$

$$3\tfrac{4}{51}+1\tfrac{16}{85} = 3\,\frac{4}{3\times17}+1\,\frac{16}{5\times17} = 4\,\frac{20+48}{3\times5\times17} = 4\,\frac{68}{3\times5\times17} = 4\tfrac{4}{15},$$

$$7\tfrac{1}{9}-4\tfrac{4}{15} = 7\tfrac{10}{9}-5\tfrac{4}{15} = 2\,\frac{50-12}{45} = 2\tfrac{38}{45} \quad \textit{Ans.}$$

EXAMPLES XLVIII.

Find the value of—

1. $2\frac{1}{2}+3\frac{1}{6}-4\frac{1}{3}$. **2.** $3\frac{1}{3}+4\frac{1}{4}-2\frac{1}{2}$. **3.** $1\frac{2}{3}+2\frac{3}{4}-\frac{5}{6}-\frac{1}{12}$.

4. $1\frac{3}{4}+2\frac{5}{6}-3\frac{1}{2}-\frac{5}{12}$. **5.** $\frac{13}{60}-\frac{13}{42}+\frac{13}{90}$. **6.** $6\frac{3}{28}+\frac{19}{35}-5\frac{19}{30}$.

7. $3\frac{2}{3}-1\frac{1}{2}+3\frac{3}{4}-4\frac{4}{5}$. **8.** $2\frac{1}{2}-3\frac{1}{3}-4\frac{1}{4}+5\frac{1}{5}$. **9.** $2\frac{3}{4}+4\frac{4}{11}-3\frac{1}{5}$.

10. $3\frac{4}{7}+3\frac{18}{35}-\frac{4}{21}$. **11.** $5\frac{1}{2}-7\frac{1}{6}+4\frac{1}{4}+3-\frac{2}{3}$. **12.** $5\frac{2}{3}-7\frac{5}{6}+4\frac{1}{2}+3-\frac{3}{4}$.

13. $10\frac{1}{2}+6\frac{1}{4}+9\frac{9}{16}-23\frac{3}{8}$. **14.** $3\frac{10}{11}+5\frac{7}{15}-2\frac{9}{22}-4\frac{18}{20}$.

15. $2\frac{1}{5}+1\frac{4}{7}-4\frac{1}{2}-\frac{3}{5}+2\frac{8}{35}$. **16.** $2\frac{3}{4}+\frac{13}{8}+\frac{23}{10}-3\frac{5}{24}+1\frac{8}{15}$.

17. $3\frac{4}{5}+1\frac{5}{16}+4\frac{3}{20}-3\frac{7}{15}-2\frac{11}{12}$. **18.** $\frac{1}{6}+\frac{21}{7}+13\frac{3}{10}+\frac{4}{21}-4\frac{33}{35}$.

19. $4\frac{2}{15}+3\frac{11}{20}-2\frac{1}{12}-3\frac{7}{18}$. **20.** $3\frac{7}{15}+4\frac{9}{20}-1\frac{11}{12}+\frac{13}{18}$.

21. $\frac{11}{12}+\frac{17}{18}+\frac{29}{30}+\frac{47}{48}+\frac{59}{60}-\frac{29}{720}$. **22.** $\frac{8}{9}+\frac{14}{15}+\frac{35}{36}+\frac{39}{40}+\frac{59}{60}-\frac{1}{45}$.

23. $6\frac{9}{34}+5\frac{2}{153}-4\frac{4}{65}-3\frac{10}{91}$. **24.** $3\frac{1}{17}+\frac{8}{153}-2\frac{2}{65}-\frac{18}{91}$.

25.* $3\frac{11}{34}-1\frac{3}{26}+7\frac{32}{85}-2\frac{22}{91}+11\frac{4}{171}+2\frac{12}{209}$.

26.* $3\frac{10}{17}-11\frac{37}{627}+7\frac{1}{85}+1\frac{1}{52}-\frac{2}{171}+2\frac{8}{91}$.

EXAMPLES XLIX.

1. What is the least number which must be added to $16\frac{7}{11}$ to make it an integer?

2. What is the least number which must be added to $15\frac{3}{5}$ to make it an odd integer?

3. What must be added to $15\frac{3}{5}$ to make it into $16\frac{7}{11}$?

4. What must be added to $15\frac{7}{11}$ to make it into $16\frac{3}{5}$?

5. By how much does the difference between 9 and $3\frac{4}{25}$ exceed the difference between $1\frac{4}{25}$ and $\frac{3}{4}$?

6. By how much does the difference between $7\frac{1}{2}$ and $8\frac{5}{6}$ fall short of the difference between $5\frac{3}{14}$ and $2\frac{10}{21}$?

7. Arrange in order of magnitude $3\frac{1}{4}-2\frac{3}{4}$, $3\frac{3}{4}-2\frac{5}{8}$, $4-2\frac{6}{7}$, $1\frac{3}{5}-\frac{11}{60}$.

8. By how much does the greatest of the four differences given in Question 7 exceed the least?

9. The difference of two numbers is $9\frac{37}{75}$, and the smaller is $3\frac{8}{15}$. What is the greater?

10. The difference of two numbers is $2\frac{31}{42}$, and the larger of them is $3\frac{2}{3}$. What is the smaller?

11. Subtract the sum of $\frac{3}{10}$, $\frac{13}{15}$, $\frac{1}{5}$, and $\frac{4}{9}$ from the sum of $2\frac{2}{3}$, $\frac{3}{5}$, 4, and $5\frac{5}{6}$.

12. Subtract the sum of $\frac{7}{8}$, $\frac{7}{12}$, $\frac{7}{16}$, and $\frac{7}{18}$ from the sum of $1\frac{3}{8}$, $\frac{1}{6}$, $\frac{5}{18}$, and $2\frac{1}{12}$.

Multiplication of Fractions by Integers.

113. I. *Proper and Improper Fractions.*

If the fraction be $\frac{5}{12}$ and the multiplier 4, the fraction indicates that unity is divided into 12 equal parts, and we take 5 of these parts. To multiply by 4 we must take 4 times as many, *i.e.* we must take 20 of these parts.

$$\therefore \quad \tfrac{5}{12} \times 4 = \tfrac{20}{12}.$$

If the fraction be a/b and the multiplier c, the fraction indicates that unity is divided into b equal parts, and we take a of these parts. To multiply by c we must take c times as many, *i.e.* we must take ac of these parts.

$$\therefore \quad \frac{a}{b} \times c = \frac{ac}{b}.$$

But we may look at the multiplication differently.

We shall have performed the multiplication if, instead of making the parts 4 times as numerous, we make them 4 times as large as they were before. But in this case unity will no longer contain 12 of them, but only $12 \div 4$, *i.e.* 3.

$$\therefore \quad \tfrac{5}{12} \times 4 = \tfrac{5}{3}.$$

We shall have performed the multiplication if, instead of making the parts c times as numerous, we make them c times as largo as they were before. But in this case unity will no longer contain b of them, but only $b \div c$.

$$\therefore \quad \frac{a}{b} \times c = \frac{a}{b \div c}.$$

These statements should be illustrated on squared paper.

Hence: *To multiply a fraction by an integer, we may* EITHER *multiply the numerator* OR *divide the denominator by the integer.*

Where possible use the second method.

II. *Mixed Numbers.*

When we have to multiply 1 *shilling* and 5 *pence* by 7, we multiply the *shillings* and the *pence* separately, getting 7 *shillings* and 35 *pence*. Then we note that 35 *pence* is 2 *shillings* and 11 *pence*; so that we write the product 9 *shillings* and 11 *pence*.

When we have to multiply 1 *unit* and 5 *twelfths* by 7, we multiply the *units* and the *twelfths* separately, getting 7 *units* and 35 *twelfths*. Then we note that 35 *twelfths* is 2 *units* and 11 *twelfths*; so that we write the product 9 *units* and 11 *twelfths*.

It follows that $1\frac{5}{12} \times 7 = 7\frac{35}{12} = 7 + 2\frac{11}{12} = 9\frac{11}{12}$.

EXAMPLES L.

Multiply—

1. $2\frac{5}{6}$ by 3, 5, 6, 7, and 10.

2. $3\frac{4}{5}$ by 3, 5, 6, 7, and 15.

3. $\frac{8}{7}$ by 5, 6, 7, 8, and 9.

4. $\frac{4}{7}$ by 3, 5, 6, 7, and 8.

5. $1\frac{13}{14}$ by 2, 5, 7, 9, and 14.

6. $1\frac{13}{15}$ by 3, 5, 7, 10, and 15.

7. $1\frac{11}{13}$ by 5, 11, 13, 50, and 91.

8. $2\frac{9}{17}$ by 5, 9, 17, 50, and 102.

9. $13\frac{1}{17}$ by 17, 20, 34, 100, and 102.

10. $17\frac{1}{13}$ by 13, 20, 39, 100, and 104.

Division of Fractions by Integers.

114. I. *Proper and Improper Fractions.*

If the fraction be $\frac{5}{12}$, and the divisor be 5, the fraction indicates that unity is divided into 12 equal parts, and we take 5 of these parts. To divide by 5 we must divide the number of parts taken by 5, *i.e.* we must take 1 part instead of 5.

$$\therefore \quad \tfrac{5}{12} \div 5 = \tfrac{1}{12}$$

If the fraction be a/b and the divisor be c, the fraction indicates that unity is divided into b equal parts, and we take a of these parts. To divide by c we must divide the number of parts taken by c, *i.e.* we must take $a \div c$ parts instead of a.

$$\therefore \quad \frac{a}{b} \div c = \frac{a \div c}{b}.$$

But we may look at the division differently.

We shall have performed the division if, instead of dividing the *number* of parts by 5, we divide the *parts themselves* each into 5 equal portions, and take the same number of these portions as we had before of the parts. But, since unity contained 12 parts, and each part contains 5 portions, unity will contain 12×5 portions, *i.e.* 60 portions. Of these we are to take 5.

$$\therefore \quad \tfrac{5}{12} \div 5 = \tfrac{5}{60}.$$

We shall have performed the division if, instead of dividing the *number* of parts by c, we divide the *parts themselves* each into c equal portions, and take the same number of these portions as we had before of the parts. But, since unity contained b parts and each part contains c portions, unity will contain $b \times c$, *i.e.* bc portions. Of these we are to take a.

$$\frac{a}{b} \div c = \frac{a}{bc}$$

Both these methods of Division should be worked out by the Diagram of § 105.

Hence: *To divide a fraction by an integer, we may* EITHER *divide the numerator* OR *multiply the denominator by the integer.*

If the integer will divide *exactly* into the numerator of the

(proper or improper) fraction, the first method should be employed; if not, the other.

II. *Mixed Numbers.*

When we have to divide 11 *shillings* and 1 *penny* by 7 we divide the 11 *shillings* by the 7, getting a quotient of 1 *shilling*, together with a remainder of 4 *shillings*, *i.e.* 48 *pence*, which, added to the 1 *penny* we had at first, makes 49 *pence*. When this is divided by 7 we get a quotient of 7 *pence*, and thus a final quotient of

1 *shilling* and 7 *pence*.

When we have to divide 11 *units* and 1 *twelfth* by 7 we divide the 11 *units* by the 7, getting a quotient of 1 *unit*, together with a remainder of 4 *units*, *i.e.* 48 *twelfths*, which, added to the 1 *twelfth* we had at first, makes 49 *twelfths*. When this is divided by 7 we get a quotient of 7 *twelfths*, and thus a final quotient of

1 *unit* and 7 *twelfths*.

It follows that $11\frac{1}{12} \div 7 = 1 + (4\frac{1}{12} \div 7)$
$= 1 + (\frac{49}{12} \div 7) = 1\frac{7}{12}$.

Ex. 1.—*Divide* $17\frac{7}{9}$ *by* 8.

We first divide the 17 by 8, the quotient being 2 and the remainder 1; therefore $17\frac{7}{9} \div 8 = 2 + (1\frac{7}{9} \div 8)$. We now convert the $1\frac{7}{9}$ into an improper fraction, obtaining $\frac{16}{9}$. This can be divided by 8 by dividing the numerator by 8; so that the complete working is

$$17\tfrac{7}{9} \div 8 = 2 + (1\tfrac{7}{9} \div 8) = 2 + (\tfrac{16}{9} \div 8)[= 2 + \tfrac{2}{9}] = 2\tfrac{2}{9}.$$

The step in [ ] is always omitted.

Ex. 2.—*Divide* $15\frac{5}{9}$ *by* 8.

We have $15\frac{5}{9} \div 8 = 1 + (7\frac{5}{9} \div 8) = 1 + (\frac{68}{9} \div 8) = 1\frac{68}{72}$ (for the numerator will not divide exactly by 8; so we multiply the denominator) $= 1\frac{17}{18}$, reducing the fraction to the lowest terms.

Ex. 3.—*Divide* 257 *by* 7.

We have $257 \div 7 = 36 + (5 \div 7) = 36\frac{5}{7}$. Thus we see that we have now another method for writing the result of an ordinary division. We may say $257 \div 7$ gives quotient 36 and remainder 5; or we may say, with equal correctness, $257 \div 7$ gives quotient $36\frac{5}{7}$ and remainder 0. A quotient thus expressed, with no remainder, is called a **Complete Quotient.**

EXAMPLES LI.

Divide the following fractions by 7, 11, 15, 18, and 21, obtaining five answers in the lowest terms for each question:—

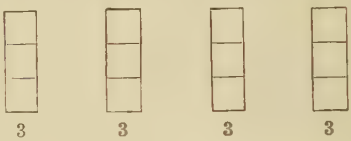
1. $3\frac{1}{7}$. **2.** $2\frac{8}{11}$. **3.** $\frac{77}{15}$. **4.** $\frac{105}{11}$. **5.** $11\frac{7}{11}$.
6. $15\frac{13}{15}$. **7.** $15\frac{9}{19}$. **8.** $17\frac{8}{17}$. **9.** $150\frac{15}{23}$. **10.** $266\frac{7}{13}$.

Multiplication by Fractions.

115. In Chap. IV. we defined Multiplication as "the process of finding the sum of a given number of repetitions of the same number"; so that to multiply 3 by 4 was to find the sum of 3 repeated 4 times, *i.e.* of $3+3+3+3$. *In this sense it is absurd to speak of multiplying by a fraction.* We cannot repeat a number "half-a-time." But, without really altering the meaning of the definition, we may put it so that it can be applied to fractions, and define thus:—

Multiplication is the process of finding the total made up of a certain number of groups (**multiplier**), *each group containing the same number* (**multiplicand**) *as every other group.*

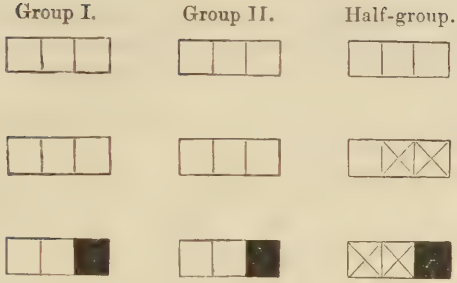
Thus, to multiply 3 by 4 we picture groups containing 3 objects, and we take 4 such groups as in the diagram. We count the result and find it to be 12.



Now, if we are asked to multiply 3 by $2\frac{1}{3}$, we are asked to picture two such groups, *and a third of a group.* The result is shown by the figure to be 7; therefore $3 \times 2\frac{1}{3} = 7$.

Suppose now that we are asked to multiply $2\frac{2}{3}$ by $2\frac{1}{2}$. How are we to represent this? Each group is to contain 2 units and two-thirds of a unit, and we are to take 2 such groups and a-half of such a group. As we have to deal with a unit which is split into three parts, we *represent the unit by three squares,* and one of the squares is one-third of a unit, *i.e.* represents $\frac{1}{3}$. When we count up the number of squares in a group we find 8, and therefore the half-group contains 4. We draw lines through the other four, to show that they are not wanted. The figure now gives us the result of multiplication. There are 5 complete units and 5 thirds, which make one unit more, with 2 thirds left.

$$\therefore \quad 2\tfrac{2}{3} \times 2\tfrac{1}{2} = 5 + \tfrac{5}{3} = 5 + 1\tfrac{2}{3} = 6\tfrac{2}{3}.$$



EXAMPLES LII.

[*No answers are given to these Examples.*]

Prove the following statements by means of figures in squared paper—

1. $3 \times 2 = 6$. **2.** $2 \times 3 = 6$. **3.** $2 \times \frac{1}{2} = 1$. **4.** $3 \times \frac{1}{3} = 1$.

5. $2\frac{1}{2} \times \frac{1}{2} = 1\frac{1}{4}$. **6.** $3\frac{1}{2} \times \frac{1}{2} = 1\frac{3}{4}$. **7.** $2\frac{1}{2} \times 2\frac{2}{3} = 6\frac{2}{3}$.* **8.** $1\frac{2}{3} \times 3\frac{1}{2} = 5\frac{5}{6}$.

9. $2\frac{1}{2} \times 2\frac{1}{2} = 6\frac{1}{4}$ **10.** $1\frac{1}{3} \times 1\frac{1}{3} = 1\frac{7}{9}$.†

116. We may now proceed to find the most convenient nethod of performing such multiplications.

I. *Proper and Improper Fractions.*

When we have to multiply $\frac{3}{4}$ by $\frac{5}{7}$ we have to take 5 sevenths of $\frac{3}{4}$, *i e.* we have to divide the $\frac{3}{4}$ into 7 equal parts and take 5 such parts. But $\frac{3}{4}$ is divided into 7 equal parts by multiplying the denominator by 7 (see § 114), and, therefore, each of the parts is $\frac{3}{4 \times 7}$, and when we take 5 of them we get $\frac{3 \times 5}{4 \times 7}$. (See § 113.)

$$\therefore \quad \frac{3}{4} \times \frac{5}{7} = \frac{3 \times 5}{4 \times 7}.$$

When we have to multiply a/b by c/d we have to take c d-ths of a/b, *i.e.* we have to divide a/b into d equal parts and take c such parts. But a/b is divided into d equal parts by multiplying the denominator by d (see § 114), and, therefore, each of the parts is $\frac{a}{b \times d}$, and when we take c of them we get $\frac{a \times c}{b \times d}$. (See § 113.)

$$\therefore \quad \frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}.$$

* This does NOT follow from the example worked in the text, namely $2\frac{2}{3} \times 2\frac{1}{2} = 6\frac{2}{3}$. It is true that we have proved that multiplications may be done in any order, but this proof *does not apply to fractions.* See § 125.

† The teacher must make other similar examples, unless the meaning of fractional multiplication is now clear to the class. He should confine himself to fractions with denominators not greater than 6.

Hence: *To multiply two fractions (proper and improper) together, we put the product of the numerators as the new numerator and the product of the denominators as the new denominator.*

It is desirable at this stage to prove the *Commutative Law* for Multiplication. [See § 125, Law I. (2).]

It is clear that this rule may be extended *to any number of fractions.*

Ex.—$\frac{3}{4} \times \frac{5}{7} \times \frac{9}{11} = \frac{3 \times 5}{4 \times 7} \times \frac{9}{11} = \frac{3 \times 5 \times 9}{4 \times 7 \times 11}$.

The answer must be brought to its lowest terms. But this method brings the answer in factors, and it is easy to see and to cancel (see § 106) any factors common to both numerator and denominator, and then to multiply the remaining factors together and, if necessary, reduce the fraction so obtained to a mixed number.

The above explanation makes it clear *that* $\frac{3}{4} \times \frac{5}{7}$ *is the same thing as* $\frac{5}{7}$ of $\frac{3}{4}$. A fraction of a fraction, such as this, is sometimes called a **Compound Fraction.**

Ex. 1.—*Multiply* $\frac{5}{8} \times \frac{2}{5} \times \frac{4}{7}$.

$$Product = \frac{\cancel{5} \times \cancel{2} \times \cancel{4}}{\cancel{8} \times \cancel{5} \times 7} = \frac{1}{7}.$$

Explanation.—We write the product down at once as the product of the numerators over the product of the denominators, then we look for any factors common to numerator and denominator. We see the 5 and cancel it by drawing a line through it thus, $\cancel{5}$, as well as through the 5 of the denominator. We see that the 2 will divide into the 8: cancel both 2 and 8, but write 4 underneath the 8, that being the quotient of 8 divided by 2. Then we see that the 4 of the numerator will cancel the 4 just written. We have now *no factor but unity* left in the numerator, and 7 left in the denominator. Therefore the answer is $\frac{1}{7}$. Note the 1 of the numerator. *Do not be misled into supposing that because all the factors are cancelled the numerator is zero.**

Ex. 2.—*Multiply* $\frac{5}{7}$ by $\frac{7}{5}$.

$$Product = \frac{\cancel{5} \times \cancel{7}}{\cancel{7} \times \cancel{5}} = 1.$$

Such fractions are called Reciprocal Fractions. **Reciprocal Fractions** *are two fractions such that the numerator of either is the denominator of the other.* Their product is always 1.

This definition applies to improper fractions. The reciprocal of $2\frac{1}{2}$, *i.e.* of $\frac{5}{2}$, is $\frac{2}{5}$. The reciprocal of 2, *i.e.* $\frac{2}{1}$, is $\frac{1}{2}$.

* The teacher would find it well to insist that the beginner should place a 1 above any factor, whether in numerator or denominator, which is cancelled completely.

II. *Mixed Numbers.*

These can be dealt with by the method just explained, provided they are first turned into improper fractions.

Ex. 3.—*Simplify* $2\frac{1}{2}$ *of* $3\frac{1}{4} \times \frac{4}{5}$.

$$\text{Expression} = 2\tfrac{1}{2} \times 3\tfrac{1}{4} \times \tfrac{4}{5} = \tfrac{5}{2} \times \tfrac{13}{4} \times \tfrac{4}{5} \left[\begin{array}{c} = \cancel{5} \times 13 \times \cancel{4} \\ 2 \times \cancel{4} \times \cancel{5} \end{array} \right] = \tfrac{13}{2} = 6\tfrac{1}{2}.$$

An advanced student may omit the step in [ ] and perform the cancelling mentally.

For the product of two large mixed numbers the algebraic method of multiplication is sometimes of considerable advantage.

Ex. 4.—*Simplify* $13\frac{1}{6} \times 6\frac{1}{13}$.

$$\begin{aligned} \text{Product} &= 13 \times 6 + 13 \times \tfrac{1}{13} + \tfrac{1}{6} \times 6 + \tfrac{1}{6} \times \tfrac{1}{13} \\ &= 78 + 1 + 1 + \tfrac{1}{78} \\ &= 80\tfrac{1}{78} \end{aligned}$$

The whole of this work can be done mentally.

The method is clearly useful when the integer part of one fraction contains factors in common with the denominator of the second fraction.

EXAMPLES LIII.

Simplify—

1. $\frac{11}{42} \times \frac{28}{53}$. **2.** $\frac{78}{17} \times \frac{51}{39}$. **3** $4\frac{13}{16} \times 1\frac{31}{33}$. **4.** $1\frac{12}{23} \times 2\frac{13}{28}$.

5. $3\frac{3}{4} \times 7\frac{11}{15}$. **6.** $7\frac{9}{11} \times 2\frac{13}{43}$. **7.** $1\frac{11}{18} \times 1\frac{5}{58}$. **8.** $7\frac{7}{44} \times 2\frac{17}{63}$.

9. $10\frac{15}{16} \times 3\frac{23}{35}$. **10.** $9\frac{5}{11}$ of $4\frac{1}{3}$. **11.** $5\frac{15}{22}$ of $1\frac{58}{85}$. **12.** $3\frac{3}{8}$ of $3\frac{43}{63}$.

13. $4\frac{12}{13} \times \frac{13}{64}$. **14.** $9\frac{3}{5} \times 2\frac{17}{24}$. **15.** $3\frac{5}{17} \times 5\frac{13}{28}$. **16.** $4\frac{43}{50} \times 12\frac{26}{27}$.

17. $6\frac{57}{64} \times 4\frac{4}{63}$. **18.** $11\frac{33}{35} \times \frac{7}{22}$. **19.** $6\frac{12}{25} \times 5\frac{5}{36}$. **20.** $6\frac{5}{34} \times 2\frac{1}{76}$.

21. $5\frac{8}{27} \times 3\frac{6}{13}$. **22.** $4\frac{4}{27}$ of $5\frac{3}{16}$. **23.** $2\frac{166}{225}$ of $7\frac{59}{38}$.

24. $105\frac{4}{35}$ of $4\frac{2}{7}$. **25.** $1\frac{11}{28} \times \frac{35}{36} \times 2\frac{10}{13}$. **26.** $\frac{180}{525} \times \frac{735}{264}$ of $\frac{66}{315}$.

27. $\frac{51}{91} \times 1\frac{27}{68}$ of $2\frac{46}{55}$. **28.** $\frac{25}{117} \times \frac{72}{77} \times 1\frac{29}{75}$.

29. $4\frac{23}{25}$ of $7\frac{5}{16} \times \frac{10}{41}$ of $2\frac{22}{39}$. **30.** $3\frac{19}{37} \times \frac{19}{64} \times 2\frac{19}{65} \times \frac{74}{149}$.

31. $6\frac{3}{8} \times \frac{19}{51} \times 2\frac{34}{111} \times \frac{37}{38}$. **32.** $\frac{209}{338} \times \frac{91}{187}$ of $\frac{17}{60} \times \frac{130}{133}$.

33. $5\frac{14}{23}$ of $1\frac{13}{56} \times 3\frac{15}{43} \times \frac{49}{243}$. **34.** $\frac{23}{60} \times 4\frac{16}{17} \times \frac{268}{69} \times \frac{51}{721}$.

35. $1\frac{6}{13} \times 2\frac{1}{19} \times 4\frac{4}{27} \times \frac{81}{112}$. **36.** $2\frac{18}{49} \times \frac{19}{255} \times 1\frac{6}{29} \times 1\frac{13}{38}$.

37. $9\frac{5}{8} \times 1\frac{71}{121} \times 1\frac{1}{104} \times 2\frac{45}{49}$. **38.** $1\frac{67}{186} \times \frac{144}{161} \times 2\frac{55}{112} \times 1\frac{97}{99}$.

39. $\frac{65}{123} \times 1\frac{5}{6} \times \frac{75}{121} \times 1\frac{44}{325}$. **40.** $\frac{93}{125} \times 1\frac{163}{187} \times \frac{22}{35} \times \frac{26}{31}$.

41. $\frac{18}{19} \times 182\frac{1}{2}$ of $\frac{11}{219}$ of $4\frac{29}{45}$. **42.** $11\frac{7}{11} \times \frac{15}{86} \times 5\frac{1}{24} \times 2\frac{11}{16}$.

43. $2\frac{14}{27} \times 1\frac{11}{15} \times 2\frac{7}{34} \times 1\frac{5}{18} \times \frac{24}{121} \times \frac{16}{43} \times 1\frac{3}{8} \times 5\frac{11}{15}$.

44. $4\frac{3}{35} \times \frac{27}{58} \times \frac{15}{44} \times 5\frac{8}{39} \times 5\frac{4}{9} \times \frac{12}{143} \times 2\frac{1}{7} \times 8\frac{29}{50}$.

Division by Fractions.

117. We defined Division in Chap. V. as "the process of finding how often one number is contained in another larger number." But a study of § 114, and particularly of the note to Ex. 3, will show you that in that paragraph we have *slightly widened the definition of division.* We speak of a number being contained in another number when it is not contained in it exactly. Thus, Ex. 3 gives us the right to say that 7 is contained in 257 36 *and five-sevenths times.* This means that, in order to obtain a group of 257 from a group of 7, we have to repeat 36 times the group 7, and then to add to the result five-sevenths of the group 7, *i.e.* 5. We may now recast the definition of division, just as we recast that of multiplication :—

Division *is the process of finding the number of times* (**quotient**) *a certain group of objects* (**divisor**) *must be repeated in order to produce the same total as in another group* (**dividend**).

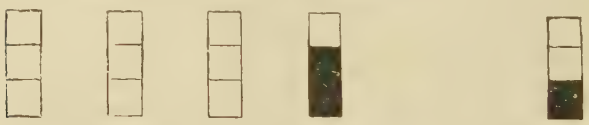
Or we may say equally well, as we have really said already in § 35 :—

To **divide** *a number* (**dividend**) *by another number* (**divisor**) *is to find what number* (**quotient**) *the divisor must be multiplied by to produce the dividend.*

Thus, to divide 27 by 3 is to find how often the group ⋮ must be repeated in order to make the group ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮, or, in other words, what number 3 must be multiplied by to make 27.

It must be noted that these definitions are *not really new.* Moreover, before proceeding further, you must convince yourself that the definitions mean the same thing. The first is given to lead up to the second.

Ex. 1.—*Divide* $3\frac{1}{3}$ *by* $\frac{2}{3}$ *by means of a diagram.*



Dividend Group. Divisor Group.

We must choose a unit for our diagram which is capable of being divided into 3 parts. We choose a 3-square unit. The dividend is represented by 10 squares, the divisor by 2 squares, and it is easily seen that we have to repeat the divisor group 5 times to make up the dividend group ; therefore the quotient is 5.

Ex. 2.—*Divide* $\frac{2}{3}$ *by* $3\frac{1}{3}$ *by means of a diagram.*

Here we are asked how many of the *left-hand groups*, regarding the four blocks as one group, in the figure above we must take to get one of the *right-hand groups*. It is clear we must take one-fifth of a group. *Ans.* $\frac{1}{5}$.

EXAMPLES LIV.

[*No answers are given to these Examples.*]

Verify the following statements by means of a diagram :—

1. $4\frac{1}{2} \div 1\frac{1}{2} = 3$.

2. $1\frac{1}{2} \div 4\frac{1}{2} = \frac{1}{3}$.

3. $\frac{1}{2} \div \frac{1}{4} = 2$.

4. $\frac{1}{3} \div \frac{2}{3} = \frac{1}{2}$.

5. $\frac{1}{2} \div \frac{1}{3} = 1\frac{1}{2}$ [Take a unit divisible into 6 parts].

6. $\frac{1}{2} \div \frac{2}{3} = \frac{3}{4}$.

7. $\frac{1}{3} \div \frac{1}{2} = \frac{2}{3}$.

8. $1\frac{1}{8} \div 1\frac{1}{2} = \frac{3}{4}$.

9. $2\frac{2}{3} \div \frac{1}{2} = 5\frac{1}{3}$.

10. $\frac{1}{2} \div 2\frac{2}{3} = \frac{3}{16}$.

118. It follows, from the second definition of § 117, that—

To divide $\frac{3}{4}$ by $\frac{5}{7}$ is to find what $\frac{5}{7}$ must be multiplied by to produce $\frac{3}{4}$.

Now $\frac{5}{7} \times \frac{7}{5} = \frac{\cancel{5} \times 7}{7 \times \cancel{5}} = 1$

$\therefore \quad \frac{5}{7} \times \frac{7}{5} \times \frac{3}{4} = 1 \times \frac{3}{4}$ *

$= \frac{3}{4}$.

It follows, therefore, that $\frac{5}{7}$ must be multiplied by $\frac{7}{5} \times \frac{3}{4}$ to produce $\frac{3}{4}$, and, therefore, that

$$\frac{3}{4} \div \frac{5}{7} = \frac{7}{5} \times \frac{3}{4}$$
$$= \frac{3}{4} \times \frac{7}{5} \dagger \quad \text{(See [§ 125.)}$$
$$= \frac{21}{20}.$$

To divide a/b by c/d is to find what c/d must be multiplied by to produce a/b.

Now $\frac{c}{d} \times \frac{d}{c} = \frac{c \times d}{d \times c} = 1$.

$\therefore \quad \frac{c}{d} \times \frac{d}{c} \times \frac{a}{b} = 1 \times \frac{a}{b}$ *

$= a/b$.

It follows, therefore, that c/d must be multiplied by $d/c \times a/b$ to produce a/b, and, therefore, that

$$\frac{a}{b} \div \frac{c}{d} = \frac{d}{c} \times \frac{a}{b}$$
$$= \frac{a}{b} \times \frac{d}{c} \dagger \quad \text{(See [§ 125.)}$$
$$= \frac{ad}{bc}.$$

Hence the following practical rule :—

To divide one fraction by a second multiply the first by the RECIPROCAL (*see* § 116, *Ex.* 2) *of the second. The result is the quotient.*

---

* The teacher must explain the nature of the axiom assumed here. The statement made is that, if two groups of things contain the same number in each, any portion (a half, a third, &c.) of one group will contain the same number as the same portion of the other.

† This step must not be assumed as "obvious." It is far from obvious.

Ex. 1.—*Divide* $\frac{3}{14}$ *by* $\frac{2}{7}$.

$$\text{Quotient} = \frac{3}{\cancel{14}_{2}} \times \frac{\cancel{7}}{2}\left[=\frac{3\times 1}{2\times 2}\right] = \frac{3}{4}. \quad \textit{Ans.}$$

Ex. 2.—*Divide* $5\frac{1}{3}$ *by* $2\frac{2}{3}$. Mixed numbers must be converted into improper fractions.

$$5\tfrac{1}{3} \div 2\tfrac{2}{3} = \tfrac{16}{3} \div \tfrac{8}{3} = \frac{\cancel{16}^{2}}{\cancel{3}} \times \frac{\cancel{3}}{\cancel{8}} = 2. \quad \textit{Ans.}$$

Ex. 3.—*Divide* 1 *by* $2\frac{2}{3}$.

$$1 \div 2\tfrac{2}{3} = 1 \div \tfrac{8}{3} = 1 \times \tfrac{3}{8} = \tfrac{3}{8}. \quad \textit{Ans.}$$

This result is important. *To divide unity by a number write the reciprocal of the number. This is the quotient.*

EXAMPLES LV.

1. $72\frac{1}{2} \div 7\frac{1}{4}$. **2.** $7\frac{1}{5} \div 1\frac{11}{25}$. **3.** $7\frac{1}{4} \div 72\frac{1}{2}$. **4.** $1\frac{11}{25} \div 7\frac{1}{5}$.

5. $6\frac{3}{14} \div 5\frac{1}{7}$. **6.** $2\frac{2}{13} \div 1\frac{9}{26}$. **7.** $7\frac{1}{4} \div 11\frac{3}{5}$. **8.** $257 \div 77\frac{1}{10}$.

9. $3\frac{7}{9} \div 8\frac{2}{21}$. **10.** $1\frac{8}{49} \div 13\frac{4}{7}$. **11.** $5\frac{1}{7} \div 6\frac{3}{14}$. **12.** $5\frac{5}{36} \div 4\frac{1}{9}$.

13. $\frac{13}{96} \div \frac{65}{84}$. **14.** $3\frac{7}{9} \div 4\frac{13}{18}$. **15.** $3\frac{17}{55} \div 3\frac{1}{121}$. **16.** $9\frac{9}{55} \div \frac{72}{121}$.

17. $8\frac{11}{56} \div 2\frac{13}{76}$. **18.** $3\frac{17}{32} \div 1\frac{43}{48}$. **19.** $8\frac{13}{15} \div 2\frac{7}{25}$. **20.** $\frac{51}{64} \div 7\frac{7}{16}$.

21. $1\frac{29}{55} \div \frac{108}{121}$. **22.** $2\frac{17}{44} \div 2\frac{73}{121}$. **23.** $105\frac{4}{35} \div 4\frac{2}{7}$. **24.** $7\frac{5}{16} \div 62\frac{1}{48}$.

25. $2\frac{30}{49} \div 2\frac{34}{63}$. **26.** $\frac{153}{308} \div 3\frac{24}{77}$. **27.** $18\frac{3}{8} \div 4\frac{13}{44}$.

28. $3\frac{17}{23} \div 1\frac{11}{161}$. **29.** $\frac{33}{52} \div 5\frac{25}{39}$. **30.** $\frac{64}{273} \div 1\frac{5}{91}$.

31. $17\frac{3}{16} \div 1\frac{239}{256}$. **32.** $1\frac{93}{232} \div 3\frac{51}{58}$. **33.** $\frac{77}{156} \div 15\frac{31}{39}$.

34. $1\frac{214}{325} \div 2\frac{57}{143}$. **35.** $33\frac{3}{25} \div 2\frac{91}{265}$. **36.** $6\frac{47}{138} \div 6\frac{21}{184}$.

37. $5\frac{44}{47} \div \frac{107}{141}$ **38.** $30\frac{2}{17} \div 4\frac{20}{51}$. **39.** $11\frac{11}{63} \div 7\frac{1}{14}$.

40. $25\frac{26}{27} \div \frac{11}{90}$.

41. How often is $\frac{164}{165}$ contained in $1\frac{27}{55}$?

42. How often is $\frac{39}{40}$ contained in $8\frac{15}{16}$?

43. By what must $2\frac{17}{54}$ be multiplied to produce $4\frac{31}{36}$?

44. By what must 171 be multiplied to produce $8\frac{1}{7}$?

45. How many groups each containing $2\frac{17}{30}$ are needed to make a group containing $1\frac{11}{24}$?

46. How many groups each containing $1\frac{1}{2}$ are needed to make a group containing $7\frac{17}{18}$?

Complex Fractions.

[*Before commencing this section you should revise* § 46.]

119. We saw in § 114, Ex. 3, that the —— in the symbol for a fraction might be regarded as the sign of division; so that, for instance, $\frac{31}{5}$ might be regarded either as "31 fifths of unity" or as "a fifth part of 31," or as "the quotient on dividing 31 by 5." Since we have now learnt the meaning of the division of a fraction by a fraction, such an expression as $\frac{3}{4}/\frac{1}{4}$, which means the quotient on dividing $\frac{3}{4}$ by $\frac{1}{4}$, can be interpreted, and its value is, of course, 3. Such an expression is called a Complex Fraction.

A **complex fraction** is a fraction whose numerator and denominator are either or both fractions.

In the simplification of complex fractions the rules given in *italics* in § 46 are most important, and to these may be added—

(4) *In complex fractions the word* **of** *has the force of a bracket.*

Thus, $\frac{2}{3} \div \frac{1}{3}$ of 2 means $\frac{2}{3} \div (\frac{1}{3} \times 2)$, *i.e.* $\frac{2}{3} \div \frac{2}{3}$, *i.e.* 1; but $\frac{2}{3} \div \frac{1}{3} \times 2$ means 2×2, *i.e.* 4, working from left to right.

Ex. 1.—*Find the values of the complex fractions—*

$$\frac{\frac{2}{3}}{\frac{4}{7}}, \quad \frac{2}{\frac{3}{\frac{4}{7}}}, \quad \frac{\frac{\frac{2}{3}}{4}}{7}, \quad \frac{\frac{2}{\frac{3}{4}}}{7}, \quad \frac{2}{\frac{\frac{3}{4}}{7}}.$$

The darkest (or the longest) line is our guide in interpreting these fractions.

First fraction $= \frac{2}{3} \div \frac{4}{7} = \frac{\cancel{2}}{3} \times \frac{7}{\cancel{4}\,2}$ (§ 118) $= \frac{7}{3 \times 2} = \frac{7}{6}$.

Second fraction $= 2 \div \left(\frac{3}{\frac{4}{7}}\right)$. Now ( ) $= 3 \div \frac{4}{7} = 3 \times \frac{7}{4} = \frac{21}{4}$.

$\therefore$ Fraction* $= 2 \div \frac{21}{4} = 2 \times \frac{4}{21} = \frac{8}{21}$.

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* The teacher must insist on this or some equivalent word. (See "Equality" in § 46.) The average schoolboy will continue the work with = only, without saying what "equals" the expressions which follow.

Third fraction $= \left(\dfrac{\frac{2}{3}}{\frac{3}{4}}\right) \div 7$. Now ( ) $= \frac{2}{3} \div 4 = \dfrac{\cancel{2}}{3} \times \dfrac{1}{\cancel{4}_{2}} = \frac{1}{6}$.

$\therefore$ Fraction * $= \frac{1}{6} \div 7 = \frac{1}{42}$ (§ 114).

Fourth fraction $= \left(\dfrac{2}{\frac{3}{4}}\right) \div 7$. Now ( ) $= 2 \div \frac{3}{4} = 2 \times \frac{4}{3} = \frac{8}{3}$.

$\therefore$ Fraction * $= \frac{8}{3} \div 7 = \frac{8}{21}$ (§ 114).

This fraction therefore means the same as the second.

Fifth fraction $= 2 \div \left(\dfrac{\frac{3}{4}}{7}\right)$. Now ( ) $= \frac{3}{4} \div 7 = \frac{3}{28}$ (§ 114).

$\therefore$ Fraction * $= 2 \div \frac{3}{28} = 2 \times \frac{28}{3} = \frac{56}{3} = 18\frac{2}{3}$.

Ex. 2.—*Find the value of* $\dfrac{3}{4\frac{1}{2} \text{ of } 1\frac{1}{3} + 4\frac{1}{4}} \times (\frac{5}{12} + \frac{4}{15})$.

Work thus :—

First fraction $= \dfrac{3}{\frac{9}{2} \times \frac{4}{3} + 4\frac{1}{4}}$ [note carefully that it is *not* $4\frac{1}{2}$ of $(1\frac{1}{3} + 4\frac{1}{4})$; see Rule (4) above].

$= \dfrac{3}{6 + 4\frac{1}{4}}$ †

$= \dfrac{3}{10\frac{1}{4}} = 3 \div 10\frac{1}{4}$ † $= 3 \div \frac{41}{4}$ † $= 3 \times \frac{4}{41} = \frac{12}{41}$.

Second fraction $= \dfrac{5}{3 \,.\, 4} + \dfrac{4}{3 \,.\, 5} = \dfrac{25 + 16}{3 \,.\, 4 \,.\, 5} = \frac{41}{60}$. It is often best to leave such denominators in factors.

$\therefore$ Whole fraction $= \dfrac{\cancel{12}}{\cancel{41}} \times \dfrac{\cancel{41}}{\cancel{3} \,.\, \cancel{4} \,.\, 5} = \frac{1}{5}$. *Ans.*

Ex. 3.—*Simplify* $\dfrac{5\frac{5}{8} + 5\frac{1}{6}}{5\frac{5}{8} - 5\frac{1}{8}} \times \dfrac{2112}{7104} \times \dfrac{2778}{25928} \div \dfrac{(4\frac{2}{3} - 4\frac{7}{12}) \text{ of } \frac{24}{33}}{4\frac{2}{3} + 4\frac{7}{12} \text{ of } \frac{24}{33}}$

The full work is as follows, and notes are added (p. 159) :—

First fraction $= \dfrac{10\dfrac{15+4}{24}\,(^1)}{\dfrac{15-4}{24}} = \dfrac{10\frac{19}{24}}{\frac{11}{24}}$ † $= \frac{259}{24} \div \frac{11}{24}$ † $= \dfrac{259}{\cancel{24}} \times \dfrac{\cancel{24}}{11}$ † $= \frac{259}{11}$

Second fraction $= \frac{176}{592}\,(^2) = \frac{44}{148}\,(^3) = \frac{11}{37}$.

Third fraction $= \frac{1389}{12964}\,(^4) = \dfrac{3 \,.\, \cancel{463}}{28 \,.\, \cancel{463}}\,(^5) = \frac{3}{28}$.

$$\begin{array}{r|l} 3\,)\ 1389 & 12964\ (\ 9\ (^6) \\ 0 & 463 \end{array}$$

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* See footnote on p. 157.

† These steps may, and after a time *should*, be omitted.

$$\textit{Fourth fraction} = \frac{\frac{8-7}{12} \times \frac{24}{33}}{4\frac{2}{3} + \frac{\overset{5}{\cancel{55}}}{\cancel{12}} \times \frac{\overset{2}{\cancel{24}}}{\underset{3}{\cancel{33}}}} = \frac{\frac{1}{\cancel{12}} \times \frac{\cancel{24}\,\dagger}{33}}{4\frac{2}{3} + \frac{10}{3}} = \frac{\frac{2}{33}}{4\frac{2}{3} + 3\frac{1}{3}} = \frac{\frac{2}{33}}{8} = \frac{\cancel{2}}{33} \times \frac{1}{\underset{4}{\cancel{8}}}$$

$$= \frac{1}{4 \times 33}.(7)$$

$$\therefore \quad \text{Whole fraction} = \frac{\overset{7}{\cancel{289}}}{\cancel{11}} \times \frac{\cancel{11}}{\cancel{37}} \times \frac{3}{\cancel{28}} \times \frac{\cancel{4} \times 33}{1} = 3 \times 33\,\dagger = 99. \quad \textit{Ans.}$$

Notes.—Every step of the work is exhibited above except what may and ought to be done mentally. Steps marked † should be omitted by the advanced pupil.

(1) Numerator and denominator are simplified simultaneously and brought to their obvious L.C.D. No part should be rewritten in exactly the same state as it was at first, unless this is its simplest state.

(2) 2112, 7104 satisfy the tests for 3 and 4 (§ 85), and therefore are divisible by 12.

(3) It is rarely worth while to test for 8.

(4) We test rapidly for 3 and 11, and, failing, find the H.C.F. The full work for this is shown at 6.

(5) We do not need to perform divisions here. The H.C.F. work shows $1389 = 3 \times 463$. It also shows that 12964 is exactly divisible by 463, and the first figure must be 2, while the last must be 8, since only 8×463 ends in 4.

(6) Exhibits all the work necessary for the H.C.F.

(7) Such a result is, in every case, best left in factors until further on.

EXAMPLES LVI.

Simplify the following :—

1. $\frac{1}{2} - \frac{4}{5}$ of $\frac{5}{6} + \frac{1}{4}$. **2.** $\frac{6}{7} - \frac{3}{13}$ of $1\frac{6}{7}$. **3.** $4\frac{2}{3} \times 5\frac{4}{7} \div 3\frac{1}{4}$.

4. $7\frac{7}{8} \times 4\frac{4}{7} \div 5\frac{1}{7}$. **5.** $3\frac{3}{17} \times 4\frac{7}{11} \div 3\frac{15}{22}$. **6.** $3\frac{1}{8} \times 2\frac{2}{5} \times 1\frac{5}{9} \div 2\frac{1}{3}$.

7. $(6\frac{1}{7}$ of $10\frac{1}{2}) \div (2\frac{1}{10}$ of $7\frac{1}{6})$. **8.** $(\frac{7}{55}$ of $\frac{13}{20}) \div (\frac{53}{33}$ of $\frac{21}{20})$.

9. $2 \times 5\frac{8}{51} - (4\frac{3}{17} + 3\frac{2}{3})$. **10.** $(2\frac{2}{3}$ of $3\frac{3}{4}) \div (2\frac{2}{3} + 3\frac{3}{4}.)$

11. $4\frac{3}{11} - 2\frac{3}{4} + 5\frac{1}{8} \times 4\frac{1}{2}$. **12.** $5\frac{1}{4} - 2\frac{1}{3} + \frac{1}{3}$ of $5\frac{1}{2}$.

13. $(3\frac{7}{8} - 2\frac{1}{4}) \div (3\frac{1}{4} \times \frac{1}{2})$. **14.** $(\frac{5}{2} - \frac{2}{5}) \div (\frac{4}{3} - \frac{3}{4})$.

15. $(\frac{2}{3} \times 4\frac{1}{4}$ of $3\frac{2}{17}) - (5\frac{1}{12} - 2\frac{3}{4})$. **16.** $5\frac{2}{5} + 3\frac{1}{5} \div 1\frac{1}{3} - 4 \div 1\frac{9}{11}$.

17. $(17 \text{ of } 4\tfrac{5}{11}) \times (\tfrac{1}{7} \text{ of } \tfrac{2}{51}) \times (\tfrac{3}{28} \div \tfrac{17}{28})$.

18. $(\tfrac{1}{4}+\tfrac{3}{8}) \times (\tfrac{5}{6}-\tfrac{5}{7}) \div (\tfrac{7}{8}-\tfrac{7}{9})$.

19. $\tfrac{11}{17} \text{ of } 21\tfrac{6}{7} - \tfrac{11}{19} \text{ of } 10\tfrac{5}{9} + \tfrac{11}{21} \div \tfrac{33}{61}$.

20. $2\tfrac{1}{12} \div 5\tfrac{5}{8} + 1\tfrac{7}{11} \times 2\tfrac{4}{9} - 2\tfrac{1}{7} \times 1\tfrac{3}{25}$.

21. $3\tfrac{1}{15} \text{ of } 2\tfrac{1}{23} \text{ of } 1\tfrac{1}{4} + 5\tfrac{1}{2} \text{ of } 1\tfrac{1}{6} \text{ of } 22$.

22. $(3\tfrac{1}{2} \times 5\tfrac{1}{7}) \div (\tfrac{16}{5} \times 3\tfrac{3}{4}) + \tfrac{1}{2}$.

23. $\left\{1\tfrac{29}{36} + 4\tfrac{1}{8} \times 1\tfrac{7}{11}\right\} \div \left\{5\tfrac{1}{9} - 7\tfrac{6}{7} \div 9\tfrac{9}{20}\right\}$

24. $\left\{10\tfrac{5}{6} + 49\tfrac{1}{2} \div 1\tfrac{2}{9}\right\} \div \left\{30\tfrac{2}{3} - 47\tfrac{1}{4} \times \tfrac{20}{189}\right\}$.

25. $\dfrac{5\tfrac{2}{3} - 2\tfrac{3}{5}}{4\tfrac{5}{8} \text{ of } \tfrac{2}{185}}$.

26. $\dfrac{3\tfrac{1}{17} + 17\tfrac{1}{3}}{\tfrac{1}{17} \text{ of } 2\tfrac{1}{6}}$.

27. $\dfrac{\tfrac{1}{3}(\tfrac{1}{2}+\tfrac{1}{5})}{\tfrac{1}{5}(\tfrac{1}{2}+\tfrac{1}{3})}$.

28. $\dfrac{\tfrac{1}{3}(\tfrac{1}{4}+\tfrac{1}{5})}{\tfrac{1}{5}(\tfrac{1}{3}-\tfrac{1}{6})}$.

29. $\dfrac{\tfrac{1}{9} \text{ of } 1\tfrac{1}{4} \text{ of } 4\tfrac{1}{2}}{\tfrac{5}{56} \text{ of } 1\tfrac{1}{8} \text{ of } 3\tfrac{1}{2}}$.

30. $\dfrac{(4\tfrac{37}{40} \div 2\tfrac{191}{200}) \text{ of } \tfrac{10}{11}}{\tfrac{5}{7} \text{ of } 1\tfrac{3}{11} \text{ of } \tfrac{2}{3}}$

31. $\dfrac{4\tfrac{2}{13} \times 2\tfrac{1}{19} \times 3\tfrac{1}{2}}{19\tfrac{2}{7} - 5\tfrac{2}{5} \times 3\tfrac{3}{14}}$.

32. $\dfrac{3\tfrac{1}{2} + 2\tfrac{1}{3} + 2\tfrac{4}{5}}{\tfrac{1}{7} \text{ of } \tfrac{11}{12} \text{ of } \tfrac{14}{55}}$.

33. $\dfrac{\tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{5}}{2\tfrac{1}{4} + 4\tfrac{1}{5} + 5\tfrac{1}{4}}$.

34. $\dfrac{2\tfrac{1}{7} \text{ of } \tfrac{8}{15} - \tfrac{1}{3}}{2\tfrac{1}{7} \text{ of } (\tfrac{8}{15} - \tfrac{1}{3})}$.

35. $\dfrac{(2\tfrac{3}{4} + \tfrac{1}{4}) \text{ of } 2\tfrac{4}{9}}{1\tfrac{1}{4} \text{ of } (\tfrac{3}{2} + 2\tfrac{1}{3})}$.

36. $\dfrac{2\tfrac{3}{4} - \tfrac{1}{3} \text{ of } \tfrac{6}{7}}{1\tfrac{2}{3} + 3 \div 4\tfrac{1}{8}}$.

37. $\dfrac{3\tfrac{1}{5} + \tfrac{11}{21}}{2\tfrac{1}{3} + 1\tfrac{29}{35}}$.

38. $\dfrac{23\tfrac{4}{15} \div 12}{7\tfrac{1}{6} + 8\tfrac{2}{5} - 9\tfrac{3}{4}}$.

39. $\dfrac{3\tfrac{5}{14} + 2\tfrac{4}{9} - 4\tfrac{19}{21}}{3\tfrac{37}{63}}$.

40. $\dfrac{3\tfrac{7}{8} - 4\tfrac{1}{3} + 2\tfrac{1}{12}}{9\tfrac{1}{36} - 7\tfrac{19}{27}}$

41. $\dfrac{2\tfrac{1}{3} \times \tfrac{3}{14} - 5\tfrac{1}{2} \div 16\tfrac{1}{2}}{16\tfrac{1}{2} - (5\tfrac{1}{2} - \tfrac{3}{14}) + 2\tfrac{1}{3}}$.

42. $\dfrac{(\tfrac{7}{8} \div 2\tfrac{1}{3}) \times (\tfrac{1}{4} \text{ of } 6\tfrac{2}{3})}{2\tfrac{1}{4} + 3\tfrac{1}{2} - 5\tfrac{7}{8} + \tfrac{23}{24}}$.

43. $\dfrac{\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4}}{\dfrac{1}{2\tfrac{1}{2}} + \dfrac{1}{3\tfrac{1}{2}} + \dfrac{1}{4\tfrac{1}{2}}}$.

44. $\dfrac{\tfrac{2}{3} + \tfrac{4}{5} - \tfrac{3}{4}}{\tfrac{2}{9} + \tfrac{3}{5} - \tfrac{2}{7}}$

45. $\dfrac{6\tfrac{2}{3} - 1\tfrac{5}{6} \text{ of } \tfrac{20}{33}}{3\tfrac{2}{7} + 4\tfrac{2}{5} \text{ of } \tfrac{20}{33}}$.

46. $\dfrac{1\tfrac{1}{2} + 4\tfrac{2}{7} - 2\tfrac{1}{3}}{1\tfrac{1}{3} + 3 - \dfrac{1}{6\tfrac{2}{3}}}$.

47. $\dfrac{1\tfrac{1}{2} \times 4\tfrac{2}{7} \times 2\tfrac{1}{5}}{1\tfrac{1}{3} \times 1\tfrac{6}{7} \times \left(\dfrac{1}{4\tfrac{1}{2}} + \dfrac{1}{6\tfrac{1}{2}}\right)}$

48. $\dfrac{(\tfrac{1}{2} + \tfrac{1}{4}) \div (\tfrac{5}{6} \text{ of } \tfrac{3}{8})}{2\tfrac{2}{3} \div (3\tfrac{1}{3} - 2\tfrac{1}{2})}$.

49. $\dfrac{5\tfrac{3}{4} - \tfrac{3}{7} \text{ of } 15\tfrac{3}{4} + 2\tfrac{2}{35} \div 1\tfrac{11}{25}}{\tfrac{3}{4} \text{ of } 7\tfrac{3}{7} - 5\tfrac{3}{5} \div 3\tfrac{4}{15}}$.

50. $\dfrac{3\tfrac{2}{3} - 5\tfrac{1}{4} \div 3\tfrac{1}{2} + 4\tfrac{5}{6}}{3\tfrac{2}{3} \times 5\tfrac{1}{4} - 3\tfrac{1}{2} \times 4\tfrac{5}{6}}$.

51. $\tfrac{29}{105} + \tfrac{4}{35} \text{ of } 1\tfrac{3}{4} - \tfrac{7}{11} \text{ of } 3\tfrac{2}{3} \text{ of } \tfrac{6}{46}$.

52. $1\frac{4}{7}$ of $5\frac{1}{11} - 1\frac{4}{23}$ of $5\frac{1}{9} + 1\frac{15}{19}$ of $2\frac{4}{17}$.

53. $\dfrac{2\frac{3}{4} + 7\frac{15}{17} - \frac{5}{34}}{2\frac{5}{11} \times 4\frac{1}{8} + \frac{7}{8}}$.

54. $\dfrac{2\frac{1}{3} - \frac{3}{14} \text{ of } 5\frac{1}{2} \div 16\frac{1}{2}}{16\frac{1}{2} \div 5\frac{1}{2} - \frac{3}{14} \text{ of } 2\frac{1}{3}}$.

55. $\dfrac{(\frac{4}{11} \div \frac{3}{5}) + (\frac{1}{11} \text{ of } 2) + \frac{1}{6}(\frac{1}{5} - \frac{1}{11})}{\dfrac{1}{4\frac{1}{2}} + \dfrac{1}{5\frac{1}{2}} + \dfrac{1}{7\frac{1}{2}}}$

56. $\dfrac{5\frac{1}{2} \times 1\frac{2}{33} \div 1\frac{3}{4} - \frac{5}{18} \div 1\frac{1}{18}}{1\frac{1}{3} \times \frac{2}{3} - \frac{22}{57} + \frac{1}{4} \div 2\frac{1}{4}}$

57. $(\frac{3}{5} + \frac{4}{7} + \frac{5}{8}) \times (17\frac{6}{17} - 11\frac{9}{16})$.

58. $\dfrac{7\frac{1}{5} \times 5\frac{1}{7}}{7\frac{1}{5} - 5\frac{1}{7}} \div \dfrac{5\frac{1}{3} \times 3\frac{1}{5}}{5\frac{1}{3} + 3\frac{1}{5}}$.

59. $\dfrac{1}{2 + \dfrac{3}{4\frac{1}{5}}} \div \dfrac{2}{3 + \dfrac{1}{4\frac{5}{6}}}$.

60. $\dfrac{3\frac{1}{3} + 8\frac{4}{9}}{6\frac{1}{12} - 3\frac{7}{8}} \div \dfrac{3\frac{1}{7} \text{ of } 9\frac{7}{9}}{3\frac{13}{14} \text{ of } 4\frac{2}{5}}$.

61. $\dfrac{3}{3 - 1 \div (3 - \frac{1}{3})} \div \dfrac{6 - 1 \div (6 - \frac{1}{6})}{6\frac{3}{8}}$.

62. $\dfrac{(\frac{1}{6} - \frac{1}{7})}{(\frac{1}{7} - \frac{1}{8})} \text{ of } \dfrac{(\frac{1}{8} - \frac{1}{9})}{(\frac{1}{9} - \frac{1}{10})} \text{ of } \dfrac{(\frac{1}{10} - \frac{1}{11})}{(\frac{1}{11} - \frac{1}{12})}$.

63. $\dfrac{7\frac{1}{9}}{4\frac{1}{6}} \text{ of } \dfrac{2\frac{1}{2} - \frac{5}{8} \times 3\frac{1}{4}}{\frac{8}{25} + \frac{2}{3} \div 3\frac{1}{8}}$.

64. $\dfrac{2\frac{1}{7} + 3\frac{2}{3} - 4\frac{5}{8}}{5\frac{16}{21} - 3\frac{26}{35}} \text{ of } \dfrac{1}{1 - \dfrac{1}{1\frac{2}{3}}}$.

65. $\dfrac{\frac{1}{2} - \frac{1}{4} + \frac{1}{8}}{1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8}} \div \dfrac{\frac{1}{3} - \frac{1}{9} + \frac{1}{27}}{1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27}}$.

66. $\dfrac{2\frac{3}{4} \text{ of } 32\frac{1}{2} - 71\frac{1}{4}}{47\frac{9}{16} - 2\frac{1}{2} \text{ of } 8\frac{7}{8}} \div \dfrac{5\frac{5}{7} \text{ of } 12\frac{1}{2}}{333\frac{1}{3} - 222\frac{2}{9}}$.

67. $\dfrac{\dfrac{1}{1\frac{1}{2}} + \dfrac{1}{1\frac{1}{3}} + \dfrac{1}{1\frac{1}{4}}}{3 - \left\{ \dfrac{1}{1 - \frac{1}{16}} - \dfrac{1}{4 - \frac{8}{17}} \right\}}$.

68. $\dfrac{\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}}{\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5}} \div \dfrac{1\frac{4}{15} - \frac{8}{9} + \frac{1}{5} - \frac{1}{3}}{1\frac{4}{5} + \frac{5}{9} - \frac{1}{3}}$.

69. $\dfrac{3\frac{1}{15} - 4\frac{3}{20} + 2\frac{1}{12}}{3\frac{1}{25} - 2\frac{1}{10}} \times \dfrac{3\frac{2}{15}}{\frac{2}{3} + \frac{3}{8} - \frac{5}{48}}$.

70. $\dfrac{10\frac{1}{5} - 2\frac{3}{7} \text{ of } 2\frac{5}{8}}{(10\frac{1}{5} - 2\frac{3}{7}) \text{ of } 2\frac{5}{8}} \div \dfrac{1}{3 + \dfrac{1}{1\frac{4}{5}}}$.

71. $\dfrac{(\frac{5}{6} + \frac{4}{5}) \text{ of } \frac{3}{4} - \frac{2}{3}}{\frac{5}{6} + \frac{4}{5} \text{ of } (\frac{3}{4} - \frac{2}{3})} \div \dfrac{\frac{5}{6} \text{ of } (\frac{4}{5} - \frac{3}{4}) \text{ of } \frac{2}{3}}{\frac{5}{6} \text{ of } \frac{4}{5} - \frac{3}{4} \text{ of } \frac{2}{3}}$.

72. $\dfrac{4\frac{1}{5} - (\frac{2}{3} \text{ of } \frac{5}{6})}{1\frac{2}{3} + 1\frac{3}{7} - \frac{2}{5}} \times \dfrac{1}{\frac{1}{2} + \dfrac{\frac{1}{6}}{\frac{3}{4} - \frac{1}{3}}}$.

73. $\dfrac{(\frac{2}{3}-\frac{2}{11}+\frac{16}{21}) \div \left(2+\dfrac{1}{2+\frac{1}{2}}\right)}{(\frac{3}{77}+\frac{2}{33}-\frac{1}{21}) \times \left(3-\dfrac{1}{3-\frac{1}{3}}\right)}$.

74. $\dfrac{\frac{5}{14}-\frac{3}{7} \text{ of } \frac{1}{2}}{\frac{5}{16}+\frac{7}{12} \text{ of } 3\frac{1}{4}-(\frac{7}{8} \text{ of } \frac{37}{21}-\frac{1}{3})} \div \dfrac{\frac{1}{3} \text{ of } \frac{1}{2}+\frac{3}{2} \text{ of } 5}{9\frac{1}{3}-1\frac{2}{3}}$.

75. $\dfrac{5}{7-\dfrac{9}{3-\frac{3}{4}}}+\dfrac{5}{16-\dfrac{11}{2-\frac{1}{6}}}$.

76. $\dfrac{39}{39-\dfrac{1}{4\frac{1}{3}}}-\dfrac{40}{40-\dfrac{1}{4\frac{3}{4}}}$.

77. $\dfrac{16\frac{1}{2}-11\frac{3}{4}}{2\frac{2}{3} \text{ of } 3\frac{4}{5}}-\dfrac{2\frac{13}{25}}{9\frac{1}{3} \text{ of } 1\frac{17}{55}}$.

78. $\dfrac{8\frac{7}{12}}{5\frac{11}{18}+5\frac{5}{6}}-\dfrac{11\frac{3}{8}}{16\frac{11}{12}+5\frac{5}{6}}$.

79. $\dfrac{\frac{1}{2}-\frac{1}{3} \text{ of } \frac{1}{4}-\frac{1}{5}}{\frac{1}{2}-\frac{1}{3}-\frac{1}{4} \text{ of } \frac{1}{5}}+19 \div \frac{7}{3}$.

80. $19\frac{3}{5} \times \dfrac{6\frac{3}{7}}{10\frac{1}{2}}+\dfrac{8\frac{2}{3}-6\frac{1}{4}}{4\frac{8}{9}-4\frac{1}{12}}$.

81. $\dfrac{\frac{1}{6} \div \frac{3}{11}}{\frac{5}{7}-\frac{2}{3}}-\dfrac{2\frac{1}{2}-1\frac{5}{6}}{7 \times \frac{16}{21} \div \frac{8}{3}}$.

82. $\dfrac{4\frac{11}{13} \text{ of } \frac{8}{95} \text{ of } 7\frac{3}{7}}{12\frac{1}{35}-2\frac{3}{7}}+\dfrac{2\frac{1}{3}+1\frac{35}{48}}{9\frac{2}{7}-3\frac{39}{112}}$.

83. $\frac{1}{6}$ of $12\frac{3}{4}$ of $\frac{27}{85}$ of $(\frac{1}{4}+\frac{3}{8}-\frac{5}{7})$ of $\frac{15}{19}$.

84. $\frac{91}{122} \times (4\frac{1}{8} \text{ of } 6\frac{2}{7}+\frac{3}{14}) \div \left\{4\frac{1}{8} \text{ of } (6\frac{2}{7}+\frac{3}{14})\right\}$.

85. $\dfrac{\dfrac{1\frac{5}{12}-\frac{5}{6}}{1\frac{1}{2}} \times \dfrac{3\frac{3}{7}}{8} \div \dfrac{2\frac{4}{9}}{5}}{1 \div \left(2+\dfrac{1}{1+\frac{1}{4}}\right) \div \frac{1}{5}}$*.

86. $(1\frac{3}{4}-1)^2+\dfrac{\frac{7}{8}+1\frac{1}{2}+25\frac{5}{16}+147\frac{3}{4}}{1+(20)^2}$.

87. $\dfrac{2\frac{11}{26} \text{ of } \frac{4}{95} \text{ of } 7\frac{3}{7}}{6\frac{1}{70}-1\frac{3}{14}}+\dfrac{4\frac{2}{3}+3\frac{11}{24}}{18\frac{4}{7}-6\frac{39}{56}}$.

88. $3+\dfrac{1\frac{3}{8}+\frac{11}{12}}{2\frac{11}{15}-1\frac{9}{20}} \text{ of } 4\frac{1}{12} \text{ of } 3\frac{3}{7}$.

89. $10 \times \dfrac{1+\frac{1}{2} \times \frac{3}{4}+\frac{2}{3} \times \frac{4}{5}}{2-\frac{1}{2} \times \frac{5}{6}-\frac{2}{3} \times \frac{3}{4}}-2\frac{8}{13}$.

90. $2\frac{4}{5}-\frac{1}{9} \text{ of } \dfrac{5\frac{1}{7}-2\frac{1}{4}}{1+\dfrac{1}{3\frac{1}{8}}} \text{ of } \dfrac{9\frac{1}{3}}{12\frac{1}{11}} \div 2\frac{1}{5}$.

91. $\dfrac{4\frac{1}{4}-3\frac{1}{3}}{4\frac{1}{4} \times 3\frac{1}{8}}+\dfrac{5\frac{1}{5}-4\frac{1}{4}}{5\frac{1}{5} \times 4\frac{1}{4}}-\dfrac{5\frac{1}{5}-3\frac{1}{3}}{5\frac{1}{5} \times 3\frac{1}{3}}$.

92. $\dfrac{7\frac{1}{8}-6\frac{1}{7}}{7\frac{1}{8} \times 6\frac{1}{7}}+\dfrac{9\frac{1}{10}-7\frac{1}{8}}{9\frac{1}{10} \times 7\frac{1}{8}}-\dfrac{9\frac{1}{10}-6\frac{1}{7}}{6\frac{1}{7} \times 9\frac{1}{10}}$.

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* By § 46 (2) $a \div b \div c$ means $(a \div b) \div c$ not $a \div (b \div c)$.

93. $\dfrac{4\frac{1}{5}}{3\frac{3}{4}-2} + \dfrac{7\frac{2}{3}}{3\frac{1}{8}-1\frac{2}{5}} - \dfrac{4\frac{1}{5}-1\frac{2}{3}}{6\frac{1}{3}}.$

94. $\dfrac{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}}{\frac{1}{2}+\frac{1}{3}-\frac{1}{4}-\frac{1}{5}} + \dfrac{\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}}{\frac{1}{4}+\frac{1}{5}-\frac{1}{6}-\frac{1}{7}} - \frac{1024}{1357}.$

95. $\frac{5}{87} - 3\frac{9}{16} \div 71\frac{1}{4} - \dfrac{\frac{8}{63} \text{ of } \frac{49}{15} \text{ of } \frac{117}{56}}{116}$

96. $1\frac{4}{7} - \frac{3}{4}\left(\frac{1}{15}+\frac{1}{5}\right) + \frac{3}{104} \div \dfrac{2\frac{3}{16}}{3-\frac{5}{8}}.$

97. $\dfrac{3\frac{1}{2}}{8\frac{3}{4}} \times \dfrac{8\frac{3}{5}-5\frac{3}{8}}{1\frac{1}{4}+\frac{2}{7}} \div \dfrac{1}{2-\dfrac{1}{1-\frac{7}{15}}}.$

98. $\left(\dfrac{2\frac{1}{5}}{3\frac{2}{3}} + \dfrac{400}{8\frac{1}{3}} - 4\frac{1}{2} \text{ of } 5\frac{1}{5}\right) \div 5\frac{5}{11} + \frac{3}{5} \text{ of } 1\frac{1}{250} + \frac{1}{250}.$

99. $\dfrac{2\frac{1}{3}\times 3\frac{1}{4}\times(1\frac{1}{2}+4\frac{1}{5})}{(2\frac{1}{3}-1\frac{1}{2})(3\frac{1}{4}-1\frac{1}{2})} - \dfrac{3\frac{1}{4}\times 1\frac{1}{2}\times(2\frac{1}{3}+4\frac{1}{5})}{(3\frac{1}{4}-2\frac{1}{3})(2\frac{1}{3}-1\frac{1}{2})} + \dfrac{1\frac{1}{2}\times 2\frac{1}{3}\times(3\frac{1}{4}+4\frac{1}{5})}{(3\frac{1}{4}-1\frac{1}{2})(3\frac{1}{4}-2\frac{1}{3})}.$

100. $\dfrac{(\frac{1}{2}+\frac{1}{5})(\frac{1}{2}+\frac{1}{6})}{(\frac{1}{2}-\frac{1}{3})(\frac{1}{2}-\frac{1}{4})} - \dfrac{(\frac{1}{8}+\frac{1}{5})(\frac{1}{3}+\frac{1}{6})}{(\frac{1}{2}-\frac{1}{3})(\frac{1}{3}-\frac{1}{4})} + \dfrac{(\frac{1}{4}+\frac{1}{5})(\frac{1}{4}+\frac{1}{6})}{(\frac{1}{2}-\frac{1}{4})(\frac{1}{3}-\frac{1}{4})}.$

101. $\left(\dfrac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{2}-\frac{1}{3}} + \dfrac{\frac{1}{3}+\frac{1}{4}}{\frac{1}{3}-\frac{1}{4}}\right) \div \left(\dfrac{\frac{1}{7}+\frac{1}{9}}{\frac{1}{7}-\frac{1}{9}} - \dfrac{\frac{1}{5}+\frac{1}{7}}{\frac{1}{5}-\frac{1}{7}}\right).$

102. $\dfrac{17\frac{5}{8}-4\frac{3}{4}}{17\frac{5}{8}+2\frac{1}{2}} \times 5\frac{3}{4} \times \dfrac{48}{2\frac{1}{2}} \times \frac{7}{103}.$

103. $\dfrac{2\frac{1}{4}}{2\frac{2}{3}} + \dfrac{2\frac{1}{2}+5\frac{1}{5}}{3\frac{1}{3}+9\frac{1}{2}} + \frac{1}{2} + \left(\frac{3}{8} \text{ of } \frac{3}{20}\right).$

104. $\frac{1}{2} \text{ of } \left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right) + 7\left(\frac{1}{35}+\frac{1}{36}\right) - \frac{1}{12} \div \left(\dfrac{11\frac{1}{2}}{15\frac{1}{3}} - \frac{1}{4}\right).$

105. $2\frac{45}{49} \text{ of } \dfrac{7}{10+\dfrac{2\frac{1}{2}}{7}} \div \dfrac{1}{8+\dfrac{7}{11-\frac{2}{3} \text{ of } 16}}.$

106. $10\frac{3}{4} + 3\frac{9}{10} + 7\frac{24}{25} - \dfrac{\frac{2}{3}\times\frac{5}{7}\times\frac{35}{9}}{\frac{1}{7}\times\frac{2}{3}\times\frac{84}{81}}.$

107. $\dfrac{7\frac{1}{2} \text{ of } (9\frac{1}{2}-5\frac{1}{14}) - \left\{2\frac{1}{7} \text{ of } 9\frac{1}{2} - 5\frac{1}{14}\right\}}{(6\frac{51}{76} \div 2\frac{14}{19}) \times \frac{16}{39}}.$

108. $\frac{1}{8} \text{ of } \left(\frac{1}{10}-\frac{1}{11}\right) \div \dfrac{\frac{1}{7}-\frac{1}{9} \div \left(\frac{4}{9}+\frac{4}{11}\right)}{\left(\frac{1}{7}+\frac{1}{3}\right) \div \left(\frac{4}{9}-\frac{4}{11}\right)} \times \dfrac{\frac{1}{3}+\frac{1}{7} \div \left(\frac{1}{7}-\frac{1}{9}\right)}{\left(\frac{1}{3}+\frac{1}{7}\right) \div \frac{1}{7}-\frac{1}{9}}.$ [See §46,(2).]

109. $\frac{66}{8} \times \frac{1\frac{17}{20} - \frac{3}{5}}{4\frac{1}{2} + 2\frac{5}{11} - 3\frac{31}{44}} \times \frac{10625}{13125} \div \left\{ \frac{1}{4\frac{3}{11}} \times 4\frac{7}{10} \times \frac{34}{39} \right\}.$

110. $\dfrac{\left[\frac{1}{21} \times 5\frac{17}{23} \times 6\frac{3}{11}\right] + \left[6\frac{19}{51} \times 1\frac{28}{49} \div 2\frac{5}{17}\right] + 1\frac{10}{49}}{\left(9\frac{16}{57} \times 1\frac{22}{23} \div 5\frac{17}{38}\right) + \left(3\frac{11}{78} \times 6\frac{17}{21} \div \bar{1}\frac{21}{52}\right)}.$

Continued Fractions.

120. Fractions of the form $\cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \dots}}}$ are called **Continued Fractions.** They are usually written

$$\frac{1}{a+}\ \frac{1}{b+}\ \frac{1}{c+} \dots .$$

It is essential that the + should be written in the denominator.

The fractions $\frac{1}{a}$, $\frac{1}{a+}\ \frac{1}{b}$, $\frac{1}{a+}\ \frac{1}{b+}\ \frac{1}{c}$, ... are called **convergents** to the continued fraction when reduced and simplified.

These fractions have many most important properties, for an account of which reference must be made to some work on algebra. [See *Tutorial Algebra*, pp. 554–569.] We shall state some of these properties without proof.

(1) Any convergent may be found at once from the two convergents which immediately precede it.

(2) Any convergent is nearer in value to the continued fraction than any preceding convergent.

(3) The convergents are alternately less and greater than the continued fraction.

(4) The difference of any two consecutive convergents is always a proper fraction with unit numerator.

(5) Convergents are necessarily in their lowest terms, and never admit of reduction.

The name of continued fraction is also given to such fractions as $\frac{l}{a+}\ \frac{m}{b+}\ \frac{n}{c+} \dots$; but the properties just stated do not all hold with regard to such fractions.

Ex. 1.—*Simplify the continued fraction* $\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4+}\,\frac{1}{5}$

This means $$\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\frac{1}{5}}}} = \cfrac{1}{2+\cfrac{1}{3+\frac{5}{21}}} = \frac{1}{2+\frac{21}{68}} = \frac{68}{157}.$$

The rule for simplification is "Begin at the bottom." We begin then with $\frac{1}{4+\frac{1}{5}}$. The safest way of dealing with this is to multiply numerator and denominator by 5. [See § 106, (1).] We get $\frac{5}{20+1}$, *i.e.* $\frac{5}{21}$, &c.

Ex. 2.—*Convert the fraction* $\frac{30}{43}$ *into a continued fraction.*

The reversal of the process of Ex. 1 is easy. In the first instance, we divide both numerator and denominator of $\frac{30}{43}$ by 30. [See § 106, (2).]

$$\frac{30}{43} = \frac{1}{\frac{43}{30}} = \frac{1}{1+\frac{13}{30}} = \frac{1}{1+\frac{1}{\frac{30}{13}}} = \frac{1}{1+\cfrac{1}{2+\frac{4}{13}}}$$

$$= \frac{1}{1+\cfrac{1}{2+\cfrac{1}{\frac{13}{4}}}} = \frac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\frac{1}{4}}}} = \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4}$$

It is easily seen that the successive divisions *are those necessary in finding the H.C.F.* of 30 and 43, and, therefore, that the whole process may be written as here, the successive partial denominators being the quotients 1, 2, 3, 4.

$$\begin{array}{r|l} 2\,)\;30 & 43\,(\,1 \\ 4\,)\;\;4 & 13\,(\,3 \\ 0 & 1 \end{array}$$

Ex. 3.—*Convert* $\frac{532}{1193}$ *into a continued fraction.*

Ans. $\frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{8+}\,\frac{1}{16}$.

$$\begin{array}{r|l} 4\,)\;532 & 1193\,(\,2 \\ 16\,)\;\;16 & 129\,(\,8 \\ 0 & 1 \end{array}$$

EXAMPLES LVII.

Simplify—

1. $5+\frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{3}$.

2. $\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4+}\,\frac{1}{5+}\,\frac{1}{6}$.

3. $\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2}$.

4. $\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{10+}\,\frac{1}{17+}\,\frac{1}{26}$

5. $\frac{1}{2+}\ \frac{2}{3+}\ \frac{3}{4+}\ \frac{4}{5}$.

6. $\frac{1}{2+}\ \frac{2}{3+}\ \frac{3}{4+}\ \frac{5}{6}$.

7. $\frac{2}{2+}\ \frac{1}{3+}\ \frac{2}{5+}\ \frac{1}{2}$.

8. $\frac{3}{3+}\ \frac{1}{2+}\ \frac{2}{2+}\ \frac{1}{5}$.

Convert the following fractions into continued fractions :—

9. $\frac{109}{637}$. **10.** $\frac{114}{981}$. **11.** $\frac{1234}{2345}$. **12.** $\frac{671}{1225}$.

Convert the following fractions into continued fractions, and find the first four convergents in their simplest form :—

13. $\frac{591}{1888}$. **14.** $\frac{233}{522}$.

15.* Show that the convergents to a continued fraction are alternately less and greater than the fraction itself.

H.C.F. (G.C.M.) AND L.C.M. OF FRACTIONS.

121. A *measure of a fraction* is a fraction contained in it a whole number of times.

A *multiple of a fraction* is a fraction (or an integer) containing it a whole number of times.

The *greatest common measure* of several fractions is the greatest fraction contained in all of them a whole number of times.

The *least common multiple* of several fractions is the least fraction (or integer) containing all of them a whole number of times.

Thus, $\frac{1}{6}$ is a measure of $\frac{1}{2}$ and also of $\frac{1}{3}$, for it is contained in the first 3 times, in the second 2 times. It is also the G.C.M.

Again, 1 is a multiple of $\frac{1}{2}$ and also of $\frac{1}{3}$, for it contains the first 2 times, the second 3 times. It is also the L.C.M.

These definitions may be very well illustrated by the diagram used for Division.

122. *To find the G.C.M. of several fractions, e.g.* $\frac{2}{3}, \frac{6}{5}, \frac{2}{9}$.

Bring them all to their L.C.D. :

$$\frac{2\times3\times5,\ 6\times9,\ 2\times5}{5\times9}.$$

The fraction $\frac{1}{45}$ is clearly contained an integral number of times in all. It is a common measure. Is it the greatest common measure?

In order that $a/45$ should divide them all integrally, a must divide all the numerators. But nothing can divide *all* the numerators except what divided them before they were altered; for, in bringing the fractions to a common denominator, *we did not multiply all the numerators by the same factor;* therefore we did not introduce any common factor.

Hence the greatest value of a is the G.C.M. of 2, 6, 2, *i.e.* 2, and the G.C.M. of the fractions is $\frac{2}{45}$, which they contain respectively 15, 27, and 5 times.

A study of the reasoning of this example will make it clear that

The G.C.M. of any number of fractions, proper or improper, if in their lowest terms, is $\dfrac{\textit{G.C.M. of numerators}}{\textit{L.C.M. of denominators}}$.

123. *To find the L.C.M. of several fractions, e.g.* $\frac{2}{3}, \frac{6}{5}, \frac{2}{9}$.

Bring them all to their L.C.N. (least common numerator):

$$\frac{6}{3 \times 3,\ 5,\ 3 \times 9}.$$

The number 6 clearly contains them all an integral number of times. It is a common multiple. Is it the least common multiple?

In order that $6/a$ should contain them all integrally, a must divide all the denominators. But nothing can divide *all* the denominators except what divided them before they were altered; for, in bringing the fractions to a common numerator, *we did not multiply all the denominators by the same factor;* therefore we did not introduce any common factor.

Hence the greatest value of a (which gives the least value of $6/a$) is the G.C.M. of 3, 5, 9, the original denominators, *i.e.* is 1, and the L.C.M. of the fractions is 6, which contains them respectively 9, 5, and 27 times.

A study of the reasoning of this example will make it clear that

The L.C.M. of any number of fractions, proper or improper, if in their lowest terms, is $\dfrac{\textit{L.C.M. of numerators}}{\textit{G.C.M. of denominators}}$.

When fractions are not in their lowest terms they must be reduced before applying the rule.

EXAMPLES LVIII.

Find the G.C.M. and L.C.M. of the following sets of fractions:—

1. $\frac{1}{2}, \frac{2}{3}$. **2.** $\frac{1}{3}, \frac{3}{4}$. **3.** $\frac{2}{5}, \frac{3}{4}, \frac{9}{20}$. **4.** $\frac{3}{5}, \frac{6}{25}, \frac{9}{20}$.

5. $1\frac{1}{2}, 1\frac{1}{3}, 1\frac{1}{4}$. **6.** $2\frac{1}{2}, 1\frac{2}{3}, 1\frac{1}{4}$. **7.** $3\frac{1}{3}, 2, 1\frac{3}{7}$. **8.** $2\frac{2}{5}, 3, 1\frac{5}{7}$.

9. $\frac{9}{45}, \frac{15}{20}, \frac{16}{20}, \frac{15}{25}$. **10.** $\frac{9}{12}, \frac{12}{16}, \frac{16}{20}, \frac{20}{25}$.

11.* Prove that the product of the G.C.M. and the L.C.M. of two fractions is equal to the product of the fractions. Is this true for three fractions?

12.* Show that the least integer which contains two or more fractions an exact number of times is the least common multiple of their numerators.

(See also Exx. 30–33 in Exx. lix.)

Miscellaneous Theorems.

124. Two theorems are of great importance in the history of fractions:

I. *If both numerator and denominator of a fraction are increased by the same quantity, the fraction is brought nearer to unity.*

E.g. $\frac{2+2}{3+2}$, *i.e.* $\frac{4}{5}$, is nearer 1 than $\frac{2}{3}$ is; so also $\frac{5+2}{4+2}$, *i.e.* $\frac{7}{6}$, is nearer 1 than $\frac{5}{4}$. The first fraction, which was less than 1, has been increased, and so brought nearer to 1; the second fraction, which was greater than 1, has been diminished, and so brought nearer to 1.

Proof.—Suppose the fraction less than 1, and let N be its numerator and D its denominator: $1-\frac{N}{D}=\frac{D}{D}-\frac{N}{D}=\frac{D-N}{D}$. Now, if both N and D are increased by the same quantity, $D-N$ is left unaltered; but, since D is increased, $\frac{D-N}{D}$ is diminished; therefore $1-\frac{N}{D}$ is diminished; therefore the difference between $\frac{N}{D}$ and 1 is diminished.

The case when the fraction is greater than 1 is treated in the same way. We then begin with $\frac{N}{D}-1$, not $1-\frac{N}{D}$.

II. *If we have two unequal fractions, the fraction produced by adding the numerators for a new numerator and adding the denominators for a new denominator lies between the fractions in value.**

E.g. $\frac{2+3}{3+4}$, *i.e.* $\frac{5}{7}$, lies between $\frac{2}{3}$ and $\frac{3}{4}$, being greater than $\frac{2}{3}$, but less than $\frac{3}{4}$.

Proof.

Let the fractions be $\frac{2}{3}$ and $\frac{4}{5}$.
Then $\frac{4}{5}$ is the greater fraction.
$\therefore$ 4 is 5 times the greater fraction.

Let the fractions be a/b and c/d, and let a/b be the greater fraction.
$\therefore$ a is b times the greater fraction.

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* This process greatly interested the older arithmeticians, and was known by the name of **Mediation** between two fractions. Thus, for the value of π, the ratio of the circumference to the diameter of a circle, Archimedes had obtained the number $\frac{22}{7}$. A Dutch mathematician had arrived at $\frac{333}{106}$ (about 1580 A.D.), and it occurred to him to apply this process of mediation, arriving at the value $\frac{333+22}{106+7}$ *i.e.* $\frac{355}{113}$, which happens to be far more accurate than either of the other values.

The theorem can easily be extended to any number of fractions. See *Tutorial Algebra*, § 68, p. 49.

Also $\frac{2}{3}$ is less than the greater fraction.

$\therefore$ 2 is less than 3 times the greater fraction.

$\therefore$ $2+4$ is less than $(5+3)$ times the greater fraction.

$\therefore$ $\dfrac{2+4}{5+3}$ is less than the greater fraction.

Also c/d is less than the greater fraction.

$\therefore$ c is less than d times the greater fraction.

$\therefore$ $a+c$ is less than $b+d$ times the greater fraction.

$\therefore$ $\dfrac{a+c}{b+d}$ is less than the greater fraction.

In just the same way it can be proved to be greater than the smaller fraction.

Laws of Arithmetic.

125. In § 22 we stated, and *proved for whole numbers*, certain laws upon which the science of Arithmetic is based. We shall now prove these laws to hold for fractional numbers. They have occasionally been assumed in the previous sections.

Law I. The Commutative Law.—$A+B=B+A$, $A\times B=B\times A$.

Let $\dfrac{a}{b}$, $\dfrac{c}{d}$ be two fractions, a, b, c, and d being integers.

(1) $$\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{cb}{db} \quad [\S\ 106]$$

$$=\frac{ad}{bd}+\frac{bc}{bd} \quad [\text{for } db=bd \text{ and } cb=bc, \text{ since } b, c, \text{ and } d \text{ are integers}]$$

$$=\frac{ad+bc}{bd}=\frac{bc+ad}{bd}$$

$$[\text{for } ad+bc=bc+ad, \text{ since } ad, bc \text{ are integers}]$$

$$=\frac{bc}{bd}+\frac{ad}{bd}=\frac{cb}{db}+\frac{ad}{bd}=\frac{c}{d}+\frac{a}{b}.$$

(2) $$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}=\frac{ca}{db}=\frac{c}{d}\times\frac{a}{b}.$$

Hence the Commutative Law is true for fractions.

Law II. The Distributive Law.—$C\times(A+B)=C\times A+C\times B$.

This is obvious from the extended definition of Multiplication. To take $a+b$ groups each containing c is clearly the same as to take a groups each containing c and then to take b other groups each containing c.

Law III. The Factor Law.—$C \times (AB) = C \times A \times B$.

Let $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$ be three fractions, a, b, c, d, e, f being integers.

We have to prove $\frac{e}{f} \times \left(\frac{ac}{bd}\right) = \frac{e}{f} \times \frac{a}{b} \times \frac{c}{d}$.

$$\frac{e}{f} \times \left(\frac{ac}{bd}\right) = \frac{eac}{fbd} \text{ [§ 116]} = \frac{ea \times c}{fb \times d} \quad \text{[for } ea, c, fb, d \text{ are integers]}$$

$$= \frac{ea}{fb} \times \frac{c}{d} = \frac{e \times a}{f \times b} \times \frac{c}{d} = \frac{e}{f} \times \frac{a}{b} \times \frac{c}{d}.$$

Hence all the laws of Arithmetic are true for fractional, as well as for whole, numbers.

Problems in Fractions.*

126. The following examples will illustrate the methods of solution of problems involving fractions:—

Ex. 1.—*The seventh part of a certain number exceeds its eighth part by* 3. *What is the number?*

$\frac{1}{7} - \frac{1}{8} = \frac{8-7}{56} = \frac{1}{56}$; therefore $\frac{1}{7}$th exceeds $\frac{1}{8}$th by $\frac{1}{56}$th; therefore $\frac{1}{56}$th of the number is 3; therefore the number is $3 \times 56 = 168$.

Ex. 2.—*If I divide a sum of money among* 4 *boys, giving* $\frac{1}{3}$ *of it to A,* $\frac{1}{4}$ *of the remainder to B,* $\frac{3}{5}$ *of the remainder to C, and the* 5s. *left to D, find the whole sum.*

When A has got $\frac{1}{3}$ of sum, $\frac{2}{3}$ of sum is left; B gets $\frac{1}{4}$ of this, and therefore leaves $\frac{3}{4}$ of it, *i.e.* leaves $\frac{\cancel{3}}{\cancel{4}_{2}}$ of $\frac{\cancel{2}}{\cancel{3}}$ of sum, *i.e.* leaves $\frac{1}{2}$ of sum; C gets $\frac{3}{5}$ of this, and therefore leaves $\frac{2}{5}$ of it, *i.e.* leaves $\frac{\cancel{2}}{5}$ of $\frac{1}{\cancel{2}}$ of sum, *i.e.* leaves $\frac{1}{5}$ of sum. But what he leaves is 5s. Hence $\frac{1}{5}$ of sum is 5s.; therefore the sum is 5×5s., *i.e.* 25s.

Ex. 3.—*Supply the missing figures which are indicated by dots in the statement*

$$(.\tfrac{1}{3}) \times (4 \div .) = 10.$$

The first figure must be 2, because 4 + a fraction is necessarily contained 2 + times in 10. We now have $2\frac{1}{3} \times 4 \div . = 10$.

$\therefore$ $4 \div . = 10 \div 2\frac{1}{3} = 10 \times \frac{3}{7} = \frac{30}{7} = 4\frac{2}{7}$. Hence the complete statement is $2\frac{1}{3} \times 4\frac{2}{7} = 10$.

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* These will frequently require a knowledge of the Unitary Method (p. 269).

EXAMPLES LIX.—PROBLEMS IN FRACTIONS.

1. The eighth part of a certain number exceeds the ninth part by 5. What is the number?

2. The ninth part of a certain number exceeds the eleventh part by 10. What is the number?

3. If $\frac{7}{11}$ of an estate be worth £4655, what is the value of the remainder?

4. If $\frac{3}{8}$ of a ship be worth £3273, what is the ship worth?

5. A person had $\frac{7}{20}$ of a lottery ticket which drew a prize of £700. How much did he get?

6. If $\frac{2}{3}$ of an estate be worth £220, find the value of $\frac{3}{11}$ of it.

7. If in $4\frac{2}{7}$ hours a train travels $105\frac{4}{35}$ miles, what is its rate per hour?

8. A man has $\frac{3}{8}$ of an estate; he gives his son $\frac{1}{3}$ of his share. What portion of the estate has he then left?

9. A ship is worth £16,000, and a person possessed of $\frac{5}{16}$ of it sells $\frac{3}{8}$ of his share. What share has he remaining, and what is it worth?

10. A person owes £1 to each of 4 creditors: to one of them he pays $\frac{1}{2}$ of what he owes him, to another $\frac{3}{4}$, to another $\frac{4}{5}$, to another $\frac{19}{20}$. What will he still be owing altogether?

11. If a quarter of a number be added to a third of the number, the result is 20 greater than one-half of the number. What is the number?

12. If $\frac{1}{5}$ of a number be subtracted from $\frac{1}{3}$ of the number, the result is 20 greater than one-twentieth of the number. What is the number?

13. If I pay away one-half of my money, then $\frac{1}{3}$ of what remains, and then $\frac{1}{4}$ of what still remains, and have £10 left, how much had I at first?

14. If I pay away a third of my money, then a half of what remains, and then a quarter of what still remains, what fraction of the whole will be left?

15. A man left half of his property to his wife, one-third of the remainder to his eldest son, and the rest to be divided equally among five younger children, each of whom got £600. How much did the wife get?

16. If I divide a sum of money among 4 boys, giving one-half of it to A, two-thirds of the remainder to B, six-sevenths of the remainder and an extra shilling to C, and the 5s. left to D, how much does B get?

17. In an election $\frac{1}{20}$ of the constituency refused to vote, and of the two candidates the one who was supported by $\frac{11}{20}$ of the whole constituency was returned by a majority of 600. How many actually voted?

18. Out of a cask of wine, of which a sixth part had leaked away, 9 gallons were drawn, and then the cask was two-thirds full. How much did it hold?

19. An apple-woman sells to the first customer half her stock and half an apple, to the second half her remaining stock and half an apple, and so also to a third, to a fourth, and to a fifth customer. She finds that she has now 3 apples left. How many had she at first?

20. A shipwrecked crew lost one-half of their total number in reaching land, and before they were finally rescued one-third of the remainder had perished through famine, 2 were drowned, and 5 were carried off by wild beasts. They now found that only one in ten of the original crew had survived. How many were there?

21. A person dies worth £30,000, and leaves $\frac{1}{3}$ of his property to his wife, $\frac{1}{2}$ to his son, and the rest to his daughter. The wife, at her death, leaves $\frac{3}{5}$ of her legacy to the son and the rest to the daughter. But the son proposes to add his whole fortune to his sister's and to give her $\frac{1}{3}$ of the whole. How much would she gain by this?

22. A boy playing at marbles lost half of what he had in the first game, lost a third of what was left in the second game, but won in the next game one-twelfth as many as he had at first. He then gave 20 away, and found that he had left one-quarter of what he had at first; how many was that?

23. Find a fraction which is as much greater than $\frac{3}{11}$ as it is less than $\frac{11}{3}$.

24. Find six fractions in their lowest terms whose denominators lie between 8 and 16, and whose values lie between $\frac{6}{8}$ and $\frac{7}{8}$.

Supply the missing figures (each indicated by a dot) in the following statements (Examples 25-29), the fractions being in their lowest terms:—

25. $6\frac{3}{\cdot} \times \cdot\frac{2}{3} = 17.$

26. $\cdot\frac{4}{9} \times \cdot\frac{1}{4} = 10.$

27. $3\frac{3}{\cdot\cdot} \times \frac{\cdot}{9} = 2\frac{\cdot}{5}.$

28. $8\frac{4}{\cdot\cdot} \div \cdot\frac{1}{27} = 7\frac{16}{17}.$

29. $4\frac{1}{5\cdot} - \cdot\frac{1}{13} = 1\frac{\cdot\cdot}{5\cdot}.$

30. What is the least number into which $6\frac{11}{18}$, $6\frac{3}{8}$, and $6\frac{4}{5}$ will exactly divide?

31. What is the greatest fraction which will exactly divide $5\frac{19}{20}$, $5\frac{7}{18}$, and $5\frac{3}{13}$?

32.* One pendulum ticks 57 times in 58 seconds, another 608 times in 609 seconds. If they are started together, how often will they have ticked together in the first hour?

33.* One pendulum ticks 58 times in 57 seconds, another 609 times in 608 seconds. If they are started together, after how many *complete* hours will they be together again?

34. In a book on arithmetic an example was printed thus :—"Add together $\frac{1}{14\frac{2}{3}}$, $\frac{1}{19\frac{1}{4}}$, $\frac{1}{}$, $\frac{1}{13\frac{3}{4}}$," the denominator of one fraction being omitted. The answer given was $\frac{11}{28}$. What was the missing denominator?

35.* Two numbers have 101 for their H.C.F. and 27573 for their L.C.M. If the greater is $24\frac{69}{101} \times \dfrac{11\frac{2}{3}}{1+\dfrac{1}{1+\frac{1}{2}}}$ of $11\frac{2}{9}+2\left(1+\dfrac{351}{3+\dfrac{3}{3+\frac{1}{3}}}\right)$, what is the less?

36.* If we have three unequal fractions, the fraction produced by adding the numerators for a new numerator and adding the denominators for a new denominator lies between the greatest and least of the three fractions.

37.* If both numerator and denominator of a fraction are diminished by the same quantity, the difference between the fraction and unity is increased.

38.* Show that the groups of fractions $\frac{1}{3}+\frac{1}{4}$, $\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}$, $\frac{1}{9}+\frac{1}{10}+\ldots$ $\ldots+\frac{1}{16}$, ... containing respectively 2, 4, 8, 16, ... fractions are all of them greater than $\frac{1}{2}$. Deduce that by taking a sufficiently large number of the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, ... in order, we may obtain as large a total as we wish.

39.* (1) Prove that the sum of two fractions in their lowest terms can only be an integer when they have the same denominator.

(2) Prove that the sum of an integer and a proper fraction cannot be the same as the sum of another integer and another proper fraction.

40.* Prove that, if two integers be added, one to the numerator, the other to the denominator, of a fraction, the value of the fraction is unaltered only when these integers are equimultiples of the numerator and of the denominator.

41.* Prove that the sum of fractions in their lowest terms whose denominators are all prime to one another cannot be an integer.

42.* Prove that the sum of fractions in their lowest terms cannot be an integer if one of the denominators contains a prime factor which does not divide any of the other denominators.

43.* Prove that, if all proper fractions in their lowest terms whose denominator is less than a given number are arranged in order of size, any two fractions at equal distances from the ends of the chain will have the same denominator and their sum will be unity.

SECTION V.

DECIMAL FRACTIONS.

CHAPTER XVI.

TERMINATING DECIMALS.

127. Scientific men in England and the United States do not measure lengths in yards, feet, and inches; but for reasons which shall be more fully explained later (see Ch. xx.) use a standard a little longer than the yard, which is called a **metre**, and divide it into 10 parts, each of which is a **decimetre.** The decimetre, in its turn, is divided into 10 parts, each of which is called a **centimetre,** and the centimetre into 10 parts, each of which is called a **millimetre**. Their long measure table is therefore

10 *millimetres* = 1 *centimetre*,
10 *centimetres* = 1 *decimetre*,
10 *decimetres* = 1 *metre*.

Now, just as £7. 5s. 6d. may be safely written £7.5.6. without the *s* and *d*, so 7 metres 5 decimetres 6 centimetres may be written M7.5.6. But, in this case, we may omit the second dot, and write equally well M7.56. We could not omit the second dot always in dealing with £. s. d., because, if we did, we should not know whether to read £7.111 as £7. 11s. 1d. or as £7. 1s. 11d. This difficulty cannot arise with metres, &c., because we cannot have more than 9 of any of these parts, since 10 of any part make the next higher part.

We should now read M7.568 as 7 *metres* 5 *decimetres* 6 *centimetres* 8 *millimetres*. But there are 10 decimetres in a metre, and, therefore, each decimetre is a tenth part of a metre; there are 10×10 centimetres in a metre, and, therefore, each centimetre is a hundredth part of a metre; and there are $10 \times 10 \times 10 = 1000$ millimetres in a metre, and, therefore, each millimetre is a thousandth part of a metre. It is clear, then, that we may interpret

M7.568 as 7 metres, 5 tenths of a metre, 6 hundredths of a metre, 8 thousandths of a metre.

The · is generally written in the middle, not on the line; thus 7·568, *not* 7.568.

There is no reason for confining this notation to metres. We may agree generally that, whatever the unit, 7·568 shall mean 7 units 5 tenths 6 hundredths 8 thousandths; and that, if we use more figures, as in 7·5689, the nine shall mean 9 ten-thousandths, just as the 8 meant 8 thousandths. Such parts of a unit are called *fractions*, since they may be considered as obtained by *breaking* the unit up (Lat. *fractus*, broken); and since, according to this method of breaking, each unit is broken into 10 parts, each of these into 10 others, and so on until the parts are as small as we need, they are further described as **Decimal Fractions** (Lat. *decem*, ten) to distinguish them from *Vulgar Fractions*, in which the unit is broken up not into 10, but into some other number of parts.

Decimal Fractions are simply an extension of the meaning of the ordinary notation. In 111 each 1 has 10 times the value of the 1 next to it on the right. If we put a · after the units' figure, thus 111·, we may continue the number to the right—$\overset{H}{1}\overset{T}{1}\overset{U}{1}\cdot\overset{t}{1}\overset{h}{1}$, and each 1 will still be ten times the value of the one on the right, $\overset{H}{1}$ ten times the value of $\overset{T}{1}$, and $\overset{U}{1}$ ten times the value of $\overset{t}{1}$; so that 1 in the *t*-place indicates a tenth part (tenth) of a unit, 1 in the *h*-place a tenth part of 1 in the *t*-place—*i.e.*, a hundredth part of a unit (hundredth), and so on.

Ex.—*Write down* 13 *units* 3 *hundredths* 7 *millionths.*

Ans. 13·030007; for, in full, we mean 13 *units* 0 *tenths* 3 *hundredths* 0 *thousandths* 0 *ten-thousandths* 0 *hundred-thousandths* 7 *millionths.* Be careful not to write 13·37, which would mean 13 *units* 3 *tenths* 7 *hundredths*, just as £13.7 would mean £13. 7s., and not £13. 0s. 7d., which we should have to write £13.0.7 if we did not put in the letters *s* and *d*.

The dot · used in this method of writing numbers and parts of numbers is called the **Decimal Point,** and 7·3 is read "7 point 3."

It is sometimes read "7 decimal 3"; but this is not to be recommended, because it is not only longer, but is inapplicable to "Radix Fractions" (see § 276), for which a similar notation is used.

Caution.—Do not confuse 7·3—*i.e.*, "7 point 3" or 7 and 3-tenths—with 7 . 3—*i.e.*, "7 times 3," or 21.

Zeros placed to the right of a decimal do not affect its value, nor do zeros placed to the left of the decimal point, *if there are no other figures to the left.* Thus, ·070, ·0700, and 0·07 are all longer ways of saying the same thing as ·07.

Figures to the right of the decimal point are called **Decimal Figures,** and the number of such figures is spoken of as the number of **Decimal Places.** If no figures exist to the left of the decimal point, the first figure to the right of the point which is not zero is called the first **Significant Figure.**

E.g.—In ·003125 the first significant figure is 3.

Addition of Decimals.

128. Ex.—*Add together* 12·13, 2·157, *and* 7·934.

12·13
2·157
7·934
22·221

Explanation. — Arrange the decimals with their "points" in a vertical line; then the numbers to the left of the points will be arranged rightly for addition—units in one column (U), tens in the next (T), and so on, and the tenths will also be in one column (t), while the hundredths will be in the next column to the right (h), the thousandths in the next (th), &c. We see now that there are 11 thousandths; but 10 thousandths make a hundredth, so we put down 1 and "carry" 1, and it is clear that the work is exactly the same as in ordinary addition.

Note.—For *horizontal* addition it is well to supply 0's at the right, so as to bring all the decimals up to the same number of decimal places. Thus, to add 12·13, 2·157, 7·934 without rewriting it, it is well to write the first as 12·130. This is neither necessary nor desirable when they are rewritten vertically.

EXAMPLES LX.

The following examples may be read either from top to bottom, or from left to right. The answers should not be written on the printed page itself:—

1.	**2.**	**3.**	**4.**	**5.**	**6.**	**7.**		
3·8	6·29	8·77	1·125	137·3	16·21	18·69	=	**9.**
28·61	3·109	3·357	3·605	41·01	4·349	48·45	=	**10.**
7·819	8·067	·543	5·492	·59	35·08	4·008	=	**11.**
			1·51	1·153	1·84	5·242	=	**8.**
.....	.....	.....	.....	.....	.....	.....	=	**12.**

Add together the following groups of numbers:—

13. 54·067, ·0003, 10·01, 6·989, 7·4.

14. 25·343, 0·80012, 10·95, 9·09658.

15. 1002·97, 50·0674, 0·735, 9·20071.

16. 3·405, 5·0762, 301·63, 909·2, 10·8768.

17. 24·341, ·0342, 420·5, 18.
18. 7·1037, ·00982, 103·846, 12·04504
19. ·197, ·0096, 378·1, 19·64.
20. ·6513, 90·42, 6·35187, ·0028.
21. 2·364, ·0089732, 43·8097, 122·369872.
22. ·06621, 492, 53·8172, 19·68739.
23. 29·4601, 190·1, 0·009763, 132·9.
24. 2·364, ·0089732, 43·8097, 300·46537.
25. ·86621, 492, 53·8172, 435·007641.
26. 43·0637, ·98764, 347·53, 3·00645.
27. 5·05346, ·0076437, 142·830642, 43·768.
28. 17·0093, 186·90714, ·000867, 53·1789652.
29. 43·587, ·60935, 501·097, 1·8329.
30. 372·0647, 41·62835, ·964738, 876.

Subtraction of Decimals.

129. Ex.—*Subtract* 17·123 *from* 73·1.

	73·100
	17·123
Ans.	55·977

Explanation.—We have to determine what quantity has to be added to the less to make up the greater. Arrange the two quantities with "point" below "point." The work now proceeds as in ordinary subtraction. There being 0 thousandths in the minuend, we increase the minuend by 10 thousandths and the subtrahend by 1 hundredth. The mental work is :—

3 and **7**, 10; 1, 3 and **7**, 10; 1, 2 and **9**, 11; 1, 8 and **5**, 13; &c.

If the reason for this is not perfectly clear, read again the sections on Subtraction (§ 18 and § 65) and think of the process as "Compound Subtraction" in metres, decimetres, &c. Get into the habit of doing without the 0's in 73·100.

EXAMPLES LXI.

[Advanced pupils should do the whole of these mentally, without rewriting them.]

In each example subtract the smaller of the two given numbers from the greater.—

1. 17·589, 6·473. **2.** 19·859, 4·637. **3.** 20·06, 19·4325.
4. 100, 17·1825. **5.** 20, 16·34597. **6.** 5·02, 3 21037.
7. 10·035, 9·76825. **8.** 1210·375, 549·8973.
9. 132·0548, 41·75012. **10.** 59·0329, 45·82736.
11. 68·2106, 9·98435. **12.** 19·01, 9·98604.
13. 34·2054, 13·98076. **14.** 50·0472, 47·76043.

15. 75·243, 64·92279.
16. 75·4017, 30·85762.
17. 203·07604, 92·64382.
18. 173·6151, 28·72193
19. 100·75107, 89·843.
20. 96·2563, 85·34725.
21. 85·62307, 49·6703.
22. 328·0975, 38·09864.
23. 100·09, 0·987001.
24. 31·010101, 29·78787
25. 7·997431, 4·619135.
26. 100, 43·538096.
27. 320·071325, 265·5973.
28. 34·162, 13·60411.
29. 31·0625, 29·45835.
30. 127·30021, 115·490679.

31. Subtract the difference of 37·593165 and 28·7643 from the difference of 221·581 and 17·67842.

32. What is the value of 10·00901 − 9·099901 + 107·00617 − 12·0360819?

Multiplication of Decimals.

I. *By Integers.*

130. Ex.—*Multiply* 7·132 *by* 10.

Explanation.—If we had to multiply 7132 by 10, we have already seen (§ 24) that we change units into tens, tens into hundreds, &c., and that this result is gained by simply putting a 0 at the end of the number and writing 71320, for because of this 0 the 2 is now in the tens place, the 3 in the hundreds place, and so on, so that each figure now means ten times as much as it did before and the value of the whole number is increased ten times. Similarly, to multiply 7·132 by 10 we have only to shift the point one place to the right, for in 71·32 the 2 means 2-hundredths, and this is ten times what it meant before, and so on with the other figures.

Rule.—*To multiply a decimal by* 10, 100, 1000, ..., *shift the decimal point* 1, 2, 3, ... *places to the right.*

Ex. 1.—*Multiply* 7·132 *by* 4.

```
 7·132
     4
 ------
28·528
```

Explanation.—Write the units figure of the multiplier (4) below the units figure (7) of the multiplicand. The work now proceeds as in ordinary multiplication. The 2 in 7·132 is 2 thousandths. Multiplied by 4 this is 8 thousandths. Hence thousandths in the product come beneath thousandths in the multiplicand, hundredths beneath hundredths, and point is underneath point.

Ex. 2.—*Multiply* 7·132 *by* 40.

```
  7·132
     40
 -------
285·28
```

Explanation.—To multiply by 40 is to multiply by 4 and then by 10, *i.e.* to multiply by 4 and then to shift the decimal place. Or it comes to the same thing if we multiply by 4 and shift the figures, leaving the point underneath the point of the multiplicand, but writing the 8 of the product underneath the 3 of the multiplier instead of underneath the 2.

Ex. 3.—*Multiply* 6·125 *by* 48.

```
 6·125
 48
 ------
245·00
 49·000
 ------
294·000
```

Explanation.—Place units figure of multiplier below that of multiplicand. Multiply first by 4. But, since this 4 is really 40, the result of multiplying the 5 by 4 must be placed under the 2, not under the 5. Then multiply by the 8. The answer is read 294, and the 0's to the right of the point are omitted.

This question may be better done by factors, since 48 = 8 × 6. The working is shown on the right.

```
 6·125
 8
 ------
 49·000
 6
 ------
 294
```

Hints.—(1) When the multiplicand ends in 5, and the multiplier is an even number, time will usually be saved by the use of factors, still more when the multiplicand ends in 25 or 75 and the multiplier is divisible by 4, or when the multiplicand ends in 125, 375, 625, or 875 and the multiplier is divisible by 8. Thus

126·5 × 22 is worked 126·5 × 2 × 11.

16·25 × 452 is worked 16·25 × 4 × 113.

1·625 × 432 is worked 1·625 × 8 × 54.

(2) The multiplications may be tested by "Nines," the points being disregarded altogether. Note, however, that this method does *not* detect displacement of the partial products.

(3) *If no* 0's *have been dropped* at the right of the decimal during the work, the number of "decimal places" (see p. 176) in the product is the same as in the multiplicand.

EXAMPLES LXII.

[Exx. 1–4 should be worked mentally.]

Multiply—

1. ·15 by 10, 1000, and one million. **2.** 2·5 by 100, 10000, and 1000000.

3. ·05 by 2, 3, 4, 5, 6, 7, 8, and 90. **4.** 1·5 by 2, 3, 4, 5, 6, 7, 8, and 90.

5. ·1234 by 23, 31, 41, and 341. **6.** 1·312 by 23, 31, 41, and 341.

7. 19·687 by 213, 310, 47300, and 10000.

8. 16·987 by 231, 130, 47030 and 10000.

9. ·6205 by 84, and ·625 by 804. **10.** 6·705 by 48, and 67·5 by 408.

Multiply by factors—

11. ·125 by 84, 484, and 1728. **12.** ·375 by 48, 484, and 888.

II. *By Decimals.**

131. We have just seen in the last section that the following results are true:—3·1415 × 700 = 2199·05, 3·1415 × 70 = 219·905, 3·1415 × 7 = 21·9905; and we notice that, where the

* The logical difficulties involved in the multiplication of one fraction by another have been fully treated in the section on Vulgar Fractions, and are not again referred to.

multiplier is in each case one-tenth as large as the previous multiplier, the *figures* composing the answer are always the same, but the decimal point is moved one place to the left in each case, so that the answer is, as it ought to be, one-tenth of what it was before. We next ask what the result of multiplying by ·7 is. This multiplier is one-tenth as great as the multiplier in $3{\cdot}1415 \times 7$. Hence the answer is only one-tenth of what it was before, therefore it is 2·19905. If the multiplier had been ·07, the product would have been ·219905. For each change the 5 of the product is being pushed one place further away from the point. We thus reach the following:—

Rule.—*To multiply by a single figure in any decimal position, place the multiplier with its point underneath that of the multiplicand*—multiply as if there were no point at all; but *place the right-hand figure of the product as far to the right of the right-hand figure of the multiplicand as the single figure of the multiplier is to the right of its point.* The point of the product will then be below the points of the multiplier and multiplicand.*

Ex. 1.—*Multiply* ·01234 *by* ·007.

```
·01234...
·007
———
·00008638
```

Here the 7 is three places to the right of the point in ·007, therefore the 8, the right-hand figure of the product, is 3 places to the right of the 4. Dots may be placed to the right of the 4 to keep the place if necessary.

Ex. 2.—*Multiply* 1380 *by* ·005.

```
1380·
  ·005
————
 6·900
```

Here the first 0 of the product (which need not be written at all) is placed 3 places to the right of the 0 in 1380, from which it arises.

---

* Many other methods are given. Two may be named—

(1) Multiply as if there were no decimal points at all, and fix the position of the decimal point in the answer by the rule that there are as many decimal places in the answer as there are in the multiplier and multiplicand put together. This favourite method deprives the partial products of all significance, and degenerates easily into an ill-understood rule-of-thumb. *It is convenient in finding continued products of several decimals.*

(2) Arrange thus

```
 ·01234
   ·007
———————
·00008638
```

placing the point of the multiplier under the last figure of the multiplicand. The right-hand figure of the product is thus under the right-hand figure of the multiplier. This valuable method a little obscures the real meaning of the process and the place-value of the figures.

It is now easy to multiply decimals by decimals of any length. Thus, to multiply by 4·368, we have only to multiply successively by 4, ·3, ·06, and ·008, and add the products.

Ex. 3.—*Multiply* 13·45 *by* 4·368.

```
13·45
 4·368
------
53·80
 4·035
  ·8070
  ·10760
--------
58·7496ø
```

Here we multiply first by 4. The partial product is 53·80. The 0 of this goes vertically below the 5 from which it comes. We then multiply by ·3. The partial product is 4·035. Since the 3 is one place to the right of the decimal, the right-hand figure of its partial product is one place to the right of the five in 13·45, from which it comes. We then multiply by 6, placing the first (*i.e.* right-hand) figure one place more to the right, and so on. *All the points are in one vertical column.* When the point is fixed in any one line, the rest follow of themselves, the multiplication beginning always one place more to the right in each line. 0's are omitted from the right of the finished product.

Ex. 4.—*Multiply* 13·40512 *by* ·040030608.

```
13·40512..
  ·040030608
-----------
  ·5362048...
      4021536..
        8043072..
         10724096
-----------------
  ·53661510391296
```

We multiply first by the 4 of the multiplier. But this is two places to the right of its point, therefore we begin two places to the right of the multiplicand, keeping these places, if necessary, by dots. The 3 of the multiplier is three places to the right of the 4, therefore we begin its partial product three more places to the right, again using dots. The real value of this partial product is ·0004021536; but the 0's at the left can be safely missed out.

Hints. — (1) Always prove your result by "Nines" or "Elevens." In doing so disregard the point; but, in proving by elevens, be careful not to miss out 0's at the end of the product.

(2) If possible, use one partial product to help in finding another. Thus, in the above examples, since 8 is twice 4, and 6 is twice 3, the fourth and third partial products may be found by doubling the first and second.

(3) The hints given in § 44 as to multiplication by 25, 125, &c., are particularly valuable in the working of decimals.

(4) Be prepared to find the partial products in any order

Ex. 5.—*Multiply* 925·648 *by* 648·125.

```
  925·648
  648·125
--------
  7405·184
592414·72
  115·706
--------
599935·610
```

Here we note that 64 is 8 × 8, therefore we multiply first by the 8. This forms the first partial product. We multiply this by 8 to form the second line, and place the right-hand figure one place to the left. For the third line we divide 925·648 by 8. Why is this the same as to multiply it by ·125?

EXAMPLES LXIII.†

[Exx. 1–4 should be worked mentally; for Exx. 23, 24 see Ex. 2, p. 191; Exx. 25–26 are constructed to illustrate the Hints.]

Multiply—

1. ·2 by ·1, ·001 and 1·1. [3 ans.]

2. ·1 by ·03, ·0004, and 1·01.

3. 13·2 by 2, ·3, ·004, and ·0005.

4. 12·3 by ·2, 3, ·04, and 0·05.

5. 17·16 by 18·25, 12·85, and 15·28.

6. 16·17 by 81·25, 21·85, and 51·82.

7. ·00124 by ·0325.

8. ·00125 by 176.

9. 9·375 by ·0028.

10. 1·625 by ·016.

11. 7·0128 by 6·025.

12. 83·04 by ·0625.

13. 57·8234 by ·00045.

14. ·1264 by 1·00875

15. ·000162575 by ·348, 34·88, 3·808, and 808·08. [4 ans.]

16. ·00617525 by ·384, 384·8, ·3088, and 80·8008. [4 ans.]

Find the squares of—

17. 4·5825, 8·8881, 12·3693, and ·140712.

18. 11·5325, ·99498, 12·4899, and ·0150665.

Find the continued product of—

19. ·1, ·02, 2·3, 4·05, ·006.

20. ·2, ·02, ·002, ·0002, 2·00002.

21. 1·3, ·013, 13000, ·0013.

22. 3·1, 3·01, ·0031, ·00031.

Simplify—**23.*** $(12·1655)^2-(11·9582)^2$.

24.* $(13·1529)^2-(12·9614)^2$.

Multiply—**25.*** ·1348 by 7·25, 497·25, and 416·125. [3 ans.]

26.* 13·888 by 9·25, 819·125 and 9081·125. [3 ans.]

Division of Decimals.

I. *By Integers.*

132. Ex. 1.—*Divide* 7·132 *by* 10.

Explanation.—If we divide 71320 by 10, our result is 7132, for, by dropping the 0 of 71320, we have brought the 2 from the tens place into the units place, and so made it only one-tenth of its former value; similarly, the 3 has been brought from the hundreds place into the tens place, and so on, so that each figure in 7132 means only one-tenth

† The teacher may obtain as many examples as he requires in this rule, either by inserting points at random in the Exx. of Ch. IV., or by altering the position of the points in those given above. The position of the point in the answer is, of course, to be checked by the rule given in the footnote on p. 180.

of what it did in 71320, and 7132 is a tenth of 71320. Similarly, to divide 7·132 by 10 we shift the decimal point one place to the left and write ·7132, the effect of which is that units become tenths, tenths become hundredths, and so on, so that ·7132 is one-tenth of 7·132.

Rule.—*To divide a decimal by* 10, 100, 1000, ..., *shift the point* 1, 2, 3, ... *places to the left.*

Ex. 2.—*Divide* 7·132 *by* 4, *by* 40, *and by* 80.

```
        4 ) 7·132          40 ) 7·132          80 ) 7·132
Ans.        1·783                ·1783                ·08915
```

Explanation.—The division by 4 almost explains itself. After dividing 7 by 4 we have a quotient of 1 unit and 3 units over. These 3 units are 30 tenths. We had 1 tenth in 7·132; therefore we have 31 tenths. Dividing by 4, we have 7 tenths, and 3 tenths, *i.e.*, 30 hundredths over, and so on.

To divide by 40 we divide first by 4 and then shift the point one place to the left so as to divide by 10.

The division by 80 is similar, but when dividing by 8 we treat the 2 as if it were followed by ciphers; so that the 4 which is over on dividing the thousandths is regarded as 40 ten-thousandths, and so on.

Ex. 3.—*Divide* ·007132 *by* 64.

```
                                  ·0001114375                          1114375
8 ) ·007132     or     64 ) ·007132     or, contracted     64 ) ·007132
8 ) ·0008915                  64                                  73
    ·0001114375                73                                  92
                               64                                  280
                                92                                  240
                                64                                   480
                                280                                   320
                                256
                                 240
                                 192
                                  480
                                  448
Ans. ·0001114375.                  320
                                   320
```

The first process is the best when practicable, and when the factors are not numerous. In the second process 0's are brought down when required from the right of the decimal. With regard to the third process see § 39.

The quotient should *always* be written above the dividend, and then the position of the decimal point fixes itself.

EXAMPLES (Mental) LXIV.

Write the *first* significant figure (p. 176), inserting the decimal point and the necessary 0's in the following division sums:—

1. ·00789 by 2, 30, 311, 3981, and 10000.

2. ·09871 by 3, 20, 211, 2981, and 100000.

3. 7·089 by 20, 390, 3001, and 100000.

4. 9·00781 by 300, 509, 9009, and 10000.

Write the complete answers to the following division sums:—

5. ·576 by 2, 3, 4, 6, 8, and 9. **6.** ·1296 by 2, 3, 4, 6, 8, and 9.

EXAMPLES LXV.*

Divide—

1. 1·2345 by 2, 3, 4, 6, 8, 12, 16, and 24.

2. 2·1435 by 2, 3, 4, 6, 8, 12, 16, and 24.

3. 3·1416 by 2, 40, 800, and 16000. **4.** 4·3116 by 2, 400, 800, and 3200.

5. 14316 by 20, 400, 8000, and 32000.

6. 34116 by 40, 800, 1600, and 32000.

7. 5·76 by 24, 360, 1440, and 5760.

8. 12·96 by 24, 360, 1440, and 12960.

9. 3·087 by 21, 140, and 4900. **10.** 3·381 by 21, 230, and 1470.

11. ·0791979 and 791·979 by 293. **12.** ·204079 and 204·079 by 437.

13. ·002375937 by 879. **14.** ·0030944135 by 557.

Find the first five decimal places, and also the first five significant figures,† in the following division sums:—

15. ·0072 by 239. **16.** ·0073 by 293. **17.** 1 by 401. **18.** 1 by 701.

---

* The teacher may add indefinitely to the number of these by varying the position of the decimal point or by using those of Exx. xiv.-xvi. with decimal points inserted at random into the dividend.

† The distinction between these phrases must be carefully noted. Thus, in ·00135426, the first five decimal places are ·00135; the first five significant figures are 13542, which would generally be written ·0013542.

II. *By Decimals.*

133. In any division sum the divisor is contained in the dividend just as often as ten times the divisor is contained in ten times the dividend.

In general a is contained in b just as often as $10a$ is contained in $10b$, or as $100a$ is contained in $100b$. Thus 9 is contained 8 times in 72, and 90 is contained 8 times in 720.

Thus, if we have to divide ·17382 by ·002, we shall get the same result if we divide ·17382 × 10 by ·002 × 10, *i.e.* 1·7382 by ·02, and the same result again if we divide 1·7382 × 100 by ·02 × 100, *i.e.* 173·82 by 2; and this result is clearly 86·91, which is the answer. We thus reach the following rule:—

To divide one decimal by another move the point to the right in the divisor until it becomes a whole number, and move it exactly the same number of places to the right in the dividend. Then divide the new dividend by the whole number by the methods explained in § 132.

Ex. 1.—*Divide* ·002 *by* ·01.

We make the divisor ·01 into a whole number. To do this we move the point two places to the right. We must do the same with the dividend. It becomes ·2. The question now is "Divide ·2 by 1," and the answer clearly is ·2.

Ex. 2.—*Divide* ·02352 *by* ·0048.

We move the points four places to the right. We now have to divide 235·2 by 48. Four ways are available, of which the first is the worst.

```
        4·9  Ans.          4·9  Ans.
48 ) 235·2         48 ) 235·2        6 ) 235·2          4 ) 235·2
     192                  43·2        8 )  39·2         12 )  58·8
      43·2                  0              4·9  Ans.          4·9  Ans.
      43·2
```

Ex. 3.—*Divide* ·041307593 by 1·3678

```
            ·0302
13678 ) 413·07593
          2·7359
               33
```

In this example only the contracted working is shown. At the end of the work the figures 33 are over. What are we to do with them? Two courses are open. We may state them as a *Remainder*, or we may continue the division by supposing 0's to the right of the dividend. If we decide to state the remainder we must be careful. It is not 33 nor ·00033; for, since the decimal point has been shifted four places in the dividend, the *true value of the remainder can only be obtained when the decimal point has been shifted back again*, and it is, therefore, ·000000033.

```
            ·0302000241
13678 ) 413·07593
          2·7359
               33000
                56440
                 17280
                  3602
```

If we decide to continue the division, the work is as shown. But, after finding ten figures of the quotient, there is still a remainder (·00000000003602), and we shall see in the next chapter that, however far we continued the work in this example, we should always have a remainder. In such a case, if no directions are given as to when to stop the work, we should abandon it at, say, the tenth place, and write the answer ·0302000241 ... the dots indicating that it is unfinished. It is not usual to state the remainder.

Note. — We shall see later that the quotient may always be expressed as either a finite or a "circulating" decimal. Much time is often wasted in the endeavour to get the sum "out," *i.e.*, to express it completely in one or the other form. In the present question recurrence commences with the 2935th figure after the decimal point. The average student has no means of discovering this fact, and is advised to abandon the effort after finding ten places of the quotient, *unless the fraction found by placing the figures of the remainder at any stage over the figures of the divisor (neglecting the decimal point altogether) should reduce to a fraction whose denominator contains no other factors than* 2, 3, or 5. In our case the fraction is $\frac{3602}{13678}$, which reduces to $\frac{1801}{6839}$, and the factors of the denominator are 7 and 977. We therefore abandon the question.

Alternative Method. — The following method avoids any difficulty in finding the remainder. The principle underlying it will be obvious after a study of the examples.

To divide one decimal by another, underline in the dividend the figure which corresponds in position with the right-hand significant figure in the divisor. Proceed as if there were no decimal point. The figure of the quotient obtained after the underlined figure of the dividend has been reached is the units figure of the quotient, and the decimal point of the quotient immediately follows it.

Ex. 1.—

```
          4·9
·0048 ) ·0235 2
```

We underline the 5 as being the fourth decimal place, and therefore corresponding to the 8 of ·0048. Now 48 divides 4 times into 235. The 5 is now reached; ∴ the 4 is the units figure and is followed by the decimal point.

Ex. 2.—

```
          0·03020
1·3678 ) ·041307593
```

Here 13678 will not divide into 413; the units-figure of the quotient is therefore 0, and is followed by the decimal point. Again, 13678 will not divide into 4130; hence the first decimal place is 0; and so on.

EXAMPLES LXVI.

[Exx. 1-4 should be worked mentally. See also footnote to Exx. lxv.]

Divide—

1. 2·4 by ·2, ·02, ·002, and ·0004. **2.** 3·6 by ·3, 30, ·003, and ·0009.

3. 4·2 by ·07, ·0014, ·042, and ·00001.

4. 6·3 by 0·7, ·0009, ·0063, and ·0001.

5. 2·5 by ·0625, ·008, and ·0064. **6.** 6·4 by 0·256, ·005, and ·0625.

7. ·000324 by ·018, ·0000018, and 1800.

8. ·0175 by 35, and 35 by ·0175. **9.** 22·104 by ·0072.

10. ·0072013 by 713. **11.** 3·381 by ·00147. **12.** 2400 by ·0192.

13. 352·95624 by ·000504, 5·04, and 5040.

14. ·0030944135 by 5·57, ·00557, and 55700.

15. ·0405594 by 5·55. **16.** 24·9216 by 128. **17.** 31·482 by 79·5.

18. 1 by ·0625. **19.** 43·104 by 5·6125. **20.** 884·576 by 320.

21. 0·35209 by 25·7. **22.** 22·7088 by 3800. **23.** 2496·96 by ·0765.

24. 5·618375 by 321·05. **25.** 908·45294 by 69·4.

26. 1·59587772 by ·43062. **27.** ·6719245 by 402·35.

28. 8·31183 by 23·05. **29.** 1512·4781335 by ·4087.

30. ·209784253 by 36·151. **31.** ·0060472785 by 8·301.

32. ·00371 by 1·28. **33.** 1·13204182 by 24·0604.

34. ·304775546 by 59·678. **35.** 41·2666368 by ·05888.

36. 246·8754211 by ·07777. **37.** ·00094809768 by ·01567.

38. 174·13238 by 5·615. **39.** 9·62054 by 38·7925.

40. 0·109476 by 9·7312.

Divide correct to four places of decimals—

41. 64·375 by 9·573. **42.** 2·873 by 48·607. **43.** 1·11 by 8·0908.

44. 4·7362 by ·0435 **45.** ·4782 by ·00136. **46.** 5·63209 by 22·37.

Divide correct to six significant figures, and give the exact remainder—

47. 1 by ·401. **48.** 31·372 by ·0071.

49. Divide ·021 by the product of 1·5 and ·28.

50. Divide ·0013133 by the product of 2·3 and 57·1.

Conversion of Decimals into Vulgar Fractions.*

134. Ex.—*Convert* ·3125 *into a vulgar fraction.*

The meaning we assigned to ·3125 was 3 tenths, 1 hundredth, 2 thousandths, and 5 ten-thousandths;

$\therefore$ $\cdot 3125 = \frac{3}{10} + \frac{1}{100} + \frac{2}{1000} + \frac{5}{10000} = \frac{3125}{10000} = \frac{625}{2000} = \frac{125}{400} = \frac{25}{80} = \frac{5}{16}$.

Hence we reach the following rule:—

To convert a decimal into a vulgar fraction write the figures to the right of the decimal point as the numerator, and for denominator 1 *followed by as many* 0*'s as there are figures to the right of the point. Reduce the fraction thus given to its lowest terms.* Remember that, since the only factors of 10 are 2 and 5, the only factors which can possibly divide the denominator are powers of 2 and powers of 5, and therefore in the reduction it is never either necessary or advisable to find the H.C.F. of numerator and denominator.

Learn the following results:—

$$\cdot 2 = \tfrac{1}{5},\ \cdot 25 = \tfrac{1}{4},\ \cdot 5 = \tfrac{1}{2},\ \cdot 75 = \tfrac{3}{4}.$$

The advanced student should learn also

$$\cdot 125 = \tfrac{1}{8},\ \cdot 375 = \tfrac{3}{8},\ \cdot 625 = \tfrac{5}{8},\ \cdot 875 = \tfrac{7}{8}.$$

Hint.—A great deal of time is saved by the following device:—

$$7\cdot 3125 = 7\cdot 3\tfrac{1}{8} = 7\frac{3\frac{1}{8}}{10} = 7\tfrac{25}{80} = 7\tfrac{5}{16}.$$

The explanation of this is that, since $\cdot 125 = \frac{1}{8}$, *i.e.* is $\frac{1}{8}$ of a unit in the units place, the 125 in ·3125 is $\frac{1}{8}$ of a unit in the tenths place; therefore $\cdot 3125 = \cdot 3\frac{1}{8}$, &c. By means of this device the majority of decimals which will simplify much can, with a little practice, be worked mentally.

EXAMPLES LXVII.

[Exx. 1, 2 should be worked mentally. The advanced student, by means of the *Hint* above, will be able to do many of the rest mentally.]

Express as vulgar fractions in their lowest terms—

1. ·4, ·45, 3·5.	**2.** ·6, ·15, 4·25.	**3.** ·425, ·864.
4. ·525, ·872.	**5.** ·0064, 7·64.	**6.** ·00128, 1·28.
7. ·03125, ·06125.	**8.** ·03375, ·06875.	**9.** ·1328125.
10. ·984375.	**11.** ·0015625.	**12.** ·00009375.

* At this point the student who has read Decimals before Vulgar Fractions must turn back to the chapter on Vulgar Fractions.

13. ·750125. **14.** ·1484375. **15.** ·609375.
16. ·89375. **17.** ·004096. **18.** ·0003072.
19. 9·0096875. **20.** 2·0428125.

Conversion of Vulgar Fractions into Decimals.

(See also § 140.)

135. We have seen (§ 114) that $\frac{3}{4}$ may be interpreted as meaning the result of dividing 3 by 4. But we have also seen (§ 132) how to express the result of such a division as a decimal. Hence the following rule :—

To express a vulgar fraction as a decimal divide the numerator by the denominator.

The problem may be looked upon in a different way. To express a fraction as a decimal, we require to write it as a fraction whose denominator is *a power of* 10. This can be done at once, whenever it can be done at all, by splitting the denominator into factors, after which it can be easily seen by what to multiply both numerator and denominator to bring the latter into a power of 10.

Ex.—*Express* $\frac{31}{32}$ *as a decimal.*

First method.—

```
        ·96875                  ·96875
32 ) 31·0        or   32 ) 31·0        or   8 ) 31·0
     28·8                  2·20                4 ) 3·875
      2·20                  280                     ·96875
      1·92                   240
        280                   160
        256
         240
         224
          160
          160
```

This process may often be much shortened thus :—At the third place the answer reads ·968 and "24" over, *i.e.* $\cdot 968\frac{24}{32}$, *i.e.* $\cdot 968\frac{3}{4}$, *i.e.* ·96875, since the results for $\frac{1}{4}$, $\frac{3}{4}$, &c., have been learnt (§ 134). The factor method is generally the best.

Second method.—$32 = 2^5$. To make this into a power of 10 we have to multiply by 5^5. The work runs thus

$$\frac{31}{32} = \frac{31}{2^5} = \frac{31 \times 5^5}{2^5 \times 5^5} = \frac{31 \times 5^5}{10^5} = \frac{775 \times 5^3}{10^5} = \frac{19375 \times 5}{10^5} = \frac{96875}{10^5} = \cdot 96875.$$

The process looks longer than it really is. Its chief value, however, is that you are enabled to predict the number of decimal places in the answer. Directly the denominator is seen to be 10^5 it is clear that there are 5 decimal places.

EXAMPLES LXVIII.

[Exx. 1–4 should be done mentally.]

Express the following fractions as decimals :—

1. $\frac{1}{4}, \frac{3}{20}, \frac{4}{25}$. **2.** $\frac{3}{4}, \frac{1}{20}, \frac{7}{25}$. **3.** $\frac{29}{5}, \frac{3}{8}, \frac{11}{125}$.

4. $\frac{37}{5}, \frac{5}{8}, \frac{13}{125}$. **5.** $\frac{3}{800}, \frac{855}{9500}, \frac{1}{8000}$. **6.** $\frac{3}{160}, \frac{1771}{2875}, \frac{21}{8000}$.

7. $\frac{821}{128}$. **8.** $\frac{17}{320}$. **9.** $9\frac{567}{1296}$. **10.** $\frac{9912}{10752}$.

11. $\frac{15}{320000}$. **12.** $\frac{3471}{3125}$. **13.** $\frac{293}{12500000}$. **14.** $\frac{54}{6250000}$.

15. $7\frac{23897}{78125}$. **16.** $14\frac{47794}{78125}$. **17.** $\frac{1}{2}+\frac{3}{4}+\frac{1}{8}+\frac{1}{16}$. **18.** $\frac{3}{2}+\frac{1}{4}+\frac{3}{8}+\frac{1}{16}$.

19. $\frac{1}{5}+\frac{2}{25}+\frac{3}{125}+\frac{4}{625}$. **20.** $\frac{4}{5}+\frac{3}{25}+\frac{2}{125}+\frac{1}{625}$.

Express the following fractions as decimals correct to 6 places :—

21. $\frac{17}{24}$. **22.** $\frac{17}{21}$. **23.** $\frac{3}{7}$. **24.** $\frac{7}{12}$.

Without converting the fractions into decimals, state how many decimal places the following fractions would contain :—

25. $\frac{37}{128}$. **26.** $\frac{73}{625}$. **27.** $\frac{13}{12500}$. **28.** $\frac{19}{6400}$. **29.** $\frac{2651}{25600}$. **30.** $\frac{21}{5120}$.

Complex Decimals and Fractions.

136. The methods of simplification are similar to those adopted with Complex Fractions (see § 119). As a general rule, in such expressions, decimals connected by the sign + or − should be added or subtracted *as decimals* and not first reduced to vulgar fractions, while decimals connected by the sign × or ÷ should be dealt with as fractions. The answer should be presented as a fraction if most parts of the given complex expression are fractions, but as a decimal if most parts are decimals.

Ex. 1.—*Simplify* $\dfrac{\cdot 0016 \times \cdot 025}{\cdot 325 \text{ of } \cdot 05} \div \dfrac{\cdot 1216 \times \cdot 105 \times \cdot 002}{\cdot 08512 \times \cdot 625 \times \cdot 039}$.

$$\text{Fraction} = \frac{\cdot 0016 \times \cdot 025 \times \cdot 08512 \times \cdot 625 \times \cdot 039}{\cdot 3250 \times \cdot 050 \times \cdot 12160 \times \cdot 105 \times \cdot 002}$$

$$= \frac{\overset{8}{\cancel{16}} \times \cancel{25} \times \overset{7}{\cancel{8512}} \times \overset{125}{\cancel{625}} \times \overset{13}{\cancel{39}}}{3250 \times \underset{2}{\cancel{50}} \times \underset{10}{\cancel{12160}} \times \underset{\underset{8}{\cancel{21}}}{\cancel{105}} \times \cancel{2}} = \frac{8 \times \cancel{125} \times \cancel{13}}{\underset{\underset{2}{\cancel{50}}}{\cancel{250}} \times 20} = \frac{8}{40} = \frac{1}{5} = \cdot 2.$$

We commence by inverting the complex fraction which follows the sign ÷ and replacing "of" by ×. We also adjust the decimals by 0's at the right so that numerator and denominator have the same number of decimal places. But when one quantity is divided by another we may shift the decimal places in both divisor and dividend equally. Hence,

since $\frac{\cdot 0016}{\cdot 3250} = \frac{16}{3250}$, &c., we may drop the points altogether. The rest of the work proceeds as usual.

Ex. 2.—*Simplify* $\frac{(\cdot 9653)^3 - (\cdot 0347)^3}{(\cdot 9653)^2 - (\cdot 0347)^2}$.

Call the decimals a and b. The fraction to be simplified is

$$\frac{a^3-b^3}{a^2-b^2} = \frac{(a-b)(a^2+ab+b^2)}{(a-b)(a+b)} = \frac{a^2+ab+b^2}{a+b} = \frac{(a+b)^2-ab}{a+b}.$$

But $$a+b = \cdot 9653 + \cdot 0347 = 1;$$

therefore the fraction

$$= 1-(\cdot 9653)(\cdot 0347) = 1 - \cdot 03349591 = \cdot 96650409.$$

EXAMPLES LXIX.

Simplify—

1. $\frac{\cdot 00281 \times \cdot 0625}{1\cdot 405}$. **2.** $\frac{849\cdot 5 \times 19\cdot 04}{\cdot 008}$. **3.** $\frac{\cdot 4356}{\cdot 009} \div (1\cdot 1)^2$.

4. $\frac{630}{\cdot 35} \div \frac{700}{1\cdot 75}$. **5.** $\frac{\cdot 504}{\cdot 00042} \times \frac{\cdot 011977}{2\cdot 03} \div \frac{1\cdot 7}{\cdot 034}$.

6. $\frac{2\cdot 86}{\cdot 013} \times \frac{64\cdot 904}{0\cdot 2135} \div \frac{4\cdot 0975}{\cdot 2980}$. **7.** $\frac{1\cdot 405}{2\cdot 1} \times \frac{6\cdot 3}{28\cdot 1}$.

8. $\frac{\cdot 0038425 - \cdot 00183}{\cdot 035}$. **9.** $\frac{\cdot 23 - \cdot 023}{\cdot 0023 \div 23000}$.

10. $\frac{3\frac{4}{11} \times (1\frac{2}{9} \text{ of } 1\cdot 08)}{1\frac{11}{13} \times (0\cdot 6 + \frac{2}{9})}$. **11.** $(3\cdot 1416)^3 - (\cdot 1416)^3$.

12. $\frac{(2\cdot 31)^3 - (1\cdot 69)^3}{(2\cdot 31)^2 - (1\cdot 69)^2}$. **13.** $\frac{(\cdot 0347)^3 + (\cdot 9653)^3}{(\cdot 0347)^2 - (\cdot 347)(\cdot 09653) + (\cdot 9653)^2}$.

14. $\frac{3}{2 + \frac{2}{3 + \frac{2}{3 + \frac{2}{3}}}} \div \cdot 00003$ **15.** $\frac{6\cdot 12}{0\cdot 0102} - \frac{\cdot 0018}{34\cdot 2} + \frac{3\cdot 9}{1900}$.

16. $\frac{1\cdot 463}{7\cdot 315} + \frac{\frac{2}{3} \text{ of } 141\cdot 75 - \frac{4}{5} \text{ of } 38\cdot 125}{5 \times 18\cdot 9 + 25 + 1\cdot 22} \times 3\cdot 39525$.

17. $(\cdot 5 + \cdot 75) \times (2\cdot 5 - \cdot 4) \div \left(\cdot 125 + \frac{1}{4\cdot 8} \right)$.

18. $\frac{(\cdot 09 + 2\cdot 069) \div (\cdot 905 - \cdot 89103)}{2\cdot 37 \times (3\cdot 27 + 3\cdot 87) \div 16\cdot 59}$. (Ans in Fraction.)

19. $\left(0\cdot 25 \times 0\cdot 266 \times 116 - \frac{107\cdot 12}{669\cdot 5} - 0\cdot 01 \right) \times \frac{46\frac{1}{4} + 17\frac{1}{2}}{46\frac{1}{4} - 17\frac{1}{2}}$.

CHAPTER XVII.

RECURRING DECIMALS.

137. When we attempt by the method of § 135 to reduce such a fraction as $\frac{1}{3}$ to a decimal, we obtain the following result:—

$$\begin{array}{r} 3\,)\,\underline{1{\cdot}0000\ldots} \\ {\cdot}3333\ldots \end{array}$$

and it is clear that, however far we continue the division, we shall always have a remainder 1, and that the only decimal value we can assign to $\frac{1}{3}$ is ·3333 ..., the 3's being continued *for ever*. The result is written $\cdot\dot{3}$, where the dot *over* the 3 indicates that the 3 has to be continually repeated.

Such a decimal as $\cdot\dot{3}$, in which the figures are continually repeated, is called a **Recurring Decimal.**

It is also called a *Circulating Decimal*, *Repeating Decimal*, or *Periodic Decimal*. The group of figures which are repeated is called the **Period.** The decimals with which we have dealt in the previous chapter are called *Terminating Decimals* to distinguish them from Recurring Decimals.

If we similarly endeavour to convert $\frac{11}{14}$ into a decimal, the work runs as follows:—

$$\begin{array}{r} {\cdot}7857142 \\ 14\,)\,\overline{11{\cdot}0000000} \\ \underline{9{\cdot}8} \\ \mathbf{1{\cdot}2}0 \\ \underline{1{\cdot}12} \\ 80 \\ \underline{70} \\ 100 \\ \underline{98} \\ 20 \\ \underline{14} \\ 60 \\ \underline{56} \\ 40 \\ \underline{28} \\ \mathbf{12} \end{array} \quad or \quad \begin{array}{r} .7857142 \\ 14\,)\,\overline{11{\cdot}0000000} \\ \mathbf{1{\cdot}2}0 \\ 80 \\ 100 \\ 20 \\ 60 \\ 40 \\ \mathbf{12} \end{array} \quad or^{*} \quad \begin{array}{l} {\cdot}7_{12}\ 8_8\ 5_{10}\ 7_2\ 1_6\ 4_4\ 2_{12} \\ \phantom{{\cdot}7_{12}\ }8_8\ 5_{10}\ 7_2\ 1_6\ 4_4\ 2_{12} \end{array}$$

* This convenient notation—due, I believe, to Dr. Mackay—exhibits in smaller type after each figure the remainder which gives rise to the next figure, and shows clearly the periodicity of the remainders.

If we examine this work, we see that, since the remainder **12**, which was the first remainder, occurs again as the seventh remainder, and since the figures to be brought down in continuing the work are all ciphers, as they were before, all the figures after the first will keep occurring again and again, and the only result we can get for $\frac{11}{14}$ is $\cdot 785714285714285714 2\ldots$. We write this, $\cdot 785714\dot{2}$, the dots indicating that the figures from 8 to 2 are to be continually repeated.

A decimal such as this, in which some figures are not repeated, is called a **Mixed Circulator** to distinguish it from **Pure Circulators,** such as $\cdot\dot{3}$, in which *all* the figures after the decimal point are repeated.

An important question must next be discussed. Have we a right to say that $\frac{1}{3}$ *equals* $\cdot\dot{3}$ or $\frac{11}{14}$ *equals* $\cdot 785714\dot{2}$, and what meaning can be given to equating finite expressions such as $\frac{1}{3}$ and $\frac{11}{14}$ to expressions such as $\cdot 333\ldots$, or $\cdot 785714285714\ldots$, which *never end?* We proceed to examine this. If we convert the decimals into fractions there is no difficulty in proving

$$\tfrac{1}{3} - \cdot 3 = \tfrac{1}{30},\ \tfrac{1}{3} - \cdot 33 = \tfrac{1}{300},\ \tfrac{1}{3} - \cdot 333 = \tfrac{1}{3000},\ \tfrac{1}{3} - \cdot 3333 = \tfrac{1}{30000},$$

and so on.

Hence it is clear that as we take more and more figures of the decimal $\cdot 333\ldots$ the difference between it and $\frac{1}{3}$ is becoming smaller and smaller. It soon becomes very small indeed. Thus the difference between $\frac{1}{3}$ and $\cdot 3333333333$ is $\frac{1}{3000000000}$, a fraction which is so small that this fraction of the earth's circumference amounts only to *one-twentieth of an inch.* If this is the case, when we take ten 3's in $\cdot 333\ldots$, still smaller will be the difference when we take a hundred 3's. In fact, it is clear that $\cdot 333\ldots$ is, as we add 3 after 3 to it, getting closer and closer to $\frac{1}{3}$, and may be made as close to it as ever we please by taking 3's enough, so that, although $\cdot 333\ldots$ is never equal to $\frac{1}{3}$, it is indistinguishable from it. The technical term used for this kind of relation is the word **Limit,** or **Limiting Value,** and we say that the **Limiting Value** of $\cdot 333\ldots$ as the number of 3's increases more and more is $\frac{1}{3}$, and we mean that $\frac{1}{3}$ is a value (1) which $\cdot 333\ldots$ can never *pass beyond*, (2) to which it is continually getting closer, (3) to which it may get as close as ever we please. The fraction $\frac{1}{3}$ may be described as the **Generating Fraction** of $\cdot\dot{3}$.

EXAMPLES LXX.

[Of three methods given above for $\frac{11}{14}$ the beginner should adopt the first, the advanced student the second or the third. The subject is resumed in § 140, and two other valuable methods are given.]

Express the following fractions as recurring decimals:—

1. $\frac{2}{3}$. **2.** $\frac{5}{9}$. **3.** $\frac{4}{15}$. **4.** $\frac{14}{15}$. **5.** $3\frac{2}{11}$. **6.** $2\frac{3}{11}$.

7. $\frac{2}{7}$.	**8.** $\frac{3}{7}$.	**9.** $\frac{19}{35}$.	**10.** $\frac{5}{28}$.	**11.** $\frac{1}{792}$.	**12.** $\frac{7}{792}$.
13. $\frac{149}{495}$.	**14.** $\frac{139}{495}$.	**15.** $\frac{3}{14}$.	**16.** $\frac{11}{13}$.	**17.** $\frac{1}{44}$.	**18.** $\frac{56}{303}$.
19. $\frac{31}{825}$.	**20.** $\frac{31}{82}$.	**21.** $\frac{413}{672}$.	**22.** $\frac{547}{924}$.	**23.** $\frac{3}{52}$.	**24.** $\frac{452}{432}$.

Determination of the Generating Fraction.

138. Ex. 1.—*Determine the generating fraction of* $\cdot\dot{4}0\dot{7}$.

Let $F = \cdot 407407\ldots$. The fraction in question is the limit of the numbers $\cdot 407$, $\cdot 407407$, $\cdot 407407407$, ... Call these fractions a, b, c, ... Multiply by 1000. We get

$$1000a = 407, \qquad 1000b = 407\cdot 407, \qquad 1000c = 407\cdot 407407.$$

But $\quad a = \cdot 407, \qquad b = \cdot 407407, \qquad c = \cdot 407407407$;

$$\therefore\ 999a = 407 - \cdot 407,\ 999b = 407 - \cdot 000407,\ 999c = 407 - \cdot 000000407.$$

Thus, as we multiply the successive decimals by 999, we get quantities differing from 407 by $\cdot 407$, $\cdot 000407$, $\cdot 000000407$, ..., each of which is 1000 times smaller than the last, and that therefore $999F$ is getting nearer and nearer to 407, and may be made as near as ever we please to it by taking terms enough of the series a, b, c, ... Hence the *limit* of $999F$ is 407. We express this by saying $F = \frac{407}{999}$. This is the generating fraction required. This argument depends on the coincidence of position of the first group of 407 in b with the second group of 407 in $1000b$; $\therefore$ our multiplier must be 10 repeated as many times as there are figures in the recurring group. This reasoning (abbreviated in Ex. 2) is perfectly general and leads to the following rule:—

To convert a PURE *circulating decimal into a vulgar fraction write the period of the decimal as the numerator, and for denominator place as many 9's as there are figures in the period. Then reduce to lowest terms.*

Caution.—$\cdot\dot{0}2\dot{7} = \frac{27}{999}$, *not* $\frac{27}{99}$.

Ex. 2.—*Determine the generating fraction of* $\cdot 12\dot{1}4285\dot{7}$.

Let $F = \cdot 12142857142857\ldots$. Multiply by 100, as there are two non-recurring figures.

$100F = 12\cdot 142857142857\ldots$. There are six figures in the period. Multiply by 10 *six* times more.

$\therefore\ 1000000{:}00F = 12142857\cdot 142857\ldots$.

But $\quad 1{:}00F = 12\cdot 142857\ldots$. Subtract.

$\therefore\ 99999900F = 12142845\ldots$. Note here that there are *six* 9's.

$\therefore\ F = \frac{12142845}{99999900} = \frac{85}{700}$ (dividing both numerator and denominator by 142857).

$= \frac{17}{140}$. This is the fraction required.

From the reasoning of this example, which is perfectly general, we deduce the following rule :—

To convert a MIXED *circulating decimal into a vulgar fraction, subtract the non-recurring part* (12 *in the Example*) *from the whole* (12142857) *and place the difference as numerator; for denominator write as many nines as there are recurring figures, and then as many ciphers as there are non-recurring figures. Reduce the fraction thus formed to the lowest terms.*

Caution. –The beginner will usually find it best to find the H.C.F. of both numerator and denominator. The common divisors, when there are any, are often very high numbers.

EXAMPLES LXXI.

Determine, in their lowest terms, the generating fractions of the following decimals :—

1. $\cdot\dot{4}$, $\cdot\dot{0}\dot{5}$, $\cdot\dot{0}\dot{5}$. **2.** $\cdot\dot{5}$, $\cdot\dot{0}\dot{4}$, $\cdot\dot{0}\dot{4}$. **3.** $\cdot\dot{5}\dot{1}$, $\cdot\dot{5}\dot{0}\dot{1}$, $\cdot\dot{5}\dot{0}\dot{1}$.

4. $\cdot\dot{8}\dot{1}$, $\cdot\dot{8}\dot{0}\dot{1}$, $\cdot\dot{8}\dot{0}\dot{1}$. **5.** $\cdot\dot{0}\dot{3}\dot{7}$, $\cdot\dot{0}\dot{0}\dot{3}\dot{7}$, $\cdot\dot{0}\dot{0}\dot{3}\dot{7}$. **6.** $\cdot\dot{4}\dot{8}\dot{1}$, $\cdot\dot{0}\dot{4}\dot{8}\dot{1}$, $\cdot\dot{0}\dot{4}\dot{8}\dot{1}$.

7. $\cdot\dot{2}8571\dot{4}$, $\cdot\dot{5}7142\dot{8}$, $\cdot\dot{7}1428\dot{5}$. **8.** $\cdot\dot{0}7692\dot{3}$, $\cdot\dot{6}9230\dot{7}$, $\cdot\dot{2}3076\dot{9}$.

9. $\cdot 0069\dot{1}3\dot{5}$. **10.** $\cdot 0103\dot{7}0\dot{2}$. **11.** $\cdot 24\dot{2}8571\dot{4}$. **12.** $\cdot 16\dot{4}2857\dot{1}$.

13. $\cdot 16\dot{1}5384\dot{6}$. **14.** $\cdot 26\dot{1}5384\dot{6}$. **15.** $\cdot 116\dot{1}5384\dot{6}$. **16.** $126\dot{1}5384\dot{6}$.

17. $\cdot\dot{1}4185\dot{7}$. **18.** $\cdot\dot{1}4275\dot{7}$. **19.** $\cdot 1353\dot{0}7692\dot{3}$. **20.** $\cdot 2468\dot{0}7692\dot{3}$.

[See also next Examples.]

139. On a second reading you are advised to commit to memory the two results—

$$\tfrac{1}{7} = \cdot\dot{1}4285\dot{7},$$

$$\tfrac{1}{13} = \cdot\dot{0}7692\dot{3}.$$

You need only learn the first three figures of these results. The others can then be obtained by subtracting from 9's. Thus, $999 - 142 = 857$. (See § 146, vi.) To assist in remembering these, note that $1 + 4 + 2 = 7$, and that $0 + 7 + 6 = 13$.

From the first of these the following may be at once written down :—

$\frac{2}{7} = \cdot\dot{2}8571\dot{4}$, $\frac{3}{7} = \cdot\dot{4}2857\dot{1}$, $\frac{4}{7} = \cdot\dot{5}7142\dot{8}$, $\frac{5}{7} = \cdot\dot{7}1428\dot{5}$, $\frac{6}{7} = \cdot\dot{8}5714\dot{2}$.

Here it must be noticed that the same figures occur every time, and that, if we place them on the circumference of a circle, we can get them all by going round the circle in the same way, starting at different points.

Hence the figures can be written down at once if we can tell where to begin in the circle. But, if $\frac{1}{7} = \cdot 14\ldots$, $\frac{3}{7} = 3 \times \frac{1}{7} = \cdot 42\ldots$; hence for $\frac{3}{7}$ we begin with the 4, and so on in the other cases.

The reason for this "*Cyclic Property*" of decimals is as follows. Adopting the third method of § 137, the process of converting $\frac{1}{7}$ into a decimal is

$$\frac{1}{7} = \cdot \mathbf{1}_3\mathbf{4}_2\mathbf{2}_6\mathbf{8}_4\mathbf{5}_5\mathbf{7}_1,$$
$$\mathbf{1}_3\mathbf{4}_2\mathbf{2}_6\mathbf{8}_4\mathbf{5}_5\mathbf{7}_1,$$
$$\&c.$$

Thus, after the first step in the process, the remainder is 3, and the work starting with this 3 is the same as if we had been decimalizing $\frac{3}{7}$—*i.e.*, the period for $\frac{3}{7}$ is $\cdot\dot{4}2857\dot{1}$. The period for $\frac{2}{7}$ similarly is that which follows the $_2$ in the above work, and so on.

Now, from $\frac{1}{13} = \cdot\dot{0}7692\dot{3}$, it follows that $\frac{1}{13} = \frac{76923}{999999}$; therefore that $\frac{2}{13} = \frac{153846}{999999} = \cdot\dot{1}5384\dot{6}$. This need not be learned, since it can be found so easily. Applying the cyclic principle to these two results, we get

$$\tfrac{1}{13} = \cdot\dot{0}7692\dot{3}\,;\ \tfrac{3}{13} = \cdot\dot{2}3076\dot{9}\,;\ \tfrac{4}{13} = \cdot\dot{3}0769\dot{2}\,;\ \tfrac{9}{13} = \cdot\dot{6}9230\dot{7}\,;$$
$$\tfrac{10}{13} = \cdot\dot{7}6923\dot{0}\,;\ \tfrac{12}{13} = \cdot\dot{9}2307\dot{6}\,;$$
$$\tfrac{2}{13} = \cdot\dot{1}5384\dot{6}\,;\ \tfrac{5}{13} = \cdot\dot{3}8461\dot{5}\,;\ \tfrac{6}{13} = \cdot\dot{4}6153\dot{8}\,;\ \tfrac{7}{13} = \cdot\dot{5}3846\dot{1}\,;$$
$$\tfrac{8}{13} = \cdot\dot{6}1538\dot{4}\,;\ \tfrac{11}{13} = \cdot\dot{8}4615\dot{3}.$$

Thus a large number of results can be recalled at once from those for $\frac{1}{7}$ and $\frac{1}{13}$. Other results, which are not so much remembered as mentally reproduced when required, are :—

$\frac{1}{3} = \cdot\dot{3}$, $\quad \frac{2}{3} = \cdot\dot{6}$;
$\frac{1}{6} = \cdot 1\dot{6}$, $\quad \frac{5}{6} = \cdot 8\dot{3}$;
$\frac{1}{9} = \cdot\dot{1}$, $\quad \frac{2}{9} = \cdot\dot{2}$; and so on. Note that $\cdot\dot{9} = 1$, so that $\cdot 1\dot{9} = \cdot 2$.
$\frac{1}{11} = \cdot\dot{0}\dot{9}$, $\quad \frac{2}{11} = \cdot\dot{1}\dot{8}$, $\quad \frac{3}{11} = \cdot\dot{2}\dot{7}$; and so on.

These results may now be used to shorten immensely the labour of determining the generating fractions of recurring decimals.

Ex. 1.— $\cdot 641\dot{6} = \cdot 64\frac{1}{6} = \frac{64\frac{1}{6}}{100} = \frac{385}{600} = \frac{77}{120}$.

Ex. 2.— $\cdot 26\dot{4}2857\dot{1} = \cdot 26\frac{3}{7} = \frac{26\frac{3}{7}}{100} = \frac{185}{700} = \frac{37}{140}$.

Ex. 3.— $\cdot 25\dot{3}8461\dot{5} = \cdot 25\frac{5}{13} = \frac{25\frac{5}{13}}{100} = \frac{5\frac{1}{13}}{20} = \frac{66}{260} = \frac{33}{130}$.

*EXAMPLES LXXII.

[The student should work again by the abbreviated methods of the last section Exx. 11–16 and 19 and 20 of the last set. Exx. 1–8 of this set are intended to be worked *mentally*.]

Determine in their lowest terms the generating fractions of—

1. $\cdot 41\dot{6}$, $\cdot 9\dot{5}\dot{4}$, $\cdot 58\dot{3}$. **2.** $\cdot 51\dot{6}$, $\cdot 8\dot{6}\dot{3}$, $\cdot 48\dot{3}$. **3.** $\cdot 1\dot{1}4285\dot{7}$.

4. $\cdot 2\dot{4}285\dot{7}\dot{1}$. **5.** $\cdot 5\dot{0}7692\dot{3}$. **6.** $\cdot 7\dot{0}7692\dot{3}$. **7.** $\cdot 1\dot{1}5384\dot{6}$.

8. $\cdot 3\dot{3}8461\dot{5}$. **9.** $\cdot 230\dot{2}3076\dot{9}$. **10.** $\cdot 307\dot{3}0769\dot{2}$.

Simplify and express in the lowest terms—

11. $\dfrac{\cdot 17\dot{3}0769\dot{2}}{\cdot 13\dot{5}7142\dot{8}}$. **12.** $\dfrac{\cdot 27\dot{3}0769\dot{2}}{\cdot 23\dot{5}7142\dot{8}}$.

13. $\dfrac{13571428571415}{17307692307675}$. **14.** $\dfrac{23571428571405}{27307692307665}$.

From the value of $\frac{1}{17}$ given in § 140, write down the values of—

15. $\frac{3}{17}$. **16.** $\frac{11}{17}$.

Conversion of Fractions into Recurring Decimals.

140. The following additional methods are worthy of note.

Ex. 1.—*Convert $\frac{1}{17}$ into a decimal.*

```
      ·0588235
17 ) 1·00
      150
       140
         40
          60
           90
            5
```

Hence $\frac{1}{17} = \cdot 0588235\frac{5}{17}$.

The rest of the work proceeds as follows:—

$$\frac{1}{17} = \cdot 0588235 = \cdot\dot{0}5882352941176\dot{4}\dot{7}.$$

$$294117_1$$

$$647058_1$$

$$\ldots$$

Explanation.—Since $\frac{1}{17} = \cdot 0588235\frac{5}{17}$ (A),

$$\frac{10}{17} = \cdot 588235\frac{5}{17}; \quad \therefore \frac{5}{17} = \cdot 294117\frac{11}{17}.$$

Substitute this value of $\frac{5}{17}$ in (A);

$$\therefore \frac{1}{17} = \cdot 05882352941177\frac{11}{17}; \quad \therefore \frac{10}{17} = \cdot 58823529411711\frac{1}{17}.$$

Again we divide by 2. Part of the quotient (printed in italics) we have already found, and do not need to work again, $\frac{5}{17} = \cdot 294117\mathbf{647058}\frac{14}{17}$. Substituting this result in (A), we get $\frac{1}{17} = \cdot 0588235294117647\mathbf{058}\frac{14}{17}$, and it is clear that the figures are beginning to repeat. The process is obviously a continuous division by 2, *the quotient being placed as soon as found at the end of the dividend, and the division continued with the quotient as the new dividend* until repetition occurs.

This process may be applied whenever a remainder is reached which is a simple submultiple of a previous remainder. Another example is added without explanation.

Ex. 2.—*Convert $\frac{1}{47}$ into a decimal.*

$$\tfrac{1}{47} = \cdot 0_{10}2_{6}1_{13}2_{36}7_{31}6_{28}5_{45}9_{27}5_{35}7_{21}4_{22}4_{32}6_{38}8_{4}.$$

Since 4 is one-ninth of a previous remainder 36, the work is concluded by dividing by 9.

$$\begin{array}{l} \tfrac{1}{47} = \cdot\dot{0}2127659574468 \\ \qquad\qquad 0851063829_{7} \\ \qquad\qquad \textit{7872340425}_{4} \\ \qquad\qquad \textit{5319148936}_{1} \\ \qquad\qquad \textit{170}212 \end{array}$$

The advanced student will note that the figures in italics, which form the second half of the period, may (see § 146, Law vi.) be obtained from the figures of the first half by subtracting these figures from 9's.

Ex. 3.—*Convert $\frac{1}{17}$ into a decimal, finding the last figure of the recurring period first, and then the last but one, and so on.*

$$\begin{array}{r|l} \ldots 7647 & \\ \hline \ldots 999999999 & (\ 17 \\ 119 & \\ \hline 88 & \\ 68 & \\ \hline 992 & \\ 102 & \\ \hline 89 & \end{array}$$

Explanation.—To convert $\frac{1}{17}$ into a recurring decimal is to find a fraction equal to $\frac{1}{17}$ whose denominator is all 9's. The numerator will therefore be the result of dividing a series of 9's by 17. This may be done directly, adding 9's until the division is exact, or† by a kind of left-handed division as shown, the principle of which is that the last figure must be that figure which, when multiplied by 17, *ends in* 9, *i.e.* must be 7. We then subtract 7×17. The number now ends in 8. The next figure must be that which, when multiplied by 17, ends in 8, *i.e.* 4; and so on.

*EXAMPLES LXXIII.

Convert the following into recurring decimals:—

1. $\frac{1}{19}$. **2.** $\frac{3}{19}$. **3.** $\frac{1}{43}$. **4.** $\frac{7}{43}$. **5.** $\frac{1}{67}$. **6.** $\frac{9}{67}$. **7.** $\frac{1}{97}$. **8.** $\frac{11}{97}$.

Find the *last* five figures of the periods of—

9. $\frac{1}{317}$. **10.** $\frac{1}{337}$. **11.** $\frac{1}{401}$. **12.** $\frac{1}{701}$. **13.** $\frac{9}{397}$. **14.** $\frac{19}{401}$.

Addition and Subtraction.

141. The principles which underlie these rules are as follows:—

(1) *A recurring decimal may always be written with a period commencing at any desired point later than that at which it originally commenced.*

Thus $\frac{1}{7} = \cdot\dot{1}4285\dot{7} = \cdot 14\vdots285714\vdots285714\vdots2857\ldots = \cdot 14\dot{2}8571\dot{4}$.

Conversely a decimal may sometimes be written with the recurring period earlier, *e.g.* $\cdot 142\dot{8}5714\dot{2} = \cdot\dot{1}4285\dot{7}$.

† Whenever the denominator of the fraction is prime to 10, but not usually otherwise.

(2) *Thus two or more recurring decimals may be made to have their periods commencing at the same point.*

Ex.—*Write* $\cdot 13\dot{1}5\dot{6}$ and $\cdot 131\dot{5}6\dot{9}$ *with periods commencing together.* $\cdot 13\dot{1}5\dot{6} = \cdot 131\vdots561\vdots561\ldots = \cdot 131\dot{5}6\dot{1}$. The periods now commence in both cases at the fourth figure.

(3) *Two or more recurring decimals may always be written with periods of the same length.*

Ex.—*Write* $\cdot\dot{1}\dot{2}$ and $\cdot\dot{1}2\dot{3}$ *with periods of the same length.*

$$\cdot\dot{1}\dot{2} = \cdot 121212\vdots121212\vdots\ldots = \dot{1}2121\dot{2}.$$

$$\cdot\dot{1}2\dot{3} = \cdot 123123\vdots123123\vdots\ldots = \dot{1}2312\dot{3}.$$

A little consideration of this example will show—

(4) *When two or more recurring decimals are written with periods of the same length, the number of places in their periods is the L.C.M. of the numbers of places in the separate periods.*

142. The rule for addition and subtraction may now be stated.

Write the decimals to be dealt with so that their periods commence at the same place and are of the same length. The figures to be added or subtracted will then be found to recur regularly.

Ex. 1.—*Add* $3\cdot125$, $4\cdot123\dot{4}\dot{5}$, $5\ 67\dot{8}9\dot{0}$.

$$\begin{array}{l} 3\cdot125\vdots \\ 4\cdot123\vdots454545\vdots4 \\ 5\cdot678\vdots908908\vdots9 \\ \hline 12\cdot927\vdots363454\vdots \end{array}$$

Explanation.—We make all the decimals recur from the 4th place (for $4\cdot123\dot{4}\dot{5}$ cannot be made to recur earlier than this), and since there are 2 and 3 recurring places, we make them have a period of 6 places (= L.C.M. of 2 and 3)—*i.e.* we write them $3\cdot125$, $4\cdot123\dot{4}5454\dot{5}$, $5\cdot678\dot{9}0890\dot{8}$. We now find that the block of figures between the dotted lines continually repeats itself, and that the *figure* 1 *has to be carried from each block to the next on the left* (since 9 and 4 make **1**3). Hence the addition is as shown, and the sum is $12\cdot927\dot{3}6345\dot{4}$.

Ex. 2.—*Subtract* $4\cdot67\dot{8}9\dot{0}$ *from* $6\cdot123\dot{4}\dot{5}$.

$$\begin{array}{l} 6\cdot123\vdots454545\vdots4 \\ 4\cdot678\vdots908908\vdots9 \\ \hline 1\cdot444\vdots545636\vdots \end{array}$$

Explanation.—As before, we write the decimals $6\cdot123\dot{4}5454\dot{5}$ and $4\cdot678\dot{9}0890\dot{8}$. The subtraction proceeds as shown. The figure 1 has to be "carried" from each block to the next on the left. The work commences 9 *and* 5 (not written), *14*, carry ***1***. *1* (and 8), 9 *and* **6**, *15*. Write the **6** and carry the *1*; and so on.

EXAMPLES LXXIV.

Write the following decimals with periods commencing at the third figure :—

1. $\cdot\dot{3}$, $\cdot\dot{3}\dot{5}$, $\cdot 00\dot{7}$, $\cdot\dot{2}09\dot{7}$, $5\cdot 9\dot{0}5\dot{9}$, $\cdot 8\dot{7}418\dot{1}$, $\cdot\dot{2}46037$, $\cdot 125$, $\cdot 123\dot{4}56\dot{3}$.

2. $\cdot 0\dot{3}$, $\cdot 0\dot{3}\dot{5}$, $\cdot 12\dot{8}$, $\cdot\dot{3}44\dot{2}$, $8\cdot 8\dot{0}7\dot{8}$, $\cdot 4\dot{2}515\dot{2}$, $\cdot\dot{3}3079\dot{4}$, $\cdot 369$, $\cdot 678\dot{9}123\dot{8}$.

Write the following groups of decimals with periods commencing at the same point in each group :—

3. $\cdot\dot{3}$, $5\cdot 9\dot{0}5\dot{9}$; $\cdot 0\dot{3}\dot{5}$, $\cdot\dot{3}\dot{5}$, $\cdot 00\dot{7}$; $\cdot 125$, $\cdot 00\dot{7}$, $\cdot\dot{2}09\dot{7}$.

4. $\cdot 0\dot{3}$, $\cdot\dot{3}44\dot{2}$; $\cdot 12\dot{8}$, $\cdot 8\dot{0}7\dot{8}$, $\cdot\dot{3}44\dot{2}$; $\cdot 369$, $\cdot 12\dot{8}$, $\cdot\dot{8}07\dot{8}$.

Write the following groups of decimals with periods of the same length in each group :—

5. $\cdot\dot{3}$, $\cdot 0\dot{3}\dot{5}$; $\cdot\dot{7}$, $\cdot\dot{0}\dot{7}$, $\cdot\dot{0}0\dot{7}$; $\cdot\dot{2}0\dot{9}$, $\cdot 0\dot{2}09\dot{7}$, $\cdot\dot{1}4285\dot{7}$.

6. $\cdot 0\dot{3}$, $\cdot\dot{3}\dot{5}$; $\cdot 0\dot{6}$, $\cdot\dot{0}\dot{6}$, $\cdot\dot{0}0\dot{6}$; $\cdot 0\dot{3}0\dot{5}$, $\cdot\dot{8}07\dot{8}$, $\cdot\dot{0}7692\dot{3}$.

Write the following groups of decimals with periods of the same length and commencing at the same place in each group :—

7. $\cdot\dot{3}$, $\cdot 0\dot{3}\dot{5}$; $\cdot 1\dot{7}\dot{2}$, $\cdot\dot{1}234\dot{5}$.

8. $\cdot 0\dot{7}$, $\cdot\dot{4}\dot{6}$; $\cdot 01\dot{2}\dot{3}$, $\cdot 0\dot{4}567\dot{8}$.

Add the following decimals :—

9. $\cdot\dot{1}$, $\cdot\dot{2}$, $\cdot\dot{3}$.

10. $\cdot\dot{1}$, $\cdot\dot{3}$, $\cdot\dot{4}$.

11. $\cdot\dot{4}$, $\cdot\dot{5}$, $\cdot\dot{6}$.

12. $\cdot\dot{7}$, $\cdot\dot{8}$, $\cdot\dot{2}$.

13. $\cdot\dot{1}\dot{2}$, $\cdot\dot{4}\dot{5}$, $\cdot\dot{7}\dot{8}$.

14. $\cdot\dot{1}\dot{3}$, $\cdot\dot{4}\dot{6}$, $\cdot\dot{7}\dot{5}$.

15. $\cdot 1\dot{2}$, $\cdot\dot{4}\dot{5}$, $\cdot\dot{7}\dot{8}$.

16. $\cdot 1\dot{3}$, $\cdot\dot{4}\dot{6}$, $\cdot\dot{7}\dot{5}$.

17. $3\cdot\dot{4}\dot{2}$, $\cdot\dot{3}\dot{6}$, $3\cdot\dot{5}$.

18. $4\cdot 3\dot{2}$, $\cdot\dot{6}\dot{3}$, $5\cdot\dot{3}$.

19. $\cdot\dot{1}$, $\cdot 0\dot{1}$, $\cdot\dot{0}\dot{1}$, $\cdot\dot{0}0\dot{1}$.

20. $\cdot 0\dot{0}\dot{1}$, $\cdot\dot{1}\dot{1}$, $\cdot 0\dot{0}0\dot{1}$.

21. $\cdot 0\dot{0}09\dot{9}$, $\cdot 1\dot{0}00\dot{2}$, $\cdot\dot{0}\dot{1}$.

22. $1\cdot 0\dot{1}\dot{2}$, $2\cdot\dot{0}1\dot{2}$, $3\cdot 0\dot{1}2\dot{3}$.

23. $1\cdot\dot{8}\dot{1}$, $\cdot 0\dot{3}2\dot{1}$, $3\cdot 20\dot{7}4638\dot{2}$.

24. $1\cdot\dot{8}$, $1\cdot\dot{9}1600\dot{6}$, $2\cdot 8\dot{5}\dot{6}$, $3\cdot\dot{4}2\dot{7}$.

25. $4\cdot 476\dot{8}$, $5\cdot\dot{0}3\dot{2}$, $17\cdot 006\dot{1}$, $\cdot 732$.

26. $\cdot\dot{1}50\dot{9}$, $\cdot 87\dot{5}\dot{6}$, $3\cdot 4\dot{2}\dot{5}$, $44\cdot\dot{2}9\dot{4}$.

Subtract—

27. $\cdot 0\dot{2}$ from $\cdot\dot{1}$.

28. $\cdot\dot{0}\dot{3}$ from $\cdot\dot{1}$.

29. $\cdot 3\dot{7}\dot{8}$ from $\cdot\dot{3}9\dot{9}$.

30. $\cdot\dot{5}8\dot{7}$ from $\cdot 6\dot{9}\dot{3}$.

31. $2\cdot 8\dot{5}\dot{6}$ from $3\cdot\dot{4}2\dot{7}$.

32. $3\cdot 4\dot{2}\dot{5}$ from $44\cdot\dot{2}9\dot{4}$.

33. $\cdot 1\dot{2}3\dot{4}$ from $\cdot 12\dot{3}\dot{4}$.

34. $\cdot 5\dot{6}\dot{3}\dot{2}$ from $\cdot 5\dot{6}\dot{3}\dot{2}$.

35. $1\cdot\dot{2}345\dot{6}$ from $2\cdot 34\dot{5}\dot{6}$.

36. $\cdot\dot{0}7692\dot{3}$ from $\cdot\dot{1}4285\dot{7}$.

Simplify and express as a decimal—

37. $8\cdot 4\dot{7}\dot{4} + 3\frac{5}{8} + 3\frac{2}{21}$.

38. $17\cdot\dot{3}\dot{6} + 4\frac{5}{9} + 1\cdot 03\dot{7}3\dot{2}$.

39. $3\cdot 4\dot{1}\dot{0} + 1\cdot\dot{0}\dot{0}19\dot{1} - \cdot 24\dot{9}7\dot{9}$.

40. $\cdot 3\dot{2}1\dot{3} + \cdot\dot{0}\dot{2} - \cdot\dot{0}0785\dot{4}$.

Multiplication.

143. Multiplication of recurring decimals by small integers should be performed directly as in Ex. 1. Multiplication by numbers of more than one figure *may* be performed directly (as in Ex. 2); but the beginner is advised to convert the decimal into a vulgar fraction before performing the mulitiplication, and subsequently, if required, to reconvert the product into a decimal. When a recurring decimal has to be multiplied by a fraction or other decimal, the decimals should all be converted into fractions before performing the multiplication. If *only the first few figures of the answer are required*, the methods of the next chapter (xviii.) are to be used.

Ex. 1. *Multiply* $7{\cdot}14\dot{1}4285\dot{7}$ *by* 9.

7·14	142857	14
9		
64·27	285714	

On performing the multiplication, we see that 1 is to be "carried" from each recurring block to the next. The answer is $64{\cdot}27\dot{2}8571\dot{4}$.

Ex. 2.—*Multiply* $7{\cdot}14\dot{1}4285\dot{7}$ *by* 139.

First Method.—

$$\text{Product} = 7{\cdot}14\tfrac{1}{7} \times 139 = 7\frac{99}{700} \times 139 = 973\,\frac{13761}{700} = 973 + 19\frac{461}{700}$$

$$= 992\,\frac{461}{700} = 992\,\frac{\frac{461}{7}}{100} = 992\,\frac{65\frac{6}{7}}{100} = 992{\cdot}65\tfrac{6}{7} = 992{\cdot}65\dot{8}5714\dot{2}.$$

Second Method.—The product $7{\cdot}14\frac{1}{7} \times 139 = 7{\cdot}14 \times 139 + {\cdot}00\frac{1}{7} \times 139$
$= 992{\cdot}46 + {\cdot}19\frac{6}{7} = 992{\cdot}65\frac{6}{7} = \ldots.$

Third Method.—

A	B	C
7·14	142857	142
139		
714·14	285714	2
214·24	285714	26
64·27	285714	278
992·65	857142	

Explanation.—Column A contains the non-recurring figures of the multiplicand and of the partial products. Column B contains the recurring block. Column C serves only to determine what figure to carry from one recurring block to the next. It is *not exact*, for the multiplicand *does not end at the* 2. The original decimal is prolonged 3 figures after the recurring period when the multiplier has 3 figures, 4 when it has 4 figures, &c.† Why are all the partial products in the B column the same?

Ex. 3.—*Multiply* $1{\cdot}\dot{1}4285\dot{7}$ *by* ·375.

Product $= 1\frac{1}{7} \times \frac{3}{8} = \frac{8}{7} \times \frac{3}{8} = \frac{3}{7} = {\cdot}\dot{4}2857\dot{1}$. *Ans.*

† A crude rule, which in rare cases might result in error, but which can be checked by noting that the figure to be carried from C to B in the answer line should be the same as that carried from B into A.

EXAMPLES LXXV.

Multiply—

1. $1\cdot\dot{3}$ by 3, 5, 7, 19, and 191.

2. $2\cdot\dot{4}$ by 5, 9, 11, 29, and 201.

3. $1\cdot\dot{1}\dot{5}$ by 4, 12, 24, and 240.

4. $3\cdot\dot{4}\dot{5}$ by 8, 16, 32, and 320.

5. $\cdot\dot{1}\dot{5}5873\dot{0}$ by 3, 7, and 21.

6. $\cdot\dot{2}\dot{6}6984\dot{1}$ by 4, 9, and 36.

7. $\cdot004\dot{1}8\dot{8}$ by 1524.

8. $\cdot00\dot{3}5\dot{2}$ by 1668.

9. $7\cdot54\dot{8}\dot{9}$ by 5480.

10. $\cdot030\dot{8}\dot{5}$ by 6240.

11. $3\cdot4\dot{9}$ by $\cdot46\dot{9}$.

12. $\cdot394\dot{9}$ by $\frac{54}{395}$.

13. $\cdot\dot{9}2307\dot{6}$ by $1\cdot04$.

14. $\cdot\dot{0}76923$ by $1\cdot69$.

15. $\cdot7\dot{7}\dot{2}$ by $1\cdot3\dot{4}$.

16. $\cdot2\dot{9}\dot{8}$ by $2\cdot\dot{0}8\dot{1}$.

17. $1\cdot31\dot{9}$ by $\cdot3\dot{8}\dot{1}$.

18. $\cdot3\dot{5}\dot{7}$ by $2\cdot145$.

19. $8\cdot418\dot{3}$ by $\cdot10\dot{0}0\dot{9}$.

20. $4\cdot\dot{3}15\dot{3}$ by $102\cdot9896$.

21. $\cdot2\dot{1}212\dot{1}$ by $1\cdot0499895$.

22. $\cdot39\dot{2}857\dot{1}\dot{4}$ by $\cdot15\dot{9}\dot{0}$.

23. $4\cdot60\dot{8}\dot{1}$ by $\cdot3\dot{5}6\dot{7}$.

24. $43\cdot\dot{3}7\dot{2}$ by $\cdot0\dot{1}33\dot{2}$.

25. What is 37 times the sum of $\cdot41\dot{6}$, $\cdot2\dot{2}3\dot{4}$, and $\cdot00\dot{4}\dot{5}$?

26. Find the continued product of $1\cdot\dot{0}5\dot{4}$, $1\cdot\dot{1}7460\dot{3}$, $1\cdot\dot{6}1538\dot{4}$.

Division.

144. Division of recurring decimals by integers may be performed directly (see Exx. 1, 2 below). When a recurring decimal has to be divided by another decimal, or *vice versa*, both decimals should be reduced to vulgar fractions, or the methods of Chap. xviii. may be employed.

Ex. 1.—*Divide* $\cdot\dot{2}9307\dot{6}$ *by* 7.

```
7 ) ·293076 |29 ...
      ·041868₀|04 ...
```

Ans. $\cdot\dot{0}4186\dot{8}$.

Ex. 2.—*Divide* $\cdot\dot{9}3076\dot{2}$ *by* 112.

```
7 ) ·9̇30762̇
4 ) ·13296613296 ...
4 ) ·03324153324 ...
      ·00831038331 ...
```

or

```
                ·00831038331
112 ) ·93076293076 ...
      347
       116
          429
           933
            370
             347
```

Ans. $\cdot008\dot{3}1038\dot{3}$.

The process of long division shows a little more clearly than the factor method when recurrence is taking place, since it indicates the remainders at each stage.

EXAMPLES LXXVI.

(Answers to be expressed in Decimals.)

Divide—

1. $1\cdot\dot{9}\dot{0}$ and $2\cdot\dot{4}\dot{5}$ (*a*) by 3, (*b*) by 15, (*c*) by 27, (*d*) by 135.

2. $98\cdot\dot{9}5\dot{9}\dot{0}$ by 123 and 111. **3.** $188\cdot\dot{9}2\dot{2}$ by 61 and 1001.

4. $6\cdot382\dot{7}\dot{9}\dot{8}$ by $1\cdot524$. **5.** $5\cdot8\dot{7}\dot{7}2\dot{3}$ by $1\cdot668$. **6.** $4\cdot1368\dot{4}\dot{6}$ by $\cdot548$.

7. $\cdot0192\dot{5}\dot{5}\dot{7}$ by $\cdot624$ **8.** $1\cdot7$ by $5\cdot\dot{6}$. **9.** $36\cdot6$ by $\cdot\dot{5}\dot{4}$.

10. $27\cdot6$ by $\cdot0\dot{4}\dot{6}$. **11.** $1\cdot86$ by $1\cdot2\dot{3}$. **12.** $1\cdot7\dot{5}\dot{3}$ by $7\cdot\dot{5}\dot{1}$.

13. $\cdot3\dot{8}\dot{8}$ by $12\cdot\dot{4}\dot{4}$. **14.** $3\cdot84\dot{9}$ by $\cdot205\dot{3}$. **15.** $\cdot23\dot{1}\dot{7}$ by $1\cdot1\dot{5}\dot{8}$.

16. $3\cdot4\dot{3}$ by $5\frac{7}{81}$. **17.** $1\frac{1}{1089}$ by $\cdot2\dot{0}\dot{2}$. **18.** $2\cdot39\dot{4}$ by $30\frac{250}{279}$.

19. $\cdot5\dot{6}0\dot{9}$ by $5\frac{17}{90}$. **20.** $4\cdot997$ by $\cdot39\dot{8}\dot{4}$. **21.** $26\cdot3\dot{3}\dot{8}$ by $1\cdot\dot{8}62\dot{6}$.

22. $23\cdot6\dot{4}\dot{8}$ by $\cdot9\dot{4}5\dot{9}$. **23.** $\cdot323\dot{6}4\dot{6}$ by $\cdot67\dot{2}5\dot{1}$.

24. $37\cdot4\dot{3}\dot{6}$ by $\cdot13\dot{9}$.

25. By what decimal must $2\cdot3902\dot{6}$ be divided to give $\cdot0\dot{7}0\dot{2}$ as a quotient?

26. Find the first 7 decimal places in the division of $1\cdot\dot{3}45678$ by 13, and write down the exact remainder.

Complex Recurring Decimals.

145. The decimals occurring in complex expressions should almost always be reduced to fractions. Additions or subtractions of decimals should be performed in decimals. The answer should be expressed in its most concise form, in decimals if the decimal notation is the shortest, in fractions if otherwise.

EXAMPLES LXXVII.

(When the letter F is appended the answer is given in fractions.)

Simplify—

1. $\dfrac{\cdot0\dot{3}-\cdot\dot{0}\dot{3}}{\cdot1\dot{2}\dot{3}}\times 6\cdot1.$ **2.** $\dfrac{3\cdot1\dot{2}-2\cdot\dot{3}\dot{1}}{\cdot\dot{8}\dot{1}}.$ **3.** $\dfrac{2\cdot4\dot{6}-2\cdot\dot{3}\dot{0}}{\cdot\dot{3}+\cdot1\dot{2}\dot{7}}+\dfrac{4\frac{1}{3}}{19}\,(F).$

4. $\dfrac{1\cdot21\dot{5}+5\cdot\dot{2}\dot{6}}{5\cdot\dot{2}\dot{4}-2\cdot00\dot{3}}.$ **5.** $\dfrac{9\cdot\dot{3}0\dot{9}\times\cdot\dot{9}3\dot{0}}{93\cdot\dot{0}9\dot{3}}.$ **6.** $\dfrac{2\cdot791\dot{6}\times3\cdot2\dot{3}\dot{7}}{1\cdot86\dot{1}\times\cdot809\dot{3}\dot{4}}.$

7. $\dfrac{\cdot35\dot{1}}{\cdot9\dot{4}\dot{5}}\times\dfrac{2+\frac{1}{19}}{4-\frac{13}{19}}\div\dfrac{\frac{1}{4}-\frac{1}{7}}{\frac{1}{8}+\frac{1}{7}}\,(F).$ **8.** $\dfrac{3}{2\cdot01\dot{2}6\dot{5}+2\cdot\dot{8}\dot{7}+3\cdot10\dot{8}55946}.$

9. $\dfrac{7\cdot3911\dot{6}3891\dot{1}+2\cdot\dot{0}\dot{8}-9\cdot\dot{0}0\dot{9}}{4\frac{8}{9}-\frac{1}{4}\text{ of }7\frac{1}{12}-\frac{13}{18}}\,(F).$

10. $\left(\dfrac{2}{6\cdot25}+\dfrac{2\cdot55}{3\frac{3}{4}}-\dfrac{4}{4\cdot9\dot{3}}\right)\times5\cdot\dot{2}8571\dot{4}.$

11. $$\frac{(\cdot\dot{3}\div\cdot5)+1\cdot5+\dfrac{1}{2\frac{2}{5}}}{(3\cdot75-1\frac{1}{2})\div\dfrac{1}{1\frac{10}{17}}}+\frac{1}{3+\dfrac{1}{1+\dfrac{1}{1\frac{44}{79}}}}.$$

12. $$\frac{4\frac{3}{4}+\frac{7}{12}}{8\cdot1\dot{6}-3\cdot7+\frac{5}{8}}\text{ of }\frac{1\frac{\frac{7}{8}}{4}}{1\frac{5}{6}+3\frac{11}{18}}-\frac{0\cdot07+\frac{1}{195}-\frac{1}{91}}{0\cdot5\dot{2}}\text{ of }\frac{1\frac{637}{5253}}{1\frac{1}{4}+0\cdot2225}.$$

Laws of Recurring Decimals.

146. We have seen (§ 135) that any fraction whose denominator contains no other factors than 2 or 5 gives rise to a *terminating decimal.*

If the denominator be 2^m5^n, the number of figures in the decimal to which it gives rise is m or n, according as m or n is the greater (see § 135).

A fraction which gives rise to a pure circulating decimal is seen by § 138 to be equivalent to a fraction whose denominator is a number made up of nines. Such a fraction cannot, when in its lowest terms, have a denominator divisible by 2 or 5. Consequently no fraction whose denominator is divisible by 2 or 5 can give rise to a pure circulator.

Law I.—If N/D be a proper fraction in its lowest terms, and D be a prime number, other than 2 and 5, then N/D gives rise to a pure circulator, and the number of places in the recurring period is either $D-1$ or one of the factors of $D-1$.

Thus, the number of places in the period of $\frac{1}{269}$ is $269-1$—*i.e.* 268, and the number of places in the period of $\frac{1}{397}$ is 99, which is a factor of $397-1$.

This important law is an immediate consequence of Fermat's theorem, which will be proved later (see § 286). It can be easily proved that the number of places cannot be greater than $D-1$; for, in the process of converting N/D into a decimal by the method of § 137, recurrence will take place as soon as any remainder is the same as any previous remainder. But, at most, there are $D-1$ different remainders—namely, the numbers $1, 2, 3, \ldots, (D-1)$; therefore, at most, there will be $D-1$ figures in the decimal before recurrence takes place.

There is no method of foretelling *which* of the factors of $D-1$ indicates the number of places in N/D. The table given in Appendix II. gives this information as far as it goes. (See also § 287.)

Law II.—If D and E be different prime numbers other than 2 and 5, and $1/D$ have a period of p figures, while $1/E$ has a period of q figures, then $1/DE$ has a period of r figures, where r

is the L.C.M. of p and q. Similarly for a denominator consisting of more factors than 2.

Thus, $\frac{1}{7}$ has a period of 6 figures, $\frac{1}{101}$ of 4 figures; therefore $\frac{1}{707}$ has a period of 12 figures for 999999999999 may be arranged as 999999,999999, and is therefore divisible by 7, since 999999 is, or may also be arranged as 9999,9999,9999, and is therefore divisible by 101; therefore it is divisible by 7×101.

Since $1/D$ has a period of p figures, D divides a number consisting of p nines. Similarly, E divides a number consisting of q nines; therefore DE divides a number consisting of r nines, for, since r is the L.C.M. of p and q, a number consisting of r nines may be arranged in groups of p nines, or in groups of q nines, and therefore is divisible both by D and by E, and therefore by DE, since they are prime to one another.

Law III.—If D be a prime number other than* 2, 3, 5, *and* 487 *and* $1/D$ *have a period of p figures, then* $1/D^2$ *will have a period of pD figures,* $1/D^3$ *of* pD^2, *and so on.*

For instance, $\frac{1}{7}$ has a period of 6 figures, $\frac{1}{7^2}$ of 6×7, *i.e.* 42, $\frac{1}{7^3}$ of $6 \times 7^2 = 294$ figures.

The proof of this law is deferred. (See § 287.)

As regards the exceptions 2, 3, 5, and 487:

If $D = 2$ or 5, the decimal is terminating, and the case has been dealt with.

If $D = 3$, $1/D$ has a period of 1 figure, $1/D^2$ has a period of 1 figure, $1/D^3$ of 3 figures, and, generally, $1/D^n$ of 3^{n-2} $(n > 2)$.

If $D = 487$, $1/D$ and $1/D^2$ have both periods of 486 figures, $1/D^3$ of 486×487, and, generally, $1/D^n$ of $486 \times 487^{n-2}$ $(n > 2)$.

It is not yet known whether there are other exceptions, nor is the law known which determines the exception $D = 487$, which was first pointed out by Desmarest. There are no other exceptions less than 1,000.

Law IV.—If D be a number prime to 2 *and* 5, *such that* $1/D$ *has a period of p figures, the fraction* $1/2^m 5^n D$ *is a mixed circulator with either m or n non-recurring figures (according as m or n is greater) and with p recurring figures.*

For D divides into p nines, and $2^m 5^n$ divides into $10^{m \text{ or } n}$; therefore $2^m 5^n D$ divides into p nines followed by m or n ciphers; therefore gives rise to a mixed circulator with p recurring and m or n non-recurring figures. (See § 138.)

Law V.—THE CYCLIC LAW.—If D be a prime number other than 2 and 5, and $1/D$ have a period of p figures, then

* This law is incorrectly stated in text-books.

the fractions $1/D$, $2/D$, $3/D$, ..., $(D-1)/D$ can be divided into $(D-1)/p$ groups, each group containing p fractions giving rise to a circulating period made up of the same figures, for all fractions of the group taken in the same order round a circle, but starting at different points in the circle.

This law has been already explained and proved. (See § 139.)

Another illustration may be given. $\frac{1}{53}$ gives rise to a period of 13 figures; therefore the fractions 1/53, 2/53, ... , 52/53 divide into $\frac{52}{13}$—*i.e.* 4 groups. The first group contains the fractions 1/53, 10/53, 13/53, 15/53, 16/53, 24/53, 28/53, 36/53, 42/53, 44/53, 46/53, 47/53, and 49/53. The periods of all these fractions may be derived from the figures 0188679245283 by "cyclical" (*i.e.* circular) changes of these figures. (See figure in § 139.) Thus $1/53 = \cdot 0188679245283$, $47/53 = \cdot 8867924528301$, &c. The first fraction in the second group is 2/53, and the periods of the second group are derived from the figures 0377358490566. The student should find the periods of the third and fourth groups.

The Cyclic Law gives rise to a remarkable series of numbers. Thus, $\frac{1}{7} = \cdot\dot{1}4285\dot{7}$ and $\frac{2}{7} = \cdot\dot{2}8571\dot{4}$; but $\frac{2}{7}$ is $2 \times \frac{1}{7}$; therefore 285714 is 2×142857. In general the product of 142857 by any of the numbers 1, 2, 3, 4, 5, 6 consists of the same figures in the same "cyclic order," but starting at different points in the cycle.

The student will find it interesting to examine what happens when 142857 is multiplied by numbers greater than 6, and should endeavour to explain the results he gets. Similarly, the product of 0588235294117647 by any number up to 16 consists of the same figures in the same cyclic order.

Law VI.—The Complementary Law.—*If N/D be a fraction in its lowest terms which gives rise to a circulator having an even number of places in its recurring period (say $2n$), then the last n figures of the recurring period may be found from the first n figures by subtracting each of these from* 9, *provided D is* either (1) *a prime*, or (2) *the product of primes none of which is capable separately of giving a period of n figures*, or (3) *the product of such primes with a power of* 2 *or* 5 *or both.*†

The proof of this law will be given later. (See § 287.)

It follows that when we have found half the figures of the period of a circulator we can usually write the remaining half at sight. Thus, in finding $\frac{1}{46}$ we get $\cdot 0217391304347\mathbf{82}\ldots$; and we see that with the figures 7, 8, 2, ... the complement of the first part is appearing, for

† These conditions are invariably stated incorrectly in the text-books. D need not be prime. Thus $\frac{1}{91} = \cdot 010989$ and obeys the complementary law, although $91 = 7 \times 13$. But $\frac{1}{21} = \cdot 047619$, and does not obey the law, because 21 is the product of 7 and 3, and $\frac{1}{3}$ may be written $\cdot\dot{3}3\dot{3}$—*i.e.* so as to give a period of 3 figures.

$217 + 782 = 999$. We infer that we are half-way through the period, and that it may be finished by writing the complements of the other figures; therefore $\frac{1}{46} = \cdot 02\dot{1}7391304347826086956\dot{5}$.

147. We add some examples in illustration of these principles :—

Ex. 1.—*How many different proper fractions are there with unit numerators which give rise to circulating decimals with* 1 *non-recurring and* 4 *recurring figures ?*

Such fractions are capable of being expressed with denominator 99990; therefore their denominators are factors of $99990 = 10 \times 99 \times 101$. Their denominators must contain the factor 101, for, if not, they would divide into 990, and could be expressed with 1 non-recurring and 2 recurring figures. Also the denominator must contain 10 or one of its factors. Otherwise they would give rise to pure circulators. Hence the possible denominators are—(1) 2×101, 5×101, 10×101, and the product of these three numbers by (2) 3, (3) 9, (4) 11, (5) 33, (6) 99. Thus, in all, there are 18 such fractions.

Ex. 2.—*Prove that the sum of the digits of all multiples of* 142857 *up to and including the* 16*th is either* 27 *or* 54.

If $N = 142857$, the first 6 multiples, *i.e.* $1N$, $2N$, $3N$, $4N$, $5N$, $6N$, consist of the *same* figures; and therefore the sum of the digits is $1 + 4 + 2 + 8 + 5 + 7 = 27$. $7N = 999999$; therefore the sum of its digits is 54. $8N = 7N + N$. Now a little consideration will make it clear that to add 999999, *i.e.* $7N$, to any 6-figure number *which does not end in* 0 is the same as to diminish the last figure by 1 and to place 1 on the extreme left; therefore the sum of the digits is unaltered. Hence 27 is the sum of the digits for $8N$, $9N$, ..., $13N$. $14N = 1999998$, and the sum of its digits is 54. To add 1999998 to any 6-figure number not ending in 0 or 1, we diminish the right-hand digit of the 6-figure number by 2 and place a 2 at the left, and therefore the sum of the digits of any such number is unaltered; therefore 27 is the sum of the digits of $15N$ and of $16N$, but not of $17N = 14N + 3N$, for $3N$ ends in 1. In fact, $17N = 2428569$, and the sum of the digits is 36.

Ex. 3.—$\frac{1}{701}$ *produces a period of* 700 *figures. Determine the* 340*th to* 350*th.*

```
30242510699
99999999999 ( 701
      9369
      9306
      9510
      9925
      9642
      9824
      9702
       830
```

The process shown is a contracted form of that explained in § 140, Ex. 3. It shows that the 690th–700th figures of the period are 30242510699. But the number of figures in the period is even and 701 is a prime number; therefore the period obeys the Complementary Law; therefore the first half of the period ends with the complementary figures of the second half (*i.e.* with the figures obtained by subtracting from 9's). Hence the 340th–350th figures are ... 69757489300

Ex. 4.—*In converting* $\frac{1}{401}$ *into a circulating decimal, the figures* ...408... *appear at a certain stage in the period. Find the next three figures.*

401) ..... (...408

```
      ...0
(a)   1604
(b)   ··600
(c)    3208
(d)    ·392

      3920 ( 977
      3609
       3110
       2807
        3030
        2807
```

Explanation.—4 × 401 gives line (*a*).

Subtract the 4 in (*a*) from the 0 above it, and we get the 6 in line (*b*). Two zeros are added because 0 is the next figure in the quotient. 8 × 401 gives line (*c*).

Line (*d*) is given by subtraction. The dot in line (*d*) must represent zero, for, if it were 1, the second dot in (*b*) would be 4, and 0 would not have been correct in the quotient. We now have the complete remainder 392, and can continue the division as indicated, showing that the next three figures are 977.

*EXAMPLES LXXVIII.

[For other examples of a similar, but more difficult, character see Exx. clviii., 17–22; clx., 26–29; and Section XI., Nos. 45–6, 56, 76–8, 96–135. The tables given in Appendix II. may be used.]

Determine the nature of the decimal fractions resulting from the following vulgar fractions, and, without evaluating the decimals, write down the number of non-recurring figures in each, and also the number of figures in the recurring period:—

1. (*a*) $\frac{1}{32}$, (*b*) $\frac{1}{33}$, (*c*) $\frac{1}{34}$, (*d*) $\frac{1}{35}$, (*e*) $\frac{1}{36}$,
(*f*) $\frac{1}{1001}$, (*g*) $\frac{1}{1002}$, (*h*) $\frac{1}{1003}$, (*i*) $\frac{1}{1004}$, (*j*) $\frac{1}{1005}$.

2. (*a*) $\frac{1}{45}$, (*b*) $\frac{1}{46}$, (*c*) $\frac{1}{47}$, (*d*) $\frac{1}{48}$, (*e*) $\frac{1}{49}$, (*f*) $\frac{1}{50}$,
(*g*) $\frac{1}{995}$, (*h*) $\frac{1}{996}$, (*i*) $\frac{1}{997}$, (*j*) $\frac{1}{998}$, (*k*) $\frac{1}{999}$.

3. A proper fraction gives rise to a decimal containing 4 non-recurring and 1 recurring figures. What is the smallest, and what is the greatest, value of the denominator?

4. What is the least proper fraction, and what is the greatest proper fraction, with unit numerator which gives rise to a pure circulator of 6 figures?

5. What is the greatest proper fraction with unit numerator which gives rise to a pure circulator of 8 figures? What is the next greatest?

6. Prove that when, on converting $\frac{576}{701}$ into a recurring decimal, the remainder 125 occurs all the remaining figures of the recurring period may be written down at once.

7. After how many places will the remainder 576 occur when converting $\frac{125}{701}$ into a decimal?

8. Prove that, if any odd number whatever, not divisible by 5, be written down, it is possible to determine another number such that, if the two be multiplied together, the product shall be made up of 1's only.

9. If N be any number whatever, prove that it is possible to make a number divisible by N of which the figures at the left are to be all 9's, those at the right all 0's.

10. If 1020 be the denominator of a fraction in its lowest terms, prove that the number of figures in the period of this fraction must be one of these numbers—1, 2, 4, 8, 16.

11. $\frac{1}{271} = \cdot\dot{0}0369\dot{9}$. Show how to deduce, *mentally*, the value of $\frac{268}{271}$.

12. What are the 40th and 45th figures of $\frac{1}{83}$ written as in Ex. 14?

13. What are the 45th and 50th figures of the period of $\frac{1}{97}$?

14. If the fractions be converted into never-ending decimals (*e.g.*, $\frac{1}{3} = \cdot 333\ldots$), what is the 400th figure of (*a*) $\frac{1}{199}$, (*b*) $\frac{1}{197}$, (*c*) $\frac{1}{811}$, (*d*) $\frac{1}{809}$?

15. If the fractions be converted into never-ending decimals, what is the 500th figure of (*a*) $\frac{1}{199}$, (*b*) $\frac{1}{983}$, (*c*) $\frac{1}{503}$, (*d*) $\frac{1}{257}$?

16. The fraction $\frac{1}{6977}$ has a period of 6976 figures. Find the 3480th to 3500th figures inclusive.

17. The fraction $\frac{1}{10007}$ has a period of 10006 figures. Find the 4990th to 5010th figures inclusive.

18. Of all proper fractions, not necessarily in their lowest terms, having 37 for numerator and an integer less than 1000 for denominator, how many give rise to (*a*) non-recurring decimals, (*b*) pure recurring decimals, (*c*) pure recurring decimals with a 2-figure period, (*d*) pure recurring decimals with a 3-figure period, (*e*) mixed recurring decimals with 5 non-recurring figures, (*f*) mixed recurring decimals with 4 non-recurring and 1 recurring figures?

19. How many fractions are there with unit numerators and denominators less than 10,000 which produce (*a*) non-recurring decimals, (*b*) pure recurring decimals, (*c*) mixed recurring decimals?

20. Find all possible denominators for fractions with unit numerators which give rise to decimals having 2 non-recurring and 3 recurring figures.

21. How many different proper fractions are there which give rise to pure circulators of four figures?

22. The reciprocals of two prime numbers are reduced to recurring decimals, and it is found that their periods contain the same number of digits. Show that each of the periods is exactly divisible by the denominator of the other fraction.

23. Without converting $\frac{1}{181}$ into a decimal find the total sum of the 1st to 5th, 11th to 15th, 21st to 25th, 31st to 35th, and so on, up to and including the 171st–175th figures of its period.

24. In converting $\frac{1}{401}$ into a circulating decimal the figures ...842... occur at a certain stage in the period. Find the next three figures.

25. With the same decimal as in 24 find the next three figures to 778.

26. Find the next three figures to 158 which occur in the period of $\frac{1}{701}$.

27. Find the next three figures to 156 which occur in the period of $\frac{1}{701}$.

28. Prove that, if in converting $\frac{17936}{19961}$ into a recurring decimal the remainder 2025 occurs, the number of places in the period must be even.

29. Prove that the sum of the digits of all multiples of 76923 up to and including the 22nd is either 27 or 54.

30. Prove that the sum of the digits of all multiples of 153846 up to and including the 17th is either 27 or 54.

CHAPTER XVIII.

APPROXIMATION.

148. No measurements are ever quite exact. When we say that an object is exactly 1 foot long, we mean only that, if we apply the foot-rule to the object the ends of the object will be as far as the eye can see one at the division of the rule marked 0, the other at the division marked at 12. But we know that, if we took a microscope and examined the two side by side, we should see that they did not *exactly* agree at either division. Moreover, our foot-rule itself would not, under the microscope, prove to agree exactly with the Standard in London.

We naturally ask, then, how far we can trust our measurements. Of course, this depends on the care with which they are taken; but we may say roughly that no measurements taken by an untrained person are ever "correct to" more than 3 figures, and that *few measurements, however skilful the man who makes them, are "correct to" more than 4 figures.*

When more than 4 figures are given in a measurement we may always suspect them.

"When I was at school the mean distance from the earth to the sun was stated as 95,142,357 miles. I wonder why furlongs and inches were not mentioned. The best knowledge we now have of this distance is that it is not greater than 93 nor less than 92½ millions of miles."—(Prof. Perry.)

Measurements of length may be made correctly to 6 or 7 figures, while measurements of weight may be made to 8 or 9 figures. Beyond 9 figures nobody has yet passed, and it is not likely that much further progress will ever be made.†

† Sir Joseph Whitworth's measuring machine may, perhaps, in skilled hands measure the millionth part of an inch; that is, it will give a measurement correct to the 6th decimal place. In Michelson's determination of the length of the metre an accuracy of one part in 20,000,000 was attained, representing measurements correct to between 6 and 7 figures. From the Interferometer of Messrs. Fabry and Perot similar results can be obtained. Weighing is the most accurate form of measurement we possess. A fine modern balance, carrying 1 kilogram on each pan, will "turn" with $\frac{1}{10}$ mgr., *i.e.* will give results correct to 7 figures. With care such a balance will give one other figure correctly. M. Thiesen has recently compared 40 national standard kilograms with the international standard, with a probable accuracy of ·0001 mgr., indicating measurements correct to 9 figures. I owe these references to the kindness of Mr. H. Stanley Allen.

149. In speaking of a measurement as "correct to 3 figures," "correct to the 6th decimal place," we have used phrases that are important, and require further explanation. Suppose the real length of an object is ·6349 ft. We may describe it as being ·6 ft. long "*correct to the first decimal place*," or correct to 1 (significant) figure," or "correct to ·1 of a foot." By the last phrase we mean that the error in taking the length as ·6 ft. is less than ·1 of a foot. So with increasing exactness we may say of an object ·6349 ft. long that its length is—

·6 ft.	"correct to 1st place"	or	"correct to ·1 ft."	or	"correct to 1 figure."
·63 ,,	,, 2nd ,,	or	,,	·01 ,, or	,,	2 figures.
·634 ,,	,, 3rd ,,	or	,,	·001 ,, or	,,	3 ,,

But in the last case we may do better. For, if an object whose real length is ·6349 ft. is described as ·63**4** ft., the error is ·0009 ft., while, if it is described as ·63**5** ft., the error is only ·0001 ft. (= ·635 − ·6349), so that the error is nine times greater in the first case than in the second.

Similarly, ·6348, ·6347, ·6346 are best given as ·635 correct to 3 places, while ·6344, ·6343, ·6342, ·6341, ·6340 would be written as ·634. With ·6345 practice varies. It is perhaps generally best to reckon this as ·635, and this course will be followed for the remainder of this book. It is important to notice that by the adoption of this rule the *possible errors of an approximate statement are almost halved.* Thus, if this rule be followed and the length of an object be stated to be ·6 ft. "correct to the first place," we know that it cannot be less than ·55 ft. (for ·54 ft., for instance, would be called ·5, "to the first place"), and is less than ·65 ft. (for ·65 would be called ·7). Consequently ·6 ft. is not more than ·05 ft. (·6 − ·55) away from the truth; whereas on the former system it might be ·09, since ·69 would be reckoned as ·6.

If, then, we speak of a number as being ·634 "correct to 3 places," we mean, strictly speaking, that it may be any number less than ·6345 and not less than ·6335. The possible error in the value of the number is ·0005 "*in excess or in defect*," *i.e.* the number may be as much as ·0005 greater than, or as much as ·0005 less than, ·6340. This may be expressed thus:

$$\cdot 634 \text{ correct to 3 places} = \cdot 634 \pm \cdot 0005.$$

The *Limits of Error* of ·634, when described as known to 3 places, are thus ±·0005. The actual error may, of course, have any value between these extremes.

Since not more than 4 figures in any measurement are usually of practical importance, it is clear that Arithmetic as applied to

problems of practical measurement may be very greatly simplified by the omission of such figures as cannot be verified by measurement.

We proceed to examine the elementary rules of addition, &c., in order to see how far our results may be simplified by the omission of all figures in the data of which we have no certain knowledge, and to determine the possible errors due to the figures so omitted.

Addition and Subtraction.

150. Ex. 1.—*The numbers* 1·312, 9·563, 3·569, 4·213 *are known correct to* 3 *decimal places. Determine their sum, and indicate the limits of error.*

A.
1·312.
9·563.
3·569.
4·213.
18·657.

The addition is performed as usual at A, the sum being 18·657. ; but this is not necessarily correct. The fourth place (indicated by dots) might be in every case 5*, and the 4 fives added would produce 20 in the fourth place, *i.e.* 2 in the third place ; therefore the third place, as shown at A, may be as much as 2 in defect. The addition thus modified is shown at B. But also 1·312 might really stand for 1·3115, and so with the other numbers of the sum. It would then take the form shown at C, and it becomes clear that the error may be as much in excess as in defect.

B.	C.
1·3125	1·3115
9·5635	9·5625
3·5695	3·5685
4·2135	4·2125
18·6590	18·6550

Note carefully that the answer *may* be correct, and would be so, if, for instance, when the fourth figure is supplied, the question becomes

$$1·3125 + 9·5625 + 3·5685 + 4·2135.$$

We express our result thus:

$$1·312 + 9·563 + 3·569 + 4·213 \text{ (to 3 places)} = 18·657 \pm ·002.$$

Ex. 2.—*Determine the limits of error in* 4·213 − 3·569 (*correct to* 3 *places*).

A.	B.	C.
4·213	4·2135	4·2125
3·569	3·5685	3·5695
·644	·6450	·6430

It might seem at first sight as if in this case A was necessarily right, because the possible error in 4·213 was destroyed by the possible error in 3·569. But a moment's thought will show that 4·213 might be understated while 3·569 was overstated, or *vice versa*, so that the actual difference (to 4 places) might be as in B, or might, again, be as in C.

Hence $$4·213 - 3·569 \text{ (to 3 places)} = ·644 \pm ·001.$$

151. In certain cases our information may extend beyond the number of places required in the answer. Two examples will be given to illustrate the method of dealing with such cases.

* Strictly, if the convention on p. 212 is followed, 1·3125 would be written 1·313, and not 1·312. But, at the same time, 1·31249 would be written 1·312, and, if 1·31249 were written correct to 4 places, it would appear as 1·3125.

Ex. 1.—*Find, correct to 4 decimal places, the sum of*

$$\frac{1}{8}+\frac{1}{8^2}+\frac{1}{8^3}+\frac{1}{8^4}+\ldots.$$

$$\begin{array}{l|l}
\frac{1}{8} = \cdot12500 & 0 \\
\frac{1}{8^2} = \cdot01562 & 5 \\
\frac{1}{8^3} = \cdot00195 & 3 \\
\frac{1}{8^4} = \cdot00024 & 4 \\
\frac{1}{8^5} = \cdot00003 & 0 \\
\frac{1}{8^6} = \cdot00000 & 4 \\
\hline
\cdot14285 &
\end{array}$$

Explanation.—We convert $\frac{1}{8}$ into a decimal; $\cdot125000\ldots.$ $\frac{1}{8^2}$ is obtained from this by dividing it by 8, $\frac{1}{8^3}$ by dividing the last result by 8, and so on. We are asked to find the answer "correct to 4 places." We ought, therefore, to know enough about the 5th place to be able to say whether it is greater or less than 5, and we retain 6 places—the 6th merely to see what figure is to be "carried" to the 5th. We do not need to carry the series beyond $\frac{1}{8^6}$ because $\frac{1}{8^7}$ does not affect the first 6 places. Its value, in fact, is $\cdot00000047\ldots.$† In $\frac{1}{8^6}$ we write the quotient 4 *rather than* 3, because 4 is nearer to the right quotient than 3. We then add up, using the 6th place, which is separated by a vertical line from the others merely to give the carrying figure to the 5th place. The answer is, therefore, $\cdot1429$ to 4 places. The student who has studied geometrical progression will see that the real answer is $\frac{1}{7}$, *i.e.* $\cdot\dot{1}4285\dot{7}$.

Caution.—It is sometimes, though rarely, necessary to take more than two extra decimal places into account, particularly if the terms are only very gradually growing smaller. No general rule can be given.

Ex. 2.—*Subtract* $7\cdot13\dot{4}\dot{5}$ *from* $17\cdot\dot{1}512\dot{3}$ *correct to* 6 *figures.*

$$\begin{array}{r|l}
17\cdot15123 & \mathbf{2} \\
7\cdot13454 & 5 \\
\hline
10\cdot01668 &
\end{array}$$

Explanation.—If the answer is to be correct to 6 figures, we retain 8 as before. Now $17\cdot\dot{1}512\dot{3}$ means $17\cdot1512315123\ldots$, and correct to 8 figures is $17\cdot15123\mathbf{2}$, not $17\cdot15123\mathbf{1}$. The rest explains itself. The answer correct to 6 figures is $10\cdot016\mathbf{7}$; correct to 5 figures would be $10\cdot017$; correct to 4 figures, $10\cdot02$; correct to 3 figures, $10\cdot0$.

---

† A difficulty really arises here. It is true that $\frac{1}{8^7}$ does not affect the first six places, still less $\frac{1}{8^8}$ or $\frac{1}{8^9}$, or any of the other terms. Is it not possible, however, that the *sum* of all these small quantities may be big enough to affect the 6th place? We cannot deal with this difficulty here. It arises in all such questions, and the student must satisfy himself by taking a few more decimal places into account that the difficulty is not *practically* serious. The series $\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\ldots$, *i.e.* $\cdot1111\ldots$, *i.e.* $\cdot\dot{1}$, will show this point more clearly.

EXAMPLES LXXIX.

The following numbers are known to be correct to the third place. To what place can their sum be relied on? What are the limits of error?

1. 1·432, 3·412, 7·851, 8·579, 3·001, 6·003.

2. 1·414, 4·143, 1·732, 7·312, 3·415, 100·888, 1·999, 9·191.

The following numbers are known to be correct to 3 places. To what place can we rely on their difference? What are the limits of error?

3. 1·432, 3·412. **4.** 7·851, 8·579.

If the numbers stated in Questions 5–10 are known correct only to the figures given, state the limits of error of the answers.

5. $12+35+53-74$. **6.** $32+15+54-73$.

7. $131{\cdot}2+273{\cdot}5-315{\cdot}6-1{\cdot}7$. **8.** $723{\cdot}3-235{\cdot}7-123{\cdot}6-145{\cdot}9$.

9. $2{\cdot}1234-{\cdot}2134-{\cdot}0007+{\cdot}0009-{\cdot}7654$.

10. $3{\cdot}2142+{\cdot}4132+{\cdot}0009-{\cdot}0009-{\cdot}5678$.

Find, *correct to* 4 *decimal places*,† the value of—

11. $1{\cdot}\dot{7}56\dot{8}+20{\cdot}\dot{7}3\dot{6}+28{\cdot}224$. **12.** $1{\cdot}\dot{6}12\dot{8}+19{\cdot}\dot{8}7\dot{2}+26{\cdot}78\dot{4}$.

13. $17{\cdot}123-7{\cdot}\dot{2}3\dot{4}$. **14.** $18{\cdot}987-9{\cdot}8\dot{7}\dot{9}$.

15. $3\frac{1}{17}+2\frac{1}{11}+1{\cdot}\dot{8}9\dot{9}+{\cdot}00387$. **16.** $\frac{1}{19}+3\frac{1}{9}+1{\cdot}80\dot{9}\dot{1}-{\cdot}1\dot{2}3\dot{4}$.

17. $\dfrac{1}{1\times2}+\dfrac{1}{3\times4}+\dfrac{1}{5\times6}+\dfrac{1}{7\times8}$. **18.** $\dfrac{1}{1\times3}+\dfrac{1}{2\times4}+\dfrac{1}{5\times7}+\dfrac{1}{6\times8}$.

19. $\dfrac{1}{1\times2}-\dfrac{1}{3\times4}+\dfrac{1}{5\times6}-\dfrac{1}{7\times8}$. **20.** $\dfrac{1}{1\times3}-\dfrac{1}{2\times4}+\dfrac{1}{5\times7}-\dfrac{1}{6\times8}$.

Calculate, *to* 6 *decimal places*,* the value of—

21. $\dfrac{1}{9}+\dfrac{1}{9^2}+\dfrac{1}{9^3}+\dots$. **22.** $\dfrac{1}{7}+\dfrac{1}{7^2}+\dfrac{1}{7^3}+\dots$.

23. $\dfrac{1}{1.2}+\dfrac{1}{1.2.3}+\dfrac{1}{1.2.3.4}+\dots$. **24.** $\dfrac{1}{1.2}-\dfrac{1}{1.2.3}+\dfrac{1}{1.2.3.4}-\dots$.

25. $\dfrac{1}{2}+\dfrac{1}{2.4}+\dfrac{1}{2.4.6}+\dots$. **26.** $\dfrac{1}{3}+\dfrac{1}{3.6}+\dfrac{1}{3.6.9}+\dots$.

27. $1-\dfrac{1}{1.2}+\dfrac{1}{1.2.3.4}-\dfrac{1}{1.2.3.4.5.6}+\dots$.

28. $1-\dfrac{1}{1.2.3}+\dfrac{1}{1.2.3.4.5}-\dfrac{1}{1.2.3.4.5.6.7}+\dots$.

29. $4\left\{\dfrac{1}{5}-\dfrac{1}{3.5^3}+\dfrac{1}{5.5^5}-\dfrac{1}{7.5^7}+\dots\right\}-\left\{\dfrac{1}{239}-\dfrac{1}{3.239^3}+\dfrac{1}{5.239^5}-\dots\right\}$.

† The answers to these questions are given correct to 10 significant figures, and in the case of No. 29 to 30 figures for the sake of those who may wish to have some practice in calculation.

MULTIPLICATION.

152. The process in regard to Multiplication is illustrated by examples:—

Ex. 1.—*If* ·234 *is known correct to* 3 *places, determine the limits of error in* ·234 × 67.

The limits of error of ·234 are ± ·0005. Clearly, then, the limits of error of 67 times ·234 are 67 × (± ·0005), *i.e.* ± ·0335. The product is therefore reliable *only to the first place*, and there is a possible error of ± 3 in the second place.

Note carefully here that the third place is *altogether* uncertain. We cannot say that its limits of error are ± 3, for when we multiply by 67 the figures in the *fifth* place of ·234 might, if we knew them, affect the third place of the product. The student should verify the following statements for himself:—

·234 × 67 = 15·678,

·2335 × 67 = 15·6445, ·23359 × 67 = 15·65053,

·2345 × 67 = 15·7115, ·23449 × 67 = 15·71083.

It might seem from these answers as if the second place could be *4* in error, and not 3, for it appears in the last two products as 15·71 and in the first as 15·67, and these differ by ·04. But when 15·678 is written *correct to two places* it becomes 15·68, and not 15·67, and the discrepancy disappears.

Ex. 2.—*The numbers* 1·7324 *and* 2·5968 *are correct to* 4 *places only. How far can their product be relied on?*

When we say that 1·7324 is "correct to 4 places," we mean that it lies between 1·73235 and 1·73245. Similarly, 2·5968 lies between 2·59675 and 2·59685. Hence the product lies between 1·73235 × 2·59675 and 1·73245 × 2·59685, *i.e.* between (1·7324 − ·00005) (2·5968 − ·00005) and (1·7324 + ·00005) (2·5968 + ·00005), *i.e.* between

1·7324 × 2·5968 − ·00005 (1·7324 + 2·5968) + ·0000000025

and 1·7324 × 2·5968 + ·00005 (1·7324 + 2·5968) + ·0000000025.

These differ from one another only in the middle term

·00005 (1·7324 + 2·5968), *i.e.* ·00005 (4·3292) = ·00021646.

Hence the product of 1·7324 and 2·5968, both of which are "correct to 4 places," is "correct to 3 places" only, but the error in the 4th place is not greater than 2. The limits of error are ± ·0002.

For further information see § 156, Ex. 1.

CONTRACTED MULTIPLICATION.

153. It will be seen from the last example that when 1·7324 and 2·5968, each of which is known to be correct only to the fourth place, are multiplied together, the first three decimal places only of the eight decimal places of the answer are

correct. We proceed to explain how it is possible to find these places without the labour of finding the necessarily incorrect figures at the end of the decimal. The process employed is known as Contracted Multiplication. We illustrate it by examples of increasing difficulty :—

Ex. 1.—*Multiply* 7·14389613 *by* 7 *correct to* 4 *decimal places.*

```
 7·14389|613
 7      |
-------------
50·00727|
```

Explanation.—We require 4 decimal places; therefore we retain 5 (otherwise the 4th might be wrong by 1, since we should not know whether the 5th was greater or less than 5, and, if greater than 5, we ought to increase the 4th by 1, as in the Example), and draw a line to the right of them to shut off the rest from the work. We *begin the work of multiplication upon the* 9, *neglecting the figures on the right except to show us what to carry from right to left of the bar.* This is clearly 4. The rest of the work proceeds as usual. The answer is 50·007**3**.

The last example may be viewed differently. It is obvious that, to multiply 7·14389613 by 7, correct to 4 places, is the same as to multiply 71·4389613 by ·7. Now 71·4389613, correct to 4 places, is 71·4390, and the error is less than ·00005. But ·7 is a fraction; therefore the error in 71·4390 × ·7 is less than ·00005 × ·7, *i.e.* less than ·000035, *i.e.* does not appear in the fourth place. Hence the required product is that of 71·4390 × ·7. This method should always be adopted with multipliers of more than one figure, and may be expressed in the following rule :—

To multiply a number by another number correct to n places, shift the decimal point in the multiplier until the first significant figure stands in the first decimal place; then shift the decimal point in the multiplicand in the opposite direction for the same number of places. Retain exactly n *places in the altered multiplicand, altering the last place if necessary.*

The further abbreviations possible are now shown in the examples.

Ex. 2.—*Multiply* 7·14389673 *by* 73, *correct to* 4 *decimal places.*

```
714·38$\dot{9}$7|
   ·7$\dot{3}$  |
-------------|-
500·0727|9
 21·4316|9
-------------|-
521·5045|
```

Explanation.—We shift the decimal point in the multiplier 73· two places to the left until it becomes ·73, and then shift the decimal point in the multiplicand two places to the right, re-writing it as we do so *correct to* 4 *places.* Draw a vertical line on the right of the multiplicand. We then multiply by ·7. The product will commence to the right of the vertical line. We next multiply by the

3 of ·73. This partial product commences two places to the right of the vertical line. We commence therefore by multiplying the 3 not into the 7, but into the 9, of the multiplicand, and place a dot over both of these figures to indicate this fact. We use the 7 only to tell us, in this case, what to carry. Our work is: 3 nines, 27, *and 2* ("carried" mentally from 3 × 7), 29; write the 9 and carry the 2. In adding the partial products, the figures to the right of the vertical line are used only to tell us what figures to carry into the fourth place. Work thus: 9 and 9, 18 (carry 2, *not* 1); 2 and 6, 8; and 7, 15; &c.

Ex. 3.—*Multiply* ·0714389673 *by itself correct to* 6 *significant figures.*

Following the rule, we write the multiplier ·714389673, so that the multiplicand becomes ·00714389673, which is rewritten correct to 6 significant figures as ·00714390. A vertical line is drawn to the right of the altered multiplicand.

·00714390		
·714389673		
·00500073	0	Ordinary multiplication by 7.
7143	9	Multiplication by 1 commences with the 9. Dots are placed over both figures.
2857	6	Dot the 4 (multiplier) and the 3 (multiplicand). Carry 4, *not* 3, from 4 × 9.
214	3	Dot the 3 (multiplier) and the 4 (multiplicand). Carry 1 from 3 × 3, which precedes.
57	1	Dot the 8 and the 1. Carry 3.
6	4	Dot the 9 and the 7. Carry 1.
	4	The dots now cover all the multiplicand, but we see easily that, if the multiplication by $\bar{6}$ were effected, it would give this line, and that the $\bar{7}$ and the 3 do not affect this line.
·00510353		

Thus the answer correct to 6 significant figures is ·00510353.

Note.—It would be easy to *construct* a product for which this *or any other method of approximation* appeared incorrect by a unit in the last figure. If, for instance, the numbers to the right of the vertical line were as indicated at A, the last place of the answer would be written 2. But, if when one place more was taken into account the numbers were as at B, then the last place of the answer would be written 3, and not 2. The danger of this error would be lessened if two places were retained to the right of the bar. It can never be entirely removed. The method shown is more than sufficiently exact in practice. These critical cases do not *actually* occur outside the limits of examination papers.

A	
	7
	6
	5
	2
	4
2	4

B	
	74
	64
	54
	24
	44
2	60

For continued products of three or more approximate numbers the work may be most expeditiously arranged thus:—

Choose as the multiplicand that number which contains most

significant figures. Shift decimal points in the multipliers until their first significant figures stand in the first decimal places; shift the decimal point in the multiplicand a corresponding number of places in the opposite direction. Retain in the altered multiplicand the exact number of places required in the answer and reject the rest. Then proceed as in Ex. 3.

Ex. 4.—*Find correct to 4 decimal places the product of*

$$\cdot 0234 \times 12\cdot 768 \times 3\cdot 129578.$$

```
31·2958|
  ·1276|8
 3·1295|8
   6259|2
   2190|7
    187|8
     25|0
 3·9958|
   ·234|
  ·7991|6
  1198|7
    159|8
  ·9350|
```

We choose 3·129578 as the multiplicand because it has 7 figures. In the multipliers we alter the decimal points 1 place to the right in ·0234, 2 to the left in 12·768, *on the whole* therefore 1 to the *left*; therefore we move the point in 3·129578 1 to the *right*. The work then proceeds as shown.

In the first product the figures to the right of the vertical line (8, 2, 7, 8, 0) total 25. We are justified in carrying 2 rather than 3 to the left of this line, however, because these figures are all in excess of the actual figures (thus 62592 is really 6259*16*, and we write *2* for the *1* of 16), and because the multiplicand is in excess. The actual product is 3·995847744, and the actual product of the three numbers is ·935028....

154. The method of Ex. 2, § 152, suggests one very important mode of abbreviation which is summarised in the following laws:—

If r, s, and t are small quantities—

(1) $(1+r)(1+s) = 1+r+s$ nearly.

(2) $(1+r)(1+s)(1+t) = 1+r+s+t$ nearly.

(3) $(1+r)^2 = 1+2r$ nearly.

(4) $(1+r)^3 = 1+3r$ nearly.

(5) $\dfrac{1+r}{1+s} = 1+r-s$ nearly.†

The proof of these laws is suggested by the treatment of Example 5.

Ex.—*Find the approximate value of* $(1\cdot 0000013) \times (1\cdot 0000027)$ *and the magnitude of the error.*

Here $r = \cdot 0000013$, $s = \cdot 0000027$, $r+s = \cdot 0000040$. Hence the rule

† This law follows from (1), for, if $(1+r)(1+s) = 1+r+s$, $\dfrac{1+r+s}{1+s} = 1+r$. Put $r+s = t$; therefore $r = t-s$ and this becomes $\dfrac{1+t}{1+s} = 1+t-s$, which is the law.

gives 1·0000040 as the approximate value. The true value of $(1+r)(1+s)$ is $1+r+s+rs$, and this differs from the approximate value $1+r+s$ by rs, *i.e.* by ·00000000000351. Hence the approximate value differs from the real value by a quantity which does not affect any decimal place until the *twelfth*.

EXAMPLES LXXX.

Find correct to 3 decimal places the products—

1. 1·356897 by 5, 6, 7, 8, and 9. **2.** 3·158679 by 4, 6, 9, 10, and 11.

3. 3·1415926 by 17, 21, 39, 41, and 60.

4. 1·3451296 by 16, 23, 38, 43, and 61.

Find the following products, correct to the number of decimal places stated in brackets after each :—

5. 11·82075 × ·003961 (5). **6.** 13·7305 × ·006507 (5).

7. 3·40604 × ·0435075 (5). **8.** 5·61023 × ·597001 (5).

9. 4·807309 × 12·086 (4). **10.** 76·035 × ·0580079 (4).

11. 48·63207 × ·09058 (4). **12.** 57·386 × ·008709 (4).

13. 43·7246 × ·24805 (3). **14.** 98·003535 × ·0042074 (3).

15. 3·12105 × ·905008 (3). **16.** 10·1032 × ·010987 (3)

17. 12·98563 × 21·3567 × ·0013579 (3).

18. 9·21345 × ·53179 × 12·5376 (3).

19. $(\cdot 497594212)^4$ (5). **20.** $(\cdot 723740285)^4$ (5).

The following numbers are known to be correct to 4 places. To what place can we rely upon (*a*) their sum, (*b*) their difference, (*c*) their product, (*d*) the quotient of the greater divided by the less, (*e*) the quotient of the less divided by the greater ? State the possible error.

21.* 119·3013, 20·6987. **22.*** 13·1234, 70·8766.

Find the approximate value of the following, and state to how many decimal places the value you give is correct :—

23.* $(1\cdot 0013)^2$. **24.*** $(1\cdot 0009)^2$.

25.* $(1\cdot 00000372)^2(1\cdot 00000273)(1\cdot 00000156)$.

26.* $(1\cdot 0000009)^2(1\cdot 0000010)(1\cdot 0000011)^2$.

27.* $(\cdot 9999983)(\cdot 9999938)^3$. **28.** $(\cdot 9999927)^2(\cdot 9999972)^2$.

Contracted Division.

155. Methods for abbreviating multiplication similar to those just explained can be applied to the multiplications which are necessary during the operation of division, and result in a great saving of labour. We will examine an instance, working, first, in full, and then asking what abbreviations are possible.

Ex. 1.—*Divide* 2·4494897427 *by* 1·4142135623 *correct to* 5 *decimal places.*

The full working is as follows :—

```
                                  1·732050
14142135623 ) 24494897 | 427
              14142135 | 623
              10352761 | 8040
               9899494 | 9361
(A)             453266 | 86790
                424264 | 06869
                 29002 | 799210
                 28284 | 271246
                   718 | 52796400
                   707 | 10678115
                    11 | 421182850
```

Now the first point to note is that the figures of the quotient are determined by the *first two or three* figures of the successive remainders, and that, if we can retain these, we can do without the others. Our answer is to be correct to 5 decimal places, *i.e.* in this case to 6 significant figures. To be sure of 6 significant figures, we must have some knowledge of what the 7th is. We will make our preparations therefore for determining 7 figures in the quotient. The figures in thick type to the left of the vertical line are those which are needed for the purpose. Now 2 figures are needed, at least, in the final remainder which gives us the last significant figure of the quotient ; there are 6 figures to be obtained before this, and therefore we must retain 6 + 2, *i.e.* 8, figures of the original dividend (for with each figure of the quotient the remainders move one place to the right, *i.e.* one more figure moves from left to right of the vertical line). We commence, then, by retaining 8 figures of the dividend, and rejecting 3 (namely, 427). We now put a dot over the 5 of the divisor to indicate that it is the last figure of the divisor which concerns us. The 623 which follows the 5 is used only to show us what figure to carry in the multiplications. In the next step we see that 7 is the partial quotient. The **9899494** consists of the first 7 figures of the product of the divisor by 7. We multiply 1414213̇5623 by 7, and, noting that we want 7 figures, put a dot over the 3, and commence the multiplication there. That finished, and the product subtracted, the remainder is **453266**. A dot is placed over the 1 ; 141421 is regarded as the divisor ; the other figures, 35623, are used only to give the right figure

to carry, and so on. The work thus shortened stands as in (B), or, if the "Italian" division is used, as in (C).

```
                    1·732050
14i̇4̇2̇1̇3̇5̇623 ) 24494897|427
                14142136|
                10352761 (a)
                 9899495
                  453266 (b)
                  424264
(B)                29002 (c)
                   28284
                     718 (d)
                     707
                      11 (e)
```

```
                    1·732050
14142135623 ) 24494897 427
              10352761|
                453266
                 29002
(C)                718
                    11
```

The saving of time involved by these processes is obvious from the fact that (A) contains 140 figures, (B) 82, (C) only 53, and that 6 other figures, namely, 623 in the divisor and 427 in the dividend, may be safely omitted in both (B) and (C), so that (B) becomes only about half as long as (A), and (C) one-third as long.

One point of importance remains to be examined. The last figure in the partial products and remainders in (B) is usually only approximate. For instance, in the first partial product 14142136 it is 6, whereas in (A) it is 5. Is it not possible that the errors in these last figures may accumulate and make the final remainder seriously wrong? To answer this question note that, if two numbers known to be correct in the last figure are subtracted, the difference contains an error not greater than 1 in the last figure. (*Cf.* Ex. 2, § 150.) There is thus a possible error in the last figure of 1 at (*a*), of $1\frac{1}{2}$ at (*b*), of 2 at (*c*), of $2\frac{1}{2}$ at (*d*), and of 3 at (*e*). Hence in the least favourable case the error is not sufficient to disturb the final figure of the quotient. These statements refer to the *general* case, and not merely to this particular example. This error could only accumulate sufficiently to be of importance if we had several more figures to find—which never *ought* to be. A little consideration will justify the following rule:—

To perform contracted division:—

Determine how many significant figures (NOT *decimal places*) *are required in the quotient. Retain 2 more than this number in the dividend, and reject the rest, increasing, if necessary, the last figure retained by 1. Of the divisor retain as many figures as are necessary to make the first partial product in the dividend extend no further than the right-hand figure retained.*

Then proceed as above.

Ex. 2. — *Evaluate* $\dfrac{1{\cdot}7876543 \times 71{\cdot}78563473}{{\cdot}0123456789}$ *correct to 2 decimal places.*

```
                 717·856347|
                   ·178765|43
                 71·785634 7
                 50·249944 3
                  5·742850|7
                   ·502499|4
                     43071|4
                      3589|3
                       287|1
                        21|5
123456789 ) 128·327898(10394·559
            4·871109          Ans.
            1·167405
               56294
                6911
                 738
                 121
```

Explanation.—It is easy to see, by performing the work mentally with only the first two figures retained in each number ($1{\cdot}8 \times 72 \div {\cdot}01$), that the answer has 5 figures to the left of the decimal point. If it is to be correct to 2 decimal places, it must, therefore, be correct to 7 significant figures. But, if the division sum is to have its quotient correct to 7 significant figures, we must retain $7+2$, *i.e.* 9, figures in the dividend to be safe. Hence the product which makes up the numerator must be correct to 9 figures, *i.e.* to 6 decimal places. The work is now as shown, and the answer is 10394·56. The *position of the decimal point* is fixed by inspection.

EXAMPLES LXXXI.

Divide correct to 2 decimal places—

1. 53·829764 by 2·56, 2·5634, 2·563434, and ·2563434.

2. 35·827964 by 6·52, 6·5234, 6·523443, and ·6523443.

3. 3·14159265 by 1·4146, 1·73241, and 17·42317.

4. 4·41512956 by 1·1464, 1·37421, and 14·72713.

Divide correct to 4 significant figures—

5. ·01378967 by 2·389605, ·32869051, and ·00148257.

6. ·03187976 by 3·289605, ·38296015, and ·00142875.

7. 13·142857142 by 7·0769230769 and 353·846153.

8. 7·0769230769 by ·0013142857 and 657·1428.

9. 1 by 1·4142136, 1·7320508, 2·2360680, and 2·4494897.

10. 1 by 1·2599210, 1·4422496, 1·5874011, and 1·7099759.

Find the value to 4 significant figures of—

11. $\dfrac{3{\cdot}14285714 \times {\cdot}0007692307}{1{\cdot}69230769}$.

12. $\dfrac{31{\cdot}42857142 \times 1{\cdot}076923076}{{\cdot}846153846}$.

Determine the limits of error of—

13. 12·967 (3 places) ÷ 35.

14. 21·769 (3 places) ÷ 53.

Relative Error.

156. The calculation of the error involved in the evaluation of a product, or a quotient comprising two or more approximate numbers, is made much easier by the introduction of what is known as **relative error.** The "error" which we have considered up to this point has been the **absolute error.** These two terms are thus defined :—

The absolute error *of an approximation is the actual difference between the approximate value and the true value.*

The relative error *of an approximation is the fraction whose numerator is the absolute error and whose denominator is the true value.*

Thus, if 3·1435 is stated correctly to 2 places, the absolute error is ·0035 [$= 3{\cdot}1435 - 3{\cdot}14$], and the relative error is ·0035/3·1435, *i.e* $\frac{35}{31435} = {\cdot}0011\ldots.$

The following theorems are easy to prove :—

(1) The limit of relative error of the product of two or more approximate factors is (approximately) equal to the sum of the limits of relative error of the two factors.

(2) The limit of relative error of the quotient of two approximate factors is (approximately) equal to the *sum* of the limits of relative error of divisor and dividend.

The proof of these theorems is necessarily algebraical. An arithmetical proof, in particular instances, may be given on the lines of Ex. 2, § 152.

Let A and B be two approximate numbers, and aA, bB the limits of their respective errors, so that aA/A (*i.e.* a) and bB/B (*i.e.* b) are the limits of their relative errors. The higher limits of the accurate numbers are therefore $A + aA$ and $B + bB$, *i.e.* $A(1+a)$ and $B(1+b)$, and the limit of the accurate product is $AB(1+a)(1+b)$. But a and b are small ;

$$\therefore \quad (1+a)(1+b) = 1 + a + b \text{ (nearly).} \quad \text{(See § 154.)}$$

$$\therefore \quad \text{limit of product is } AB(1+a+b) \text{ nearly.}$$

But we take the product as AB ; therefore the limit of error of the product is $AB(a+b)$, and the limit of relative error is

$$\frac{AB(a+b)}{AB(1+a)(1+b)} = \frac{a+b}{1+a+b} \text{ (nearly)} = a + b \text{ (nearly),}$$

for the denominator of the fraction differs only slightly from 1.

In practice the relative error is usually found by taking the approximate value AB, instead of the true value, for the denominator.

The proof of the second theorem is similar, but it will be easily seen that the higher limit of the true values of the quotient is

$$\frac{A(1+a)}{B(1-b)} = \frac{A}{B}(1+a+b) \text{ (nearly), and not } \frac{A(1+a)}{B(1+b)}, \text{ i.e. } \frac{A}{B}(1+a-b).$$

as might at first be supposed.

Ex. 1.—*The numbers* 1·7324 *and* 2·5968 *are known correct to* 4 *places. How far can their product be relied on?*

We have already solved this problem in Ex. 2, § 152. The methods of the present paragraph provide another solution. If the numbers are in defect, their absolute error is in both cases less than ·00005; therefore their relative errors are less than ·00005/1·7324 = ·000029 and ·00005/2·5968 = ·000019, respectively, calculating these to the nearest second significant figure; therefore the relative error of the product is less than ·000029 + ·000019, *i.e.* ·000048; therefore the limit of absolute error of the product is about ·00005 × 1·7324 × 2·5968, the value to one significant figure of which is easily found to be ·0002. If the approximate values are in excess of their true values, it is easily seen that the approximate product is about ·0002 in excess of the true product.

Ex. 2.—*To how many places may* $\frac{3{\cdot}1413 \times 37{\cdot}7896}{{\cdot}1213}$ *usefully be evaluated if the numbers are all of them known accurately only as far as stated?*

The limits of relative errors are

$$\cdot 00005 \div 3{\cdot}1413 = \cdot 000016\ldots;$$

$$\cdot 00005 \div 37{\cdot}7896 = \cdot 000001\ldots;$$

$$\cdot 00005 \div {\cdot}1213 = \cdot 000412.$$

The sum of these limits is ·00043.... Hence the approximate limit of absolute error of the expression is ·00043 × the expression. The first figure of this is readily found to be ·4. It follows therefore that *it is no use to attempt to evaluate the above expression beyond the units figure*, and that this may be 1 in error, from the 4 which is a possible error in the 1st place.

EXAMPLES LXXXI. (*continued*).

Calculate, as briefly as you can, the values correct to 2 significant figures of—

15. (37·1235 − 37·123) ÷ (1) 37·123, (2) 37·1235.

16. (98·7642 − 98·7637) ÷ (1) 98·7642, (2) 98·764.

Determine in the following questions (1) the limit of relative error of each factor to 2 significant figures, (2) the limit of absolute error of the product or quotient to 1 significant figure, it being assumed that each number is known correct to the places given :—

17. $27{\cdot}13 \times 72{\cdot}31$.

18. $19{\cdot}37 \times 65{\cdot}80$.

19. $11{\cdot}82075 \times {\cdot}003961$.

20. $13{\cdot}7305 \times {\cdot}006507$.

21. $53{\cdot}829764 \div$ (*a*) $2{\cdot}56$, (*b*) $2{\cdot}5634$, (*c*) $2{\cdot}563434$, (*d*) ${\cdot}2563434$.

22. $35{\cdot}827964 \div$ (*a*) $6{\cdot}52$, (*b*) $6{\cdot}5234$, (*c*) $6{\cdot}523443$, (*d*) ${\cdot}6523443$.

If each number is known correct to the figures given, determine the number of significant figures to which we can correctly evaluate :—

23. $\dfrac{3{\cdot}14285714 \times {\cdot}0007692307}{1{\cdot}69230769}$.

24. $\dfrac{31{\cdot}42857142 \times 1{\cdot}076923076}{{\cdot}846153846}$.

SECTION VI.

CONCRETE FRACTIONS.

CHAPTER XIX.

FRACTIONS OF CONCRETE QUANTITIES.

(i.) Vulgar Fractions.

157. The method of finding the value of fractions of concrete quantities has been already referred to (§ 102), and will be further explained by examples.

Ex. 1.—*Find the value of* $\frac{7}{11}$ *of* £3 9s. 8d

	£	s.	d.	
11)	3	9	8	
		6	4	
			7	
	£2	4	4	*Ans.*

Explanation.—To find seven-elevenths of a quantity, we must divide the quantity into 11 equal parts, *i.e.* divide by 11, and take 7 of them, *i.e.* multiply by 7. Thus the answer is £2 4s. 4d.

Note.—It is clear from § 113 that the result would be the same if we had first multiplied by 7 and then divided by 11, but this will generally lead to larger numbers and increased work.

Ex. 2.—*Find the value of* $8\frac{5}{11}$ *of* £14 0s. 7d. + $3\frac{2}{7}$ *of* 5 *guineas.*

	£	s.	d.			£	s.	d.	
11)	14	0	7	× 8	=	112	4	8	
	1	5	$6\frac{1}{11}$	× 5	=	6	7	$6\frac{5}{11}$	
7)	5	5	0	× 3	=	15	15	0	
		15	0	× 2	=	1	10	0	
						£135	17	$2\frac{5}{11}$	*Ans.*

Note.—It is sometimes, though rarely, advisable to reduce the mixed number to an improper fraction before dealing with it. Thus: £17 2s. 6d. × $6\frac{10}{13}$ is best worked £17 2s. 6d. × $\frac{88}{13}$ *i.e.* £17 2s. 6d. × 8 × 11 ÷ 13. The reason for this is that the first multiplication by 8 will get rid of the shillings and pence. The gain, however, is very small.

EXAMPLES LXXXII.

Find the value of—

1. $\frac{11}{16}$ of 1 guinea. **2.** $2\frac{8}{9}$ of 21s. **3.** $\frac{13}{17}$ of £1 6s. 11d

4. $\frac{9}{50}$ of £2 7s. 6d. **5.** $\frac{3}{7}$ of £5 18s. 5d. **6.** $\frac{7}{10}$ of £3 6s. 8d.

7. $4\frac{1}{2}$ of £3 7s. 5d. **8.** $3\frac{2}{3}$ of £7 6s. $8\frac{1}{2}$d. **9.** $\frac{3}{8}$ of £5 13s. 9d.

10. $\frac{9}{22}$ of £9 8s. 10d. **11.** $\frac{5}{21}$ of £6 11s. 3d. **12.** $\frac{8}{15}$ of £3 8s. 3d.

13. £13 15s. 4d. × $4\frac{5}{8}$. **14.** £18 17s. × $4\frac{7}{8}$.

15. £2 10s. $6\frac{3}{4}$d. × $3\frac{3}{4}$. **16.** £10 11s. $2\frac{1}{4}$d. × $3\frac{3}{4}$.

17. $\frac{5}{28}$ of 10s. 6d. + $\frac{5}{8}$s. + $\frac{1}{12}$ of 2s. 6d.

18. £$3\frac{5}{8}$ + $9\frac{3}{7}$ guineas + $2\frac{4}{5}$ half-crowns.

19. $\frac{2}{5}$ of $\frac{3}{8}$ of a sovereign − $\frac{1}{7}$ of $\frac{5}{6}$ of a guinea.

20. $\frac{1}{4}$ of a guinea + $\frac{1}{8}$ of a crown − $\frac{3}{4}$ of 2d.

21. £$\frac{3}{5}$ + $\frac{5}{16}$s. + $\frac{2}{9}$ of 1 guinea.

22. $\frac{8}{9}$ of 1 guinea − $\frac{3}{8}$ of 5s. + $\frac{1}{12}$ of £1 − $\frac{1}{28}$ of 2 guineas.

23. $1\frac{5}{11}$ of £7 14s. $5\frac{1}{2}$d. + $\frac{5}{21}$ of £6 2s. 6d.

24. $1\frac{5}{13}$ of £5 5s. 1d. + $\frac{6}{7}$ of £12 13s. $5\frac{1}{2}$d.

25. $1\frac{9}{22}$ of £12 7s. 2d. + $1\frac{10}{11}$ of £6 1s. 4d.

26. $9\frac{3}{5}$ of £2 17s. 1d. + $3\frac{5}{9}$ of £2 17s. 9d.

27. $\frac{13}{15}$ of £1 + $1\frac{4}{7}$ of 10s. 6d. + $3\frac{15}{16}$ of £1 1s. 4d.

28. $\frac{3}{4}$ of £5 7s. 11d. + $\frac{4}{3}$ of £20 5s. $9\frac{3}{4}$d. − $\frac{16}{9}$ of £11 2s. 6d.

29. £$3\frac{43}{96}$ + $(9\frac{1}{12} - 2\frac{7}{8})$ of 5s.

30. $\dfrac{3\frac{1}{12} - 1\frac{5}{8} \text{ of } 1\frac{2}{7}}{4\frac{1}{7} - 2\frac{4}{5} \text{ of } 1\frac{1}{6}}$ of £39 17s. 4d. **31.** $\dfrac{4\frac{4}{5} - 1\frac{1}{4} \times (2\frac{1}{8} - \frac{5}{6})}{(4\frac{4}{5} + 1\frac{1}{4}) \div 2\frac{1}{8} - \frac{5}{6}}$ × £9 2s. 4d.

32. $\frac{1664}{1408}$ of $\left\{\frac{7}{10} \text{ of £}3\frac{3}{14} + 6\frac{2}{3} \text{ of £3 0s. 9d.} - 4\frac{13}{31} \text{ of £3 2s.}\right\}$.

EXAMPLES LXXXIII.

Find the value of—

1. $\frac{3}{4}$ of 1 ton 13 cwts. 15 lbs. **2.** $\frac{11}{14}$ of 3 cwts. 1 qr.

3. $\frac{2}{49}$ cwt. + $\frac{3}{7}$ lb. **4.** $1\frac{3}{5}$ tons + $1\frac{5}{8}$ cwts. + $1\frac{2}{3}$ lbs.

5. $3\frac{1}{5}$ of 6 cwts. 1 qr. 1 lb. 9 oz. + $5\frac{1}{3}$ of 3 cwts. 1 qr. 10 lbs. 1 oz.

6. $2\frac{2}{5}$ of 7 cwts. 3 qrs. 7 lbs. + $3\frac{7}{9}$ of 3 tons 5 cwts. 10 lbs.

7. $\dfrac{7\frac{1}{8}}{6\frac{1}{3}}$ of 3 tons 2 cwts. 14 lbs. **8.** $\dfrac{71\frac{1}{14}}{41\frac{17}{28}}$ of 1 qr. 1 lb. 2 oz

9. $\frac{2}{7}$ of 3 mls. − $\frac{8}{21}$ of a furlong + $\frac{24}{35}$ pl. + $2\frac{2}{15}$ yds.

10. $\frac{3}{25}$ of a mile + $\frac{16}{21}$ of a furlong − $11\frac{19}{35}$ pls. in feet.

11. $\frac{3}{29}$ × ($\frac{7}{33}$ of 1 ml. 5 furls. + $\frac{5}{12}$ of 4 pls. 2 yds. + $\frac{12}{17}$ of 2 yds. 2 ft. 6 ins.) in yards.

12. $\frac{17}{63}$ × ($\frac{7}{33}$ of 17 pls. 4 yds. 1 ft. 9 ins. + $\frac{5}{13}$ of 29 pls. 5 yds. 2 ft. 8 ins. − $\frac{8}{15}$ of 23 pls. 2 yds. 9 ins.) in inches.

To express one concrete quantity as a fraction of another concrete quantity of the same kind, it is only necessary to reduce both quantities to the same denomination, and then the number of units in the first gives the numerator, and the number of units in the second gives the denominator, of the fraction required. This fraction must then be reduced to its lowest terms.

The method is the same as in § 74, which should be carefully studied again. Note particularly the remark in Ex. 1 of § 74.

Ex. 1.—*What fraction of* £1 *is* 13s. 6d. *?*

Reduce both to sixpences. 13s. 6d. = 27 sixpences; £1 = 40 sixpences. *Ans.* $\frac{27}{40}$.

Ex. 2.—*What fraction of* £3 13s. 4d. *is* £2 13s. 4d. *?*

$\left.\begin{array}{l}\text{£2 13s. 4d.} = 53\frac{1}{3}\text{ shillings}\\ \text{£3 13s. 4d.} = 73\frac{1}{3}\text{ shillings}\end{array}\right\}$; therefore the answer is $\frac{53\frac{1}{3}}{73\frac{1}{3}} = \frac{160}{220} = \frac{8}{11}$.

Ex. 3.—*Reduce* $\frac{11}{12}$ *of* £7 5s. 6d. *to the fraction of* $3\frac{5}{7}$ *of* £12 13s. 9d.

By the methods of § 156, we find $\frac{11}{12}$ of £7 5s. 6d. = £6 13s. $4\frac{1}{2}$d., and $3\frac{5}{7}$ of £12 13s. 9d. = £47 2s. 6d. The highest denomination in which we can express both these sums is $1\frac{1}{2}$d. Reducing, we get

£6 13s. $4\frac{1}{2}$d. = 1067 times $1\frac{1}{2}$d. and £47 2s. 6d. = 7540 times $1\frac{1}{2}$d.

Therefore the fraction required is $\frac{1067}{7540}$, which is found to be in its lowest terms. *Ans.* $\frac{1067}{7540}$.

EXAMPLES LXXXIV.

Express—

1. $2\frac{1}{2}$d. as the fraction of £1.

2. 1s. $0\frac{1}{2}$d. as the fraction of £1.

3. $2\frac{1}{2}$ guineas as the fraction of £$1\frac{3}{4}$.

4. $2\frac{1}{4}$ guineas as the fraction of £$4\frac{1}{2}$.

5. 3s. 4d. as the fraction of £1 3s. 4d.

6. 3s. 6d. as the fraction of £1 5s.

What fraction is—

7. £30 15s. 5d. of £123 1s. 8d.?

8. £1 17s. $1\frac{1}{2}$d. of £22 5s. 6d.?

9. 17s. $0\frac{3}{4}$d. of £1 11s. 6d.?

10. £1 15s. $2\frac{1}{2}$d. of £2 19s. 7d.?

11. 2s. 11d. of £1 14s. $3\frac{1}{4}$d.?

12. 12s. $4\frac{3}{4}$d. of £1 10s. $1\frac{1}{4}$d.?

13. £1 4s. $0\frac{3}{4}$d. of £33?

14. $\frac{5}{11}$ × £33 13s. 9d. of £26 5s.?

15. 3s. $7\frac{1}{2}$d. of £1 3s. $4\frac{1}{2}$d.?

16. £4 7s. $6\frac{3}{4}$d. of £1 7s.?

17. Reduce $1\frac{1}{4}$ of £3 13s 6d. to the fraction of half-a-guinea.

18. Reduce $2\frac{2}{5}$ of £6 to the fraction of £1 13s.

19. Reduce $\frac{2}{9}$ of $\frac{5}{13}$ of 19s. 6d. to the fraction of $\frac{3}{5}$ of $\frac{4}{17}$ of £1 8s. 4d.

20. Reduce $\frac{3}{7}$ of $\frac{5}{11}$ of £8 6s. 10d. to the fraction of $\frac{4}{9}$ of $\frac{5}{13}$ of £12 13s. 6d.

21. Express $\frac{69}{70}$ of £2 17s. $7\frac{1}{4}$d. + $\frac{65}{66}$ of £2 14s. $3\frac{3}{4}$d. as the fraction of $\frac{125}{126}$ of £5 3s. $8\frac{1}{4}$d. + $\frac{143}{144}$ of £5 18s. 6d.

22. Express $\frac{68}{69}$ of £2 16s. $0\frac{3}{4}$d. + $\frac{64}{65}$ of £2 12s. $9\frac{3}{4}$d. as the fraction of $\frac{124}{125}$ of £5 1s. $6\frac{3}{4}$d. + $\frac{140}{143}$ of £5 16s. $2\frac{1}{4}$d.

23. If $\frac{35}{36}$ of £9 12s. be taken from $\frac{48}{49}$ of £13. 1s. 4d., what fraction of $\frac{17}{25}$ of £6 13s. 4d. will be left?

24. If $\frac{17}{49}$ of £20 8s. 4d. be taken from $\frac{45}{64}$ of £26 13s. 4d., what fraction of $\frac{14}{15}$ of six times £6 5s. will be left?

25. Reduce $(9\frac{19}{365} - 7\frac{17}{165})$ of £5233 3s. 9d. to the fraction of $\frac{1103}{1105}$ of £10939 10s.

EXAMPLES LXXXV.

What fraction is—

1. 1 cwt. 1 lb. 12 oz. of a ton? **2.** 3 qrs. 14 lbs. of 3 cwts. 1 qr.?

3. 1 ton 4 cwts. of 15 cwts. 1 qr. 20 lbs.?

4. 4 cwts. 5 lbs 1 oz. of 5 cwts. 3 qrs. 15 lbs.?

5. 1 ton of 3 cwts. 3 qrs. 21 lbs.? **6.** 3 cwts. 2 qrs. 3 lbs. of 1 ton?

7. $11\frac{2}{3} \times 14$ tons 4 cwts. 2 qrs. $18\frac{2}{3}$ lbs. of 21 tons 7 cwts.?

8. $\frac{2}{7} \times 5$ cwts. 10 lbs. of 2 cwts. 1 qr. 4 lbs. 8 oz.?

9. $\frac{4}{7} \times 2$ cwts. 2 qrs. 5 lbs. of $\frac{8}{9} \times 2$ cwts. 1 qr. 4 lbs. 8 oz.?

10. $\dfrac{13\frac{2}{3} \div 1\frac{1}{2}}{1 - \dfrac{\frac{2}{3} \text{ of } \frac{4}{5}}{6}} \times 5$ cwt. of $(\frac{2}{3} \times 4\frac{1}{2} \div 4) \times 1$ ton?

11. Reduce 11 furls. to the fraction of 2 mls.

12. Express $\frac{2}{3}$ of a furlong as the fraction of $\frac{3}{5}$ of a mile.

13. What part of 27 fathoms is $1\frac{1}{14}$ of $12\frac{5}{6}$ yds.?

14. What fraction of 1 furl. 28 pls. is 30 pls. 5 yds.?

15. Reduce 7 hours 12 minutes to the fraction of a day.

16. Express 14 hours 15 minutes as the fraction of $3\frac{1}{2}$ days

(ii.) Decimalisation of Money.

158. In the application of Arithmetic to problems of commerce it is of the utmost importance to be able to convert any sum of money rapidly into the decimal of a pound. The rules of this paragraph should be practised until they are entirely familiar.

Rule 1. *Decimalisation of shillings.*—Since 1s. = £$\frac{1}{20}$ = £·05, *any number of shillings may be decimalised* at once *by multiplying by 5 and marking off two decimal places.*

Ex. 1.—7s. = £·35; £12 18s. = £12·90; and so on.

Rule II. *Decimalisation of farthings.*—Since 6d. $= £\frac{1}{40}$ $= £\cdot025$, and 6d. contains 24 farthings, therefore 1 farthing $= \frac{1}{24}$ of $£\cdot025 = £\cdot001\frac{1}{24}$. Hence *any number of farthings may be decimalised by considering them as in the third decimal place and adding one twenty-fourth of that number.*†

Ex. 2.—$1\frac{1}{2}$d. = 6 farthings = $£\cdot006\frac{6}{24} = \cdot£006\frac{1}{4} = £\cdot00625$;
$3\frac{1}{4}$d. = 13 farthings = $£\cdot013\frac{13}{24} = £\cdot0135416\dot{}$.

Putting these two rules together, we can immediately decimalise any sum whatever. The method of performing the mental work must be carefully studied in the following examples. [See also § 162.]

Ex. 3.—*Express* £7 14s. 1d. *as a decimal of* £1.

£7·70	represents the decimal of £7 14s. written at once by Rule I.;	
·004	,,	·001 for each farthing;
·00016̇	,,	$\frac{1}{24}$ of the previous line
£7·70416̇	,,	£7 14s. 1d.

The only difficulty lies in dividing mentally by 24. Advanced students will be able to do this at once, but most will find it easier to divide first by 2 and then by 12. Thus, in the previous example we should divide the second line *mentally* by 2, getting ·002, and, keeping this in our head, we should divide it by 12, and thus write down the third line at once.† The three are then added together, and with a little practice it is easy to do *all* the steps mentally and write the final answer at once.

Ex. 4.—*Express* £4 17s. $9\frac{3}{4}$d. *as a decimal of* £1.

The work is as follows:—

£4·85	Multiply 17 by 5 (Rule I.).
39	farthings. Mentally divide by 2, getting 195.
1625	*i.e.* result of dividing 195 by 12.
£4·890625	*Ans.*

It is often sufficient to express three decimal places of the answer, and the rule may be thus stated:—

Rule III. To express a sum of money as a decimal of a pound correctly to three places—

Write the pounds followed by a decimal point;
Multiply the shillings by 5, and consider the product as in the second decimal place;
Reduce the pence to farthings, and consider the result as in the third decimal place;

† These devices are due to Messrs. Sonnenschein and Nesbitt.

If the number of farthings is 12 *or more, add* 1 *to the third place;*

If the number of farthings is 36 *or more, add* 2 *to the third place.*

For, if the number of farthings is, say 17, the fraction of the third place which the exact rule (Rule II.) requires is $\frac{17}{24}$. This is greater than $\frac{1}{2}$; therefore, to find the *nearest* third place we must increase the third place by 1 (see § 149). So, if it is $\frac{37}{24} = 1\frac{13}{24}$, this is greater than $1\frac{1}{2}$ and we add 2 to the third place.

Ex. 5.—*Express* £5 13s. $4\frac{3}{4}$d. *as a decimal of* £1 *to three places.*

We first deal with the farthings. There are 19. This is more than 12 ("12 or more"), therefore we increase it by 1, making 20. Write the **0** and carry the 2. Multiply the shillings by 5—65, and add in the 2 we have carried—**67**, and we obtain the answer £5·670.

EXAMPLES LXXXVI. (MENTAL.)

(N.B.—The answers in the following examples are given *exactly*. You are recommended first to work them correct to three places by Rule III., and then to work them exactly by Rule II.)

Other examples may be obtained from the answers to Exx. lxxxvii.

Decimalise—

1. £1 18s., £2 17s., £3 16s. **2.** £3 15s., £4 14s., £5 13s.

3. £4 7s. 3d., £3 15s. 9d., £2 8s. 6d.

4. £2 17s. 3d., £3 16s. 9d., £4 18s. 6d.

5. £9 6s. $2\frac{3}{4}$d., £8 11s. $4\frac{1}{2}$d., £2 19s. 6d.

6. £4 12s. $8\frac{1}{4}$d., £5 11s. $6\frac{1}{2}$d., £7 9s. 10d.

7. £3 13s. 4d., £5 12s. $9\frac{1}{4}$d., £2 15s. $8\frac{1}{2}$d.

8. £5 6s. 8d., £1 7s. $2\frac{1}{4}$d., £8 3s. $0\frac{1}{2}$d.

9. £9 0s. $1\frac{1}{2}$d., £5 6s. $7\frac{1}{2}$d., £4 17s. $5\frac{1}{4}$d.

10. £5 0s. $0\frac{3}{4}$d., £6 8s. $11\frac{3}{4}$d., £6 10s. $10\frac{1}{2}$d.

159. To reconvert a decimal of £1 into shillings and pence. —The process is similar to that of reduction.

Ex.—*Express* £1·234567 *in shillings and pence.*

£**1**·234567	
20	
s. **4**·691340	
12	
d. **8**·29608	

The process explains itself. The multiplication is not applied to the darker figures (**1**, **4**). The answer is £1 4s. 8·29608d. or £1 4s. $8\frac{1}{4}$d. nearly. If the answer is required only to the nearest farthing, we retain only four figures of the decimal, *i.e.* write it at once as £1·2346.

It is important, however, to be able to perform this operation mentally, so as to bring the answer correct to the nearest farthing. This object is gained by

Rule IV. To convert a decimal of a pound into shillings, &c., correct to the nearest farthing—

Write the decimal correct to four places (§ 149);
Remove from the first two places the largest possible multiple of 5, *and divide this multiple by* 5 *to find the shillings;*
Mark off one decimal place in the number remaining, subtract $\frac{1}{25}$*th of itself, and the nearest integer to the remainder is the number of farthings.*

This rule may be roughly proved as follows:—£(·001) = £$\frac{1}{1000}$ = $\frac{960}{1000}$ farthings = $(1-\frac{40}{1000})$ farthings = $(1-\frac{1}{25})$ farthings; therefore £(·00N) = $\left(N-\frac{N}{25}\right)$ farthings; therefore any number in the third decimal place gives the same number of farthings less one twenty-fifth. If we do not take the fourth place into account, there is a *possible* error of a farthing. The advanced student should examine this, and find the explanation.

Ex.—£1·234567 = £**1**·2346 correct to four places. The largest multiple of 5 in the first two places (23) is 20; therefore 20 ÷ 5, *i.e.* **4** is the number of shillings. Remove this 20 and we have 346 left. Mark off one decimal place, 34·6. Subtract $\frac{1}{25}$ of itself, *i.e.* 1·4, and we get 33·2. The nearest integer is 33; therefore we have 33 farthings, *i.e.* **8**$\frac{1}{4}$d. *Ans.* £1 4s. 8$\frac{1}{4}$d.

After a little practice there is no great difficulty in doing the whole of this mentally.

EXAMPLES LXXXVII.

(The advanced student should do all these mentally. Other examples may be obtained from the answers to Exx. lxxxvi.)

Convert into shillings and pence, correct to the nearest farthing, the following decimals of £1:—

1. £2·327, 3·879, 1·122.
2. £2·345, 3·978, 1·221.
3. £3·22, 5·026, 8·548.
4. £4·33, 6·062, 5·848.
5. £5·1446, 1·6643, 3·2709.
6. £1·1918, 2·1445, 9·9375.
7. £8·67054, 7·37186, 5·64163.
8. £9·48021, 8·87194, 9·62896.
9. £9·334195, 8·293038, 6·849691.
10. £7·130703, 6·399648, 5·390761.

Find the cost of 1000, and also of 100,000, articles at—

11. 1s. 0$\frac{3}{4}$d. each.
12. 11$\frac{1}{4}$d. each.
13. 7s. 7$\frac{1}{2}$d. each.
14. 6s. 9$\frac{3}{4}$d. each.

160. To express one quantity as a decimal of another quantity of the same kind, we act as in § 157, expressing both quantities in the same denomination. The fraction obtained by the method of § 157 is then reduced to a decimal.

Ex. 1.—*Express* 5s. *as a decimal of* 6s. 8d.

Reduce to fourpences: 5s. = 15 times 4d.; 6s. 8d. = 20 times 4d. Therefore the fraction is $\frac{15}{20}$, *i.e.* $\frac{3}{4}$, and this becomes the decimal ·75. *Ans.*

Or, $$\frac{5\text{s.}}{6\text{s. }8\text{d.}} = \frac{5\text{s.}}{6\frac{2}{3}\text{s.}} = \frac{5}{6\frac{2}{3}} = 5 \times \frac{3}{20} = \frac{15}{20} = \frac{3}{4} = \cdot 75.$$

Ex. 2.—*Express* £3 4s. $5\frac{1}{4}$d. *as a decimal of* £5.

By the rules of § 158, £3 4s. $5\frac{1}{4}$d. = £3·221875; therefore the decimal required is 3·221875 ÷ 5 = ·644375. *Ans.*

For another method, in which the working is shown in full, see § 163, Ex. 2.

Ex. 3.—*Express* £3 4s. $6\frac{1}{4}$d. *as a decimal of* £4 3s. $2\frac{1}{2}$d. *to four places.*

Reduce both to decimals by § 158 and the problem becomes: Find, to four places of decimals, the value of $3{\cdot}2260416 \div 4{\cdot}160416$. Referring to § 155, we see that, to find the quotient to four places, we shall need six significant figures in the dividend. The work proceeds as in the margin, using the fullest contraction. The answer is ·7754.

```
        ·7754
4160416 ) 322604
           31375
            2252
             172
               6
```

Note.—It is often shorter, however, to reduce both sums to farthings

EXAMPLES LXXXVIII.

In the following examples, reduce the first of the two given quantities to the decimal of the second:—

1. 2s. 6d. : £2.
2. 3s. 4d. : £3.
3. 7s. 6d. : £1 13s. 4d.
4. £1 3s. 4d. : £1 2s. 6d.
5. 18s. $4\frac{1}{2}$d. : 2 guineas.
6. £3 18s. 9d. : 10 guineas.
7. 11s. $0\frac{3}{4}$d. : £8 17s.
8. 11s. $7\frac{1}{2}$d. : £10 9s. 3d.
9. 4s. $3\frac{1}{2}$d. : £1 10s. $0\frac{1}{2}$d.
10. £3 12s. $2\frac{1}{4}$d. : £3 17s.
11. £11 12s. $4\frac{1}{2}$d. : £20.
12. £8 12s. $6\frac{3}{4}$d. : £5.
13. $\frac{3}{7}$ of £10 4s. $5\frac{1}{2}$d. : £5.
14. $\frac{5}{6}$ of £3 2s. $7\frac{1}{2}$d. : £5.
15. 1·258 of £1 9s. $4\frac{1}{2}$d. : £5.
16. ·3275 of $5\frac{3}{8}$ of 2 guineas : £6 11s.
17. $\cdot\dot{0}\dot{6}$ of a guinea : £1.
18. 15s. $7\frac{1}{2}$d. : $\frac{1}{27}$ of £168 15s.
19. $2\frac{1}{5}$ of £6 16s. $4\frac{1}{4}$d. : £8 3s. $7\frac{1}{2}$d.
20. $3\frac{37}{43}$ of £3 12s. $6\frac{3}{4}$d. : £35 0s. $3\frac{3}{4}$d.

Express the first of the following sums of money as a decimal (correct to four places) of the second:—

21. £4 6s. $7\frac{1}{4}$d. : £6 1s. 3d.
22. £2 2s. $10\frac{3}{4}$d. : £9 18s. 9d.
23. £7 7s. $5\frac{1}{4}$d. : £9 12s. 7d.
24. £8 5s. $10\frac{1}{4}$d. : £9 9s. $7\frac{1}{4}$d.

161. Ex. 1 —*Find the value of* $1\cdot3\dot{4}2857\dot{1}$ *of* £8 16s. $5\frac{1}{2}$d.

	£	s.	d.
7)	8	16	$5\frac{1}{2}$
5)	1	5	$2\frac{1}{2}$
		5	$0\frac{1}{2}$
			12
	3	0	6
	8	16	$5\frac{1}{2}$
	£11	16	$11\frac{1}{2}$ *Ans.*

Recurring decimals in such questions should always be reduced to fractions.

$$1\cdot3\dot{4}2857\dot{1} = 1\cdot3\tfrac{3}{7} = 1\tfrac{24}{70} = 1\tfrac{12}{35}.$$

The value of $1\frac{12}{35}$ of £8 16s. $5\frac{1}{2}$d. is found at once.

If an approximate answer only is required, use Method ii. of Ex. 3. An exact answer never *ought* to be required to a question of this kind.

Ex. 2.—*Find the value of* 1·2375 of £2 6s. 8d.

1·2375
2
2·4750
·4125
£2·8875

Here the sum of money reduces to a very simple fraction of a £, and we proceed thus:

$$£2\tfrac{1}{3} \times 1\cdot2375 = £(1\cdot2375 \times 2\tfrac{1}{3}) = £2\cdot8875$$
$$= £2\ 17s.\ 9d.\ (\S\ 159) \quad \textit{Ans.}$$

Ex. 3.—*Find the value of* 1·34567 *of* £3 8s. $4\frac{1}{2}$d.

Method (i.).—

	1·34567	$= A$
	3	
	£4·03*701*	
	20	
	·74*020*	
	10·76*536*	$= A \times 8$
s.	**11**·50*556*	
	12	
	6·06*672*	
	5·38*268*	$= A \times 4$
	·67*2835*	$= A \times \frac{1}{2}$
d.	**12**·122*235*	

Ans. (to the nearest farthing), £4 11s. 12d., *i.e.* £4 12s.

If the decimal of a penny were not required, the figures printed in italics in the above work might have been omitted. You should practise the work, omitting these figures.

Method (ii.). — Converting the money to the decimal of £1, it becomes 3·41875. If the answer is required *exactly*, this is multiplied into 1·34567, and the product, which is a decimal of £1, is reduced by the method of § 159. But Method (i.) should always be used if an exact answer is required. If an answer to the nearest farthing only is required, the multiplication is effected so as to give four places in the answer, *i.e.* keeping five figures.

£3·41875
1·34567
3·41875
1·02562
13675
1709
205
24
£4·6005

Reconverting into shillings and pence, £4 12s.

Method (iii.)—The £3 8s. $4\frac{1}{2}$d. might be reduced to halfpennies, multiplied by 1·34567, and the product again reduced to £ s. d.

It is impossible to give infallible rules for choice between these methods. Only experience can decide; but, speaking generally, we may say that for recurring decimals the method is that of Ex. 1; for sums

of money involving simple fractions of a pound that of Ex. 2; for complicated cases when an exact answer is required, Ex. 3 (i.); when an approximate answer only, Ex. 3 (ii.).

162. The following rule for the *Decimalisation of Money* will be found simpler by many, for mental work, than that explained in § 158.

The first decimal place is the number of complete florins.
The next two places (2nd and 3rd) are the number of farthings in the remainder increased by 1 *for every complete* 24.
The next two places (4th and 5th) are found by subtracting from the last two any complete multiple of 25 *and multiplying the remainder by* 4 *again, adding* 1 *for every complete* 24.
This process is to be repeated until the decimal recurs or terminates.

Ex.—Decimalise 17*s.* 3¼*d.* There are 8 florins. We write **·8** for the first place. The remainder is 1s. 3¼d., *i.e.* 61 farthings. We increase this by 2, for there are 2 complete 24's in 61, and write **63** for the next two places. For the next two we subtract 50 from 63, getting 13, multiply by 4, 52, and increase by 2, **54**, &c. *Ans.* **·86354** ....

The proof of this rule is left as an exercise. It depends upon the repeated application of the identity $\frac{1}{24} = (1\frac{1}{24}) \times \frac{4}{100}$.

EXAMPLES LXXXIX.

Find the value of—

1. ·125 of £8 8s. 8d.
2. 0·125 of £8 4s. 8d.
3. 2·25 of £3 7s. 4d
4. 17·25 of 7s. 6d.
5. ·0625 of £3 4s.
6. ·0625 of a guinea.
7. ·06875 of £20.
8. ·3125 of £4 10s.
9. ·1875 of £4 12s. 8d.
10. ·725 of £114 16s. 8d.
11. $\cdot\dot{1}\dot{6}$ of £3 4s. 10½d.
12. $\cdot\dot{2}\dot{7}$ of £1.
13. ·421875 of £5.
14. 22·6875 of £1 16s. 8d.
15. ·21125 of £5.
16. ·0053125 of £20.
17. ·15625 of £2 10s.
18. 1·62 of £26·09375.
19. $\cdot 02\dot{4}2857\dot{1}$ of 100 guineas.
20. $\cdot\dot{3}$ of $\cdot\dot{6}\dot{3}$ of £6 2s. 3d.
21. $\cdot 058\dot{3}$ of £1.
22. $\cdot 5\dot{2}\dot{3}$ of £2 1s. 3d.
23. ·1636 of 16s. 8d.
24. ·3641 of £5.
25. ·017 of £2 1s. 8d.
26. $2\cdot 3\dot{0}\dot{9}$ of £1 7s. 6d.

Find, to the nearest farthing, the value of—

27. ·8773713 of £5 12s. 9¼d.
28. ·9670797 of £4 17s. 5¼d.
29. ·1714339 of £2 15s. 8½d.
30. ·2725378 of £8 3s. 0½d.

(iii.) Concrete Decimals (Weights and Measures).

163. The methods used in dealing with decimals of the various weights and measures are so similar to those employed

for money that it will be sufficient to illustrate them by examples.

Ex. 1.—*Find the value of* ·78125 *of a ton.*

```
 ·78125 tons
      20
 15·6250  cwts
       4
  2·500   qrs.
     28
 14       lbs.
```

The process is similar to that of reduction. (*Cf.* § 159.)

The successive multiplications are not applied to the figures printed in darker type (**15**, **2**) which give the answer, 15 cwts. 2 qrs. 14 lbs.

If the decimal of 1 qr. is a long one, the multiplication by 28 is done in two parts: first by 4, then by 7.

Note.—For a recurring decimal, convert into vulgar fractions. Thus:

$\cdot 3\dot{5}7142\dot{8}$ of a ton = $\cdot 3\frac{4}{7}$ of a ton = $\frac{25}{70}$ of a ton = $\frac{5}{14}$ ton.

Then, see § 156.

Ex. 2.—*Express* 3 *cwts.* 1 *qr.* 6 *lbs.* 2 *oz. as a decimal of* 5 *tons.*

```
16 { 4 | 2      oz.
   { 4 |  ·5
28 { 4 | 6·125  lbs.
   { 7 | 1·53125
     4 | 1·21875  qrs.
    20 | 3·3046875  cwts.
     5 |  ·165234375  tons
       |  ·033046875 of 5 tons
```

The process is an exact converse of that in Ex. 1. By dividing by 16 we reduce the 2 oz. to decimal of 1 lb.; it is ·125 lb. We thus have **6**·125 lbs., the **6** being now inserted from the given sum. This is divided first by 4 and then by 7 to reduce to quarters; and so on.

Ex. 3.—*Find the value of* ·87225 *of* 2 *tons* 4 *cwts.* 3 *qrs.* 8 *lbs.*

```
 ·87225          = A
      2
 1·74450 tons    = 2 × A
     20
 14·89000 cwts.
 3·48900 cwts.   = 4 × A
 18·37900 cwts.
      4
 1·51600 qrs.
 2·61675 qrs.    = 3 × A
 4·13275 qrs.
      4
 ·53100
      7
 3·71700 lbs.
 6·97800 lbs.    = 8 × A
 10·695  lbs.
```

This line is ·87225 of 2 tons.

This line is the value in hundredweights of the decimal part of the last.

This line is ·87225 of 4 cwts.

Thus the answer is

1 ton **18** cwts. **4** qrs. **10·695** lbs.

i.e. 1 ton 19 cwts. 0 qrs. 10·695 lbs.

If the answer had been required "correct to the nearest pound," the figures printed in italics might have been omitted. Reproduce this work for yourself, omitting these figures from the first.

Note.—For a recurring decimal, reduce to vulgar fractions *unless an approximate answer is required.*

See also § 161, Ex. 3, Method ii.

Ex. 4.—*Express* 3 *cwts.* 1 *qr.* 7 *lbs. as a decimal of* 4 *cwts.* 2 *qrs.* 21 *lbs.*

Since the decimalisation of most weights and measures can only be accomplished by the method of Ex. 2, we cannot conveniently use the method of Ex. 3, § 160, and must resort to that of Ex. 1, § 160, and reduce both quantities to the same common denomination. We choose as high a denomination as possible, in this case 7 lbs., and the work proceeds as follows:—

cwts.	qr.	lbs.	cwts.	qr.	lbs.
3	1	7	4	2	21
13 qrs.			18 qrs.		
53 (7-lbs.)			75 (7-lbs.)		

Hence the required fraction is $\frac{53}{75}$, or, in decimals, $\cdot 70\dot{6}$.

EXAMPLES XC.

Find the value (in tons ... oz.) of—

1. ·125 ton. **2.** ·375 cwt. **3.** $\cdot\dot{5}7142\dot{8}$ qr.
4. $\cdot\dot{2}8571\dot{4}$ cwt. **5.** ·24140625 ton. **6.** ·04375 ton.
7. 199·0625 qrs. **8.** ·123725 ton. **9.** ·8125 of 2 tons 4 cwts.
10. ·22 of 3 qrs. 15 lbs. **11.** ·034375 of 2 tons.
12. ·325 of $1\frac{1}{2}$ tons.

Reduce the first of the quantities named in Exx. 13–28 to the decimal of the second:—

13. 1 qr. $10\frac{1}{2}$ lbs. : 1 cwt. **14.** 3 qrs. 3 lbs. 8 oz. : 1 cwt.
15. 17 cwts. 2 qrs. 7 lbs. : 1 ton.
16. 16 cwts. 3 qrs. 10 lbs. 8 oz. : 1 ton.
17. 3 cwts. 2 qrs. 14 lbs. : $\frac{1}{2}$ ton.
18. ·1357 of 1 cwt. 2 qrs. 11 lbs. : 1 oz.
19. ·03042 of 3 tons 15 cwts. 75 lbs. : 1 oz.
20. ·0752 of 1 ton 3 cwt. $15\frac{1}{2}$ lbs. : 1 lb.
21. $\frac{5}{7}$ of ·476 of 3 cwts. 1 qr. 7 lbs. $2\frac{1}{2}$ oz. : 1 oz.
22. $\frac{11}{17}$ of $1\cdot\dot{1}\dot{3}$ of 2 cwts. 5 lbs. 8 oz. : 1 ton.
23. $\cdot 03\dot{9}\dot{5}$ of $5\frac{1}{2}$ lbs. : $\frac{7}{11}$ cwt.
24. ·0125 of 4 tons 19 cwts. 6 lbs. : 3 cwts. 1 qr. 11 lbs.
25. 6 cwts. 1 qr. 4 lbs. : 11 cwts. 3 qrs. 4 lbs.
26. 2 cwts. 1 qr. 27 lbs. : 4 cwts. 1 qr. 20 lbs.
27. 11 cwts. 2 qrs. 20 lbs. : 13 cwts. 3 qrs. 3 lbs. (four places).
28. 15 cwts. 3 qrs. 13 lbs. : 16 cwts. 3 qrs. 13 lbs. (four places).

EXAMPLES XCI.

Reduce the first of the quantities named in Exx. 1–8 to the decimal of the second:—

1. $1\frac{1}{2}$ pls. : 5 furls. **2.** 2 mls. 1100 yds. : 1 league.
3. 3 furls. 3 pls. : 3 mls. 24 pls. **4.** 20 pls. : 1 ml.

5. 3 furls. 33 yds. : 1 ml.

6. 3 yds. 2 ft. 2 in. : $2\frac{1}{2}$ pls.

7. 33 yds. : 1 ml.

8. $3\frac{1}{2}$ ins. : $\frac{1}{4}$ ml.

9. Express in feet and inches $\cdot 01\dot{2}\dot{5}$ of 5 mls.

10. Express ·04375 of 5 furls. in poles, yards, feet, &c.

11. Find, in yards, feet, and inches, the value of 2·0637 pls.

12. Express 2·1234 mls. in furlongs, poles, yards, and the decimal of a yard.

13. What decimal of a mile is 3149·9424 ins.?

14. Express $2\frac{1}{8}$ ins. as the decimal of $2\frac{1}{5}$ mls. (six places).

15. Reduce ·5123 of 3 mls. 1 furl. $15\frac{1}{2}$ yds. to inches and the decimal of an inch.

16. Reduce ·6125 of 4 mls. 4 furls. $3\frac{1}{2}$ ft. to inches and the decimal of an inch.

17. Find the value of ·176 of 1 furl. 36 pls. 2 yds. 5 ins.

18. Find the value of 32·156 of 3 mls. 330 yds. in miles, yards, and inches.

EXAMPLES XCII.

1. Express 1 day 4 hours as the decimal of a week.

2. Express 1 hour 20 minutes as the decimal of a day.

3. How many days, hours, &c., are there in ·51623 of a week?

4. Express ·4125 of 1 week 2 days 16 hours in days, hours, &c.

5. Reduce 1·047 of 2 weeks 5 hours to minutes and the decimal of a minute.

6. Reduce 1 hour 1 minute 1 second to the decimal of 10 days.

7. Express 3 days 3 hours as the decimal of 2 hours 15 minutes.

8. Express 1 day 4 hours 9 minutes 36 seconds as the decimal of 9 days 9 hours 16 minutes 48 seconds.

9. Reduce ·4065 of 3 acs. 2 rs. 7 pls. to square feet and the decimal of a square foot.

10. Find the value of ·390625 of 6 acs. 1 r. 24 pls.

11. What decimal fraction of 4 acs. 3 rs. 32 pls. is equal to 1 ac. 3 rs. 8 pls.?

12. What decimal fraction of 7 acs. is 24 pls. 15 sq.yds. 1 sq.ft. 18 sq.ins.?

13. Reduce 5 rs. 17 pls. 15 sq.yds. 1 sq.ft. 18 sq.ins. to the decimal of 5 acs.

14. Find the value, in cubic feet and inches, of $\cdot 226\dot{8}\dot{5}\dot{1}$ of a cubic yard.

15. Reduce ·0438 of 3 bushs. 1 gall. to pints and the decimal of a pint.

16. Reduce ·61251 of 4 bushs. 3 galls. $1\frac{1}{2}$ qts. to pints and the decimal of a pint.

17. Reduce 2 qrs. 5 bushs. 2 pks. 1 gall. 3 qts. 1 pt. to the decimal of a load (= 5 qrs.).

18. What decimal of 1 qr. 5 bushs. 1 pk. is 3 pks. 2 qts. 1 pt.?

19. Express as a decimal 8 dwts. 6 grs. $\div 2\frac{1}{2}$ oz.

20. Express 5 oz. 11 dwts. $13\frac{1}{2}$ grs. as the decimal of $2\frac{1}{2}$ lbs.

21. Find the value of ·073 of 7 lbs. 5 oz. in ounces, pennyweights, grains, and the decimal of a grain.

22. Find the value of ·021875 of 212 lbs. Troy.

23. Express 1·403 of 3 lbs. 2 oz. $3\frac{1}{2}$ dwts. in grains and the decimal of a grain.

24. Express 1 oz. Avoirdupois as the decimal of 1 oz. Troy.

(iv.) Complex Concrete Decimals, &c.

164. Complicated combinations of the expressions dealt with in the previous sections of this chapter may be made, though they never occur in practical life. The methods of dealing with them will be shown by two examples.

Ex. 1.—*Find the value of* £·634375 + ·025 *of* 25s. + ·31$\dot{6}$ *of* 30s.

$$\begin{aligned}
\text{Expression} &= (\cdot 634375 \times 20)\text{s.} + (\cdot 025 \times 25)\text{s.} + (\cdot 31\dot{6} \times 30)\text{s.} \\
&= (\cdot 634375 \times 20 \quad + \cdot 025 \times 25 \quad + \cdot 31\tfrac{2}{3} \times 30) \text{ shillings} \\
&= (12\cdot 6875 + \cdot 625 + 9\cdot 5) \text{ shillings} \\
&= \mathbf{22}\cdot 8125 \text{ shillings} \\
&= \text{£}1\ 2\text{s. } 9\tfrac{3}{4}\text{d.}
\end{aligned}$$

$$\left[\begin{array}{ll} \cdot 8125 & \text{shillings,} \\ \mathbf{9}\cdot 75 & \text{pence,} \\ \mathbf{3}\cdot 00 & \text{farthings.} \end{array}\right.$$

If the decimals, or fractions, involved be very simple, it may be best to reduce each expression separately, as in Ex. 2, instead of bringing them all to a common denomination, as in Ex. 1.

Ex. 2.—*Find the value of* $\frac{5}{6}$ *of* £2 9s. 6d. − ·6 *of* £4 + $2\frac{1}{2}$ *of* $3\frac{1}{2}$ *of* 21s.

		£	s.	d.
$\frac{5}{6}$ of £2 9s. 6d. =	5×8s. 3d. =	2	1	3
$2\frac{1}{2} \times 3\frac{1}{2} \times 21\text{s.} = \frac{5}{2} \times \frac{7}{2} \times 21\text{s.} = \frac{5 \times 7 \times 21}{4}\text{s.} = \frac{735}{4}\text{s.} = 183\frac{3}{4}\text{s.} =$		9	3	9
		11	5	0
·6 of £4 = £2·4 =		2	8	0
	Ans.	£8	17	0

EXAMPLES XCIII.

Find the values of—

1. ·5 of £1 + ·25 of 1s. + 1·25 of 1d.

2. ·$\dot{3}$ of £1 + 1·$\dot{3}$ of 1s. + 2·5 of 1d.

3. ·68125 of £1 + ·375 of 13s. 4d. + ·605 of £3 2s. 6d.

4. $3\frac{7}{8}$ of 6s. 8d. − ·4097$\dot{2}$ of a guinea + 2·75 of £30.

5. ·25 of £1 + ·125 of 10s. + 1·625 of 7s. 6d.

6. ·8752 of a guinea − ·52521 of a pound sterling.

7. ·$\dot{1}$4285$\dot{7}$ of 2 guineas + 1·25 of 2s.

8. 3·0$\dot{6}$9$\dot{3}$ of 8s. 5d. − £·641$\dot{6}$.

9. ·6 of £2 2s. + ·75 of £2 + ·4$\dot{5}$ of 7s. 6d. − ·2 of 2s. 1d.

10. 22·25 of £2 2s. 6d. + $9\frac{3}{7}$ guineas + 9·0$\dot{3}$ half-crowns.

11. Subtract ·0523 of £1 11s. 6d. from ·29 of 15s. 3d., and give the answer in pence and the decimal of a penny.

12. Add together $\frac{5}{9}$ of 7s. 6d., 2·07 of £1 8s. 2d., and $\frac{3}{4}$ of ·0671 of 16s. 8d., and express the answer in pence and the decimal of a penny.

13. Find the sum of ·0975 of £6 5s. and ·3125 of £1 10s., and reduce your result to the decimal of £3.

14. Find the value of 3·$\dot{6}$ of ·9$\dot{5}\dot{4}$ of ·$\dot{4}$2857$\dot{1}$ of 18s. 6d.

15. Simplify ·6$\dot{9}$1$\dot{8}$ of £8 9s. 7d. − ·05$\dot{8}\dot{1}$ of £90 4s. $8\frac{1}{4}$d. + ·46875 of 4 guineas.

16. Add together ·0125 of £5 12s. 6d., 3·0057 of 15s., and ·15 of 8d., and express the result as the decimal of £1 (four places of decimals).

17. ·1$\dot{2}\dot{3}$ of 8s. 3d. + $\frac{2}{3}$ of $\frac{3}{4}$ of $4\frac{7}{9}$ of 1s. 6d. + 2·8d.

18. 2·8680$\dot{5}$ of 3s. + ·8$\dot{3}$ of 4s. − 1·8 of 5s.

19. 1·75 of £1 + ·$\dot{6}$ of 7s. 6d. − $\frac{1}{2}$ of $\frac{2}{5}$ of 4s. 2d.

20. Find the value of the sum of £·5$\dot{7}$, £·00$\dot{2}$0$\dot{1}$, £·9$\dot{0}$1$\dot{2}$, to the nearest penny.

21. Simplify ·$\dot{4}\dot{5}$ of 16s. 6d. + ·2$\dot{2}\dot{7}$ of 9s. 2d.

22. Reduce the sum of ·625 of £3 and ·175 of £3 2s. 6d. to the decimal of £6.

23. Simplify $\frac{3}{10}$ of £2 7s. 11d. + $\frac{4}{17}$ of ·1$\dot{8}$ of 1s. 3d. − $\frac{1}{6}$ of 2s. 7d.

24. Add together 1·47 of 3s. 6d., ·053 of £1 4s., and ·125 of $7\frac{1}{2}$d., and express the answer in pence and the decimal of a penny.

25. Add ·$\dot{2}$4$\dot{6}$ of 9s. 3d. + ·25$\dot{9}$ of 25s. + ·0$\dot{2}$ of £3 7s. 6d. Reduce the result to the decimal of £3.

26. Add $\frac{21}{26}$ of $\frac{39}{49}$ of 8s. 2d. to ·0625 of £6 13s. 4d., and express the sum as the decimal of £2 0s. 9d.

27. Add together $\frac{7}{40}$ of £3 7s. 6d., ·0753 of $8\frac{1}{2}$ guineas, and $\frac{3}{7}$ of $\frac{4}{5}$ of $10\frac{1}{2}$d., and express the answer in pence and the decimal of a penny.

28. Add ·$\dot{4}$2857$\dot{1}$ of a guinea and a half + ·125 of a sovereign + ·4$\dot{9}$ of half-a-crown + $1\frac{7}{17}$ of $2\frac{3}{7}$ of $1\frac{3}{4}$ of $7\frac{1}{2}$d.

29. Find the value of 6·12 of £1 + $\frac{2}{7}$ of 10s. + 6·8 of 6d., and reduce result to decimal of £10 13s. 4d. correct to six decimal places.

30. Find value in £ s. d. of $\cdot\dot{0}61728395$ of £617 2s. $4\frac{1}{2}$d. + $\cdot291\dot{6}$ of £182 17s., and reduce the result to the decimal of £53000.

31. Find value of $\cdot00651041\dot{6}$ of £8 + ·0015625 of £40 + $\cdot01380208\dot{3}$ of £16.

32. What fraction of £21 5s. 6d. is $\dfrac{0\cdot04255 \times 0\cdot32}{0\cdot0016}$ of £1 11s. 3d.?

33. Reduce $\dfrac{2\frac{1}{7} \text{ of } (3-1\frac{1}{5})}{2+\dfrac{1}{1+\dfrac{1}{2+\frac{1}{3}}}}$ of 3s. 6d. to the decimal of $\frac{2744}{8087}$ of £100.

34. Find the value of $\dfrac{1}{2+\dfrac{1}{3+\dfrac{1}{4+\frac{1}{5}}}}$ of £2 12s. 4d. − $\cdot5\dot{7}\dot{2}$ of £1 7s. 6d. − $3\frac{6}{17}$ of $1\frac{2}{19}$ of $2\frac{3}{7}$ of 9d.

35. Simplify $\dfrac{\frac{1}{3} \text{ of } \frac{2\frac{1}{2}}{2} + \frac{3}{8} \text{ of } 11\,(3-\cdot\dot{0}\dot{9}) - \frac{1}{7} \text{ of } \frac{13\frac{2}{3}}{2\frac{1}{2}}}{3 \times \dfrac{1}{5 \times 2\frac{1}{2} \times 1\frac{5}{6}} - \dfrac{\frac{1}{5}}{6\,(2\frac{1}{2}-\frac{1}{6})} + \dfrac{\frac{3}{7}}{\frac{3}{5} \text{ of } 32}}$ of £3 19s. 3d.

EXAMPLES XCIV.

1. Express as a decimal of 1 cwt. the difference between ·063 of a quarter and ·42 of 1 lb.

2. Subtract ·046 of 2 cwt. 1 qr. from 1·3 of 9 lbs., and give the answer as a decimal of 5 oz.

3. Add together 1·901 of a hundredweight, ·059 of a quarter, and ·061 of a pound, and give the result in ounces.

4. Add together $\frac{3}{8}$ of $7\frac{1}{2}$ cwts. and 2·07 of $3\frac{1}{2}$ qrs., and express the answer in ounces and the decimal of an ounce.

5. Subtract from 2 cwts. the sum of ·3125 of 6 cwts. and ·032 of 3 cwts. 2 qrs. 14 lbs. 4 oz.

6. Add $4\frac{2}{7}$ of ·1625 of 3 lbs. 8 oz. + 2·16 of $11\frac{1}{2}$ of 3 oz. 2 drs.

7. Add ·21875 of 2 cwts. + ·625 of 2 qrs.

8. Add 0·04375 ton + 0·3775 cwt. + 0·72 lb.

9. Find the number of ounces in ·0015625 of a ton + $\cdot03\dot{5}7142\dot{8}$ of a hundredweight.

10. Find the number of pounds in ·175 of a ton + $\cdot10\dot{7}1428\dot{5}$ of a hundredweight + ·15 of a pound.

11. Add together $5\cdot\dot{7}1428\dot{5}$ of 12 cwts. 3 qrs. 14 lbs. and $4\cdot\dot{2}3076\dot{9}$ of 11 cwts. 2 qrs. 12 lbs. Reduce the result to the decimal of 140 tons 11 cwts. 1 qr. 15 lbs.

12. Add together ·01425 of 1 cwt. 12 lbs., ·4756 of 1 qr. 15 lbs., and 5·42 of 7 lbs. 4 oz., and express the result in pounds and the decimal of a pound.

13. Add together $\frac{5}{17}$ of 9 lbs. 9 oz. and ·275 of 15 oz. 13 drs., and express the answer in drams and the decimal of a dram.

14. Add together ·073 of a ton, ·695 of a hundredweight, and ·83 of a pound, and express the answer in pounds and the decimal of a pound.

15. Find the value of 1·9514 of a ton + 1·5345 of a hundredweight + 1·3125 of a quarter expressed in tons, hundredweights, &c.

16.* If ·12775 of $\cdot\dot{2}1917808\dot{0}$ of 1 cwt. 13 lbs. of coal costs ·39375d., what is the price per ton?

17. Find, in miles, the difference between ·99999 of a degree and 59 mls. 2 furls. (1 degree = $69\frac{1}{9}$ mls.)

18. Express, in inches, the difference between $\frac{3}{8}$ of a yard and ·875 of a foot.

19. Subtract ·0437 of 2 mls. 3 pls. from 3·27 of 5 furls. 7 yds., and express the answer in feet and the decimal of a foot.

20. Reduce $\cdot\dot{4}8\dot{1}$ of $\cdot\dot{5}1\dot{8}$ of $\cdot0\dot{8}\dot{1}$ of ·06875 of 405 mls. to yards.

21. Reduce $\cdot\dot{9}6\dot{2}$ of $\cdot\dot{2}5\dot{9}$ of $\cdot1\dot{6}3\dot{6}$ of $\cdot\dot{0}3296\dot{7}$ of ·6875 of 27 mls. to yards.

22. Add together ·029 of 1 ac. 1 r., ·45 of $12\frac{1}{2}$ pls., and ·89 of 2 sq. yds., and express the answer in square feet and the decimal of a square foot.

23. Subtract ·406 of 2 acs. 1 r. from 5·25 of 1 r. 13 pls., and express the answer in poles and the decimal of a pole.

24. Subtract ·657 of 2 acs. from 7·042 of 1 r., and express the answer in square yards and the decimal of a square yard.

25. Subtract $\frac{7}{11}$ of 5 furls. 121 yds. 1 ft. 10 ins. from ·873 of 4 furls. 185·72 yds., and express the result in inches and the decimal of an inch.

26. Add together ·029 of 1 day 3 hours, ·45 of 11 hours 10 minutes, and ·89 of an hour, and express the answer in minutes and the decimal of a minute.

27. Add together ·007 of a week, ·572 of a day, and 5·73 of 3 hours, and express the answer in minutes and the decimal of a minute.

28. Subtract ·0523 of 11 weeks from ·932 of 6 days, and give the answer in minutes and the decimal of a minute.

29. Subtract ·075 of 3 bushs. 1 pk. from 2 pks. 3 qts., and express the answer in pints and the decimal of a pint.

30. Subtract ·00107 of a quarter from ·1751 of 3 bushs. $2\frac{1}{2}$ galls., and express the result in pints and the decimal of a pint.

31. Add together the following weights of gold: — 5776 grs., 258 dwts., 22 oz., 1·08 lbs. Express the answer in pounds, ounces, &c.

32. Add together ·113 of 2 lbs. 11 dwts., 3·75 of 16 oz. 3 grs., and ·17 of 14 dwts. 9 grs., and express the result in grains and the decimal of a grain.

CHAPTER XX.

THE METRIC SYSTEM.†

165. **Historical.**—The difference between the systems of weights and measures used in the various countries of Europe has always been a serious hindrance to commerce. Attempts at uniformity were made as early as the time of Charles the Great (800 A.D.), but met with no permanent success. At various times in the past the term "pound" has been used for several hundred different weights, and the word "foot" for at least two hundred different lengths. It was not until 1790 that any important step was taken. On May 8th of that year "the National Assembly of France requested Louis XVI. to concert with the King of England, so that a society of scientific men from the two countries might determine, at latitude 45°, the length of the pendulum beating the second, in order that this length might be adopted as the linear measure from which all the other standards should be derived" (Delbos). Political events rendered the co-operation of England impossible, but the French Commission was appointed, and on March 19th, 1791, presented their report, which was almost immediately adopted. They recommended that the standard of length should be the 10,000,000th part of the distance from the earth's equator to its pole, and that the other standards should be derived from this in the manner explained below. Two French mathematicians, Delambre and Méchain, measured the distance from Dunkirk to Barcelona, and "a 'Commission of Weights and Measures,' composed of twenty-two members chosen from various countries in Europe, then undertook the final work based on the calculations of Delambre and Méchain," and deduced the length of the new unit, the metre, and the sizes of the various derived units named below. The "metric system" was made compulsory in France in 1802, and such are its advantages that, despite international jealousy, it has spread all over Europe, and is now used by more than 400,000,000 people (H. O. Arnold-Forster). Russia and England are of civilised countries the last to fall into line with the rest, and the Russian Government in 1898 announced its intention of adopting it; so that England stands alone, and is behind even India, which partially adopted it in 1870. In England it was not until 1898 legal to use the metric system at all, and it is still, unfortunately, very little understood. The consequence is that at least half a year of school

† We must acknowledge our obligations in this section particularly to the *The Metric System*, by L. Delbos.

life is entirely wasted for every English boy in learning the arithmetical devices necessary for managing the "weights and measures" previously explained.

166. *Metrical units.*

I. LENGTH.—The unit is the **metre.**

The metre was intended to be, as explained above, $\frac{1}{10000000}$th part of the distance from the pole to the equator, and a bar supposed to represent this length, and made of iridio-platinum, is kept at the Palais des Archives in Paris. The measurements of the earth made in 1791 were

not, however, quite exact; and the distance between pole and equator is not really 10,000,000 metres, but 10,000,856 ($\frac{5}{6}$) metres. Thus the metre is an arbitrary standard like our yard. The latest determination of the length of this standard metre (1896) makes it 39·370113 ins., but the last two figures are quite uncertain.

To denote the multiples and parts of the metre, and of all derived units, the following prefixes are used:—

	10	100	1000	[10000]
Multiples (Greek prefixes):	**deca-**	**hecto-**	**kilo-**	[**myria-**]
	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$	
Sub-multiples (Latin prefixes):	**deci-**	**centi-**	**milli-**	

Thus $\frac{1}{10}$ of a metre is called a *decimetre*, while 10 metres are a *decametre*. The Metric "Long Measure Table" thus becomes 10 millimetres = 1 centimetre, 10 centimetres = 1 decimetre, 10 decimetres = 1 metre, &c. The names are abbreviated—*Multiples:* Dm., Hm., Km., Mm.; *Submultiples:* dm., cm., mm.

Learn: **1 metre = 39·37 inches;**

1 centimetre = $\frac{2}{5}$ inch (nearly),

note in the figure how nearly 5 cms. coincides with 2 ins.;

1 kilometre = $\frac{5}{8}$ mile (nearly).

The breadth of the nail of the little finger on an average hand is about 1 cm., and the length of the nail-joint of that finger is about an inch.

Decametres and hectometres are rarely used.

II. Area is measured by scientific men in square metres, &c. That is, the unit is a square each side of which is 1 metre. It is easily seen (see § 52) that [$10^2 =$] 100 sq.mm. = 1 sq.cm.; 100 sq.cm. = 1 sq.dm.; 100 sq.dm. = 1 sq.m.; &c. The best abbreviations are Dm.^{2}, Hm.^{2}, Km.^{2}, dm.^{2}, cm.^{2}, mm.^{2}

Land is generally measured in a unit called **are,** of which the *hectare* (= 100 ares) and the *centiare* (= $\frac{1}{100}$ are) are the only derivatives in use.

Learn: **1 are = 100 sq. metres**
= $\frac{1}{40}$ acre (nearly);
1 sq. metre = $1\frac{1}{5}$ sq. yards (nearly).

III. Volume is occasionally measured in cubic metres (*stere*). Only the *decastere* and the *decistere* are used. It may also be measured in *cubic metres* (m.^{3}), *decimetres* (dm.^{3}), *centimetres* (cm.^{3}), &c. The cubic decimetre is always used for liquids, and is called a **litre.** This is abbreviated *l*.

The *stere* is a name used now only by timber merchants. Scientific men use the *cubic centimetre* (c.cm. or cm.^{3}), and for liquids and dry measure the *litre* is universal. The *millilitre* is never used, and the *kilolitre* rarely.

Learn: **1 litre = 1 cubic decimetre**
= $1\frac{3}{4}$ pints (nearly).

IV. Weight.—The unit is a **gram** (sometimes spelt, as in French, *gramme*). It is the weight of 1 cm.^{3} (c.cm.) of pure water. It is abbreviated g.

This weight must be taken at a definite temperature (4° C.) because water changes its volume when the temperature changes. Really, however, a *gram* is the weight of a piece of metal in the Palais des Archives at Paris, and not exactly the weight of 1 cm.^{3} of water, which is found to be 1·000013 g. For heavy weights, *quintal* (= 100 kilograms) and *tonne* (= 1000 kilograms) are used.

Learn: **1 gram** = *weight of* **1 cubic centimetre** *of pure water*
= 15·432 grains;
1 kilogram = $2\frac{1}{5}$ pounds (nearly).

167. If this system were adopted, the so-called *Compound Rules* would disappear from Arithmetic.

Reduction.—7 kilometres = 7000 metres, 5 hectometres = 500 metres, 3 decametres = 30 metres; hence a length 7 Km. 5 Hm. 3 Dm. 9 m. becomes $7000+500+30+9$ metres, *i.e.* 7539 metres, and reduction to any denomination is effected by simply removing the names of the previous denominations. Thus, to reduce to metres, we strike out Km., Hm., Dm., and the answer is 7539 m. So 5 decimetres $=\frac{5}{10}$ metre $=\cdot5$ metre, 6 centimetres $=\frac{6}{100}$ metre $=\cdot06$ metre; thus 5 dm. 6 cm. $=\cdot56$ m. So, again,

7 Km. 5 Hm. 3 Dm. 9 m. 5 dm. 6 cm. = 7539·56 m.,
or, since 1 m. = 10 dm., = 75395·6 dm.
= 753956 cm.

Thus, again, reduction is effected by the correct placing of a decimal point.

Addition.

Ex.—*Add* 325 *grams*, 1536 *milligrams*, 1·36 *kilograms*.

```
  g.
 325·
   1·536    for 1536 milligrams = 1536/1000 grams = 1·536 ;
1360·       for 1·36 kilograms  = 1·36 × 1000 grams = 1360 grams.
1686·536 g.  Ans.
```

The answer may be, but rarely is, written

1 Kg. 6 Hg. 8 Dg. 6 g. 5 dg. 3 cg. 6 mg.

There is no need to write it thus in full, because *the position relative to the decimal point at once shows the value of the figure.*

Subtraction and Multiplication.

Ex.—*Subtract* 3·25 *m. from* ·536 *Km., and multiply the difference by* ·123, *correctly to the nearest centimetre.*

```
  m.
 536
   3·25
 532·75 m.
    ·123
  53·275
 10·655
  1·598
 65·528 m.
```

for ·536 Km. = ·536 × 1000 m. = 536 m. This is now an ordinary example of contracted multiplication. "Correctly to the nearest centimetre" means correctly to the second place, if we work in metres. The answer is 65·53 metres, or 6553 centimetres.

EXAMPLES XCV. (MENTAL.)

1. Write 1 Kg. 1 g. 1 mg. in grams and also in decigrams.

2. Write 1 Dm. 1 dm. 1 mm. in metres and also in centimetres.

3. Write 10 Kg. 10 Hg. 10 Dg. in centigrams and also in kilograms.

4. Write 13 Dg. 13 dg. 13 mg. in grams and also in kilograms.

5. How many centimetres in $\frac{1}{3}$ of a metre?

6. How many litres in $3\frac{1}{3}$ hectolitres?

7. Express, roughly, in inches 15, 26, 25, and 36 cm.

8. Express, roughly, in inches 14, 20, 24, and 30 cm.

9. Express in miles (approx.) 10, 14, 18, and 20 Km.

10. Express in miles (approx.) 12, 16, 22, and 26 Km.

11. Express in pounds (approx.) 10, 25, 30, and 45 Kg.

12. Express in pounds (approx.) 15, 35, 40, and 50 Kg.

13. Express in pints (approx.) 7, 10, 13, and 16 litres.

14. Express in litres (approx.) 7, 10, 13, and 16 pts.

15. How many litres in 5 cm.^{3}?

16. How many cubic centimetres in 5 l.?

17. How many square decimetres in 100 m.^{2}?

18. How many square metres in 100 dm.^{2}?

19. Which is greater, 40 litres or 68 pts., and by how many pints?

20. Multiply the smaller of the two quantities 4 ins. and 11 cm. by 2.

EXAMPLES XCVI.

1. Add together 359 l., 467 dl., and 586 cl.

2. Add together 1137 g., 1173 cg., and 1183 Kg.

3. 13 m. 567 dm. 675 cm. 117 mm. *minus* 12 m. 444 dm. 576 cm. 1 mm.

4. 31 l. 765 dl. 576 cl. 171 ml. *minus* 22 l. 67 dl. 756 cl. 17 ml.

5. Divide 2·345 m. by 5, 7, and 35, giving all the answers in centimetres.

6. Divide 14·28 g. by 4, 7, and 28, giving all the answers in milligrams.

7. Reduce 1 Km. 2 Hm. 3 Dm. 4 m. 5 dm. 9 cm. to decimal of 7 Km.

8. Reduce 9 Kg. 8 Hg. 7 Dg. 6 g. 5 dg. 1 cg. to decimal of 9000 mg.

9. What decimal is 17 l. of 27·2 Hl.?

10. What decimal is 16 mm. of 176 Km.?

11. Multiply 3 Kg. 7 g. 5 dg. by 34, and express the answer in grams.

12. Divide 3 Km. 6 Dm. 9 m. by 18, and express the answer in metres.

13. Multiply 29 g. 37 cg. by 300, and express the answer in kilograms and grams.

14. Divide 30 m. 7 dm. by 100, and express the result in millimetres.

15. Find the value of 0·04375 Kg. + 0·3775 g. + 0·72 mg. as the decimal of a gram.

Decimal Coinage.

168. Every country in Europe (except England) has adopted a decimal system of coinage—*i.e.* a system in which the unit is divided into 10 or 100 parts. Though not really part of the Metric System, it is convenient to mention here the systems adopted in France (along with Belgium, Italy, Greece, Switzerland), Germany, and the United States. Fuller details will be given in Chap. xxviii.

French Coinage.—The unit is the **franc,** which is divided into **100 centimes.**

Learn: **25 francs = £1.**

The correct figure (which varies slightly; see Chap. xxviii.) is 25·22 francs = £1. The so-called "French pennies," which used to be common in England, are 10-centime pieces. Their value is slightly less than 1d. The *sou*, or "French halfpenny," is the 5-centime piece.

German Coinage.—The unit is the **mark,** which is divided into **100 pfennig,** and is of about the same value as a shilling.

The correct figure (which varies slightly; see Chap. xxviii.) is 20·43 marks = £1.

American Coinage.—The unit is the **dollar** (about 4s.), which is divided into **100 cents.** A **dime** is 10 cents.

English Decimal Coinage.—For England it has been proposed to divide the £ into 10 parts (*florins*, as at present), the *florin* into 10 parts to be called *cents*, and the *cent* into 10 parts to be called *mils*. The value of our present coins in mils is: sovereign (1000), half-sovereign (500), crown (250), double-florin (200), half-crown (125), florin (100), shilling (50), sixpence (25), threepence ($12\frac{1}{2}$), penny ($4\frac{1}{6}$), half-penny ($2\frac{1}{12}$), farthing ($1\frac{1}{24}$). Thus, for a complete and satisfactory decimal system, it would be only necessary to "call in" the last four coins, and issue four new ones, which might be "cent" (10), a silver coin of about the size of a threepenny bit, "penny" (5), a new coin (2), and "mil" (1).

We add an illustration:

Ex.—*If a litre of wine costs in France* 2 fr. 10 cent., *find approximately the value in London of* 5 *gallons of the wine, the customs duty being* 1s. 3d. *per gallon and the cost of carriage* 6d. *per gallon.*

A litre is $1\frac{3}{4}$ pints = $\frac{7}{4}$ pints; therefore $\frac{7}{4}$ pints cost 2·1 fr.; therefore $\frac{1}{4}$ pint costs ·3 fr.; therefore 1 pint costs 1·2 fr. But 5 gallons contain 40 pints, and therefore cost 40 × 1·2 fr., *i.e.* 48 fr., *i.e.* roughly £$\frac{48}{25}$, *i.e.* £1. 18s. 5d. nearly. To this must be added 1s. 3d. + 6d. per gallon, *i.e.* 5 × 1s. 9d. altogether, *i.e.* 8s. 9d. Hence the total cost in London is about £2. 7s. 2d.

The closeness of the approximations employed may be seen from the fact that using the more accurate values of franc and litre the cost works out as £2. 6s. 7d.

EXAMPLES XCVII.

(A few of these questions involve a knowledge of the Unitary Method.)

1. What is the cost of 24 m. 8 cm. of silk at 4 fr. 25 cent. per metre?

2. What will be the rental of 30 Ha. 75 a. of land at 3 fr. 5 cent. per are?

3. Find the cost of 8 Km. 25 m. of silk at 5 fr. 75 cent. per metre.

4. Find the cost of 82 Kg. 125 g. of butter at 3 fr. 44 cent. per kilogram.

5. What weight of copper is worth 723 fr. 69 cent., if 1 Kg. be worth 1 fr. 29 cent.?

6. A merchant buys 485 m. 70 cm. of silk, and pays for it 3764 fr. 20 cent. Find, correct to the nearest centime, the cost per metre.

7. If a metre be 39·37011 ins., find the difference between 5 mls. and 8 Km. in yards.

8. A man rides a bicycle at the rate of 5·75 m. per second. How long will he take to ride 100 Km.?

9. A wine merchant buys 10,000 l. of wine at 2 fr. a litre, and immediately sells it for cash at 2s. 6d. per quart. Find (in English money) the amount of his gross profit. (1 l. = $1\frac{3}{4}$ pt., 1 fr. = $9\frac{1}{2}$d.)

10. How many kilometres can I travel for 35 fr. 50 cent. at the rate of 1d. per mile? (100 fr. = £3 19s. 4d., 1 Km. = $1093\frac{2}{5}$ yds.)

11. Find the value of 23·8 Kg. of copper at 2·50 fr. per kilogram.

12. Find the value of 17·6 m. of cloth at 6·25 marks per metre.

13. Three-sevenths of the length of a rod is painted white; the rest, 80 cm., is black. How long is the rod?

14. What is the weight in kilograms of 35 Hl. of water?

15. If water weighs 750 times as much as the same volume of air, what is the weight in kilograms of the air contained in a hall whose capacity is 1500 m.^{3}?

16. How many revolutions a second are made by the wheels of a locomotive travelling 45 Km. per hour, given that they measure 5·40 m. in circumference?

17. The actual value of a metre is 39·370 ins.; approximate statements are 1 cm. = $\frac{2}{5}$ in., 1 m. = $39\frac{1}{3}$ ins., 1 Km. = $\frac{5}{8}$ ml. Find, to the nearest centimetre, the length of a mile as given by each of these four data.

18. The actual value of a gram is 15·432 grs.; approximate statements are 1 g. = $15\frac{1}{2}$ grs., 1 Kg. = $2\frac{1}{5}$ lbs. Av., 1 cwt. = 50 Kg. Find, to the nearest gram, the value of 1 lb. Av. as given by each of these four data.

19. The war indemnity paid in 1871 by France to Germany was 5,000,000,000 fr. Supposing that this was paid wholly in gold pieces of 20 fr., and that the diameter of each 20-fr. piece is 2·1 cm., what would be the length in miles (1 Km. = $\frac{5}{8}$ ml.) of the line formed by placing these side by side.

20. Assuming the data of Question 19, and that each 20-fr. piece weighs 99·5 grs., and that a railway-wagon can carry 5 tonnes, find

how many wagons would be required to transport the indemnity to Germany. (Take 1 Kg. = $2\frac{1}{5}$ lbs.)

21. If glass is $2\frac{1}{2}$ times as heavy as water, what is the weight in pounds of a cubic foot of glass? (1 cm. = $\frac{2}{5}$ in., 1 g. = $15\frac{1}{2}$ gr.)

22. If land is worth £100 an acre, how many francs should one give per hectare? (£1 = 25 fr., 1 Km. = $\frac{5}{8}$ ml.)

23. The length of the Amazon is 6480 Km., that of the Congo is 4200 Km. Express these lengths in miles. (1 Km. = $\frac{5}{8}$ ml.)

24. The Nile drains an area of 2,800,000 Km.^{2}. Express this in hectares, and also in square miles. (1 are = $\frac{1}{40}$ acre.)

25. If coal is worth 30s. a ton, how many francs should be paid for a tonne? (£1 = 25 fr., 1 Kg. = $2\frac{1}{5}$ lbs.)

26. A cyclist, setting out from Bordeaux at 5 hours 1 minute 28 seconds a.m., reached Paris, 590 Km. away, at 7 hours 36 minutes 25 seconds a.m. on the next day. What was his speed in miles per hour? (1 Km. = $\frac{5}{8}$ ml.)

27. Find, to the nearest £, the cost of 5122·7 Kg. at 3·43 fr. per kilogram, reckoning 25·42 fr. to £1.

28. Express $\frac{4}{45}$ of £4 7s. $1\frac{1}{2}$d. as the fraction of 0·315 of 97·35 fr., reckoning a shilling as worth 1·26 fr.

29. The area of the principality of Monaco is 22 Km.^{2}, and there are 12,548 inhabitants. How many are there per square mile? (1 are = $\frac{1}{40}$ acre.)

30. The area of the Republic of Andorra is 452 Km.^{2}, and there are 6000 inhabitants. How many are there per square mile? (1 Km. = $\frac{5}{8}$ ml.)

31. The Jungfrau is 4617 m. high, the Matterhorn 4432 m., and Mont Blanc 4810 m. Express these heights to the nearest foot. (1 m. = 39·370 ins.)

32. The greatest depth of the Mediterranean Sea is 4000 m., that of the Pacific Ocean 8960 m., and of the deepest mine 2003·3 m. Express these depths to the nearest foot. (1 m. = 39·370 ins.)

33. One metre contains 39·3 ins.; one litre 1·76 pts. Find, to three decimal places, the number of cubic inches in a gallon.

Given that 1 m. = 39·3701 ins., 1 gall. = 277·274 cub. ins., and 1 g. = 15·4323 grs., convert, by contracted methods, correct to four significant figures:—

34.* 1 pt. into litres;

35. 1 oz. Troy into grams;

36. 1 sq. ft. into square metres;

37. 1 sq. m. into square yards;

38.* The weight of a gallon of water into pounds Avoirdupois.

39.* A litre of good milk weighs 1·032 Kg. My servant took $5\frac{1}{2}$ l. for me one morning, but I found that the weight was only 5·628 Kg. How many cubic centimetres of water had the milkman added?

40.* A barrel, when empty, weighs 17·180 Kg.; when full with 230 l. of Bordeaux wine it weighs 232 Kg. A certain quantity of wine is drawn off and replaced by water. The barrel now weighs 234·31 Kg. How many litres of wine were drawn off?

CHAPTER XXI.

AVERAGES.

169. In the daily papers of Monday, August 13th, 1900, we read that K. S. Ranjitsinhji had in first-class cricket, up to date, scored 2454 runs for 27 completed innings, and had an "average of 90·88." Of course his score in the different innings varied greatly, from 275 downwards; but, by adding all the scores and dividing by the number of completed innings, we obtain a figure 90·88 ($= 2454 \div 27$), which fairly represents his average performance during the season. So, too, C. B. Fry on the same date had scored 1939 runs for 28 completed innings, and had an average of 69·25.

Similarly, if on a walking tour I walk 15 mls. the first day, 25 the next, then 22, 0, 30, 15, and 35 mls., I have walked altogether 142 mls. in the 7 days and "averaged" $20\frac{2}{7}$ ($= \frac{142}{7}$) mls. per day.

If a series of *quantities* of the same kind are added together, and the sum is divided by the number of quantities concerned, the quotient is called the **average** or **mean** of the quantities.

Hint.—In adding these numbers, which are often very nearly equal, time is frequently saved by neglecting the part common to all the numbers, averaging the remainders, and then adding this common part to the result. Thus, to average 19**3**, 19**0**, 19**7**, 19**5**, 19**9**, we average only 3, 0, 7, 5, 9, total 24, average **4·8**, and write at once 19**4·8** for the required average.

Ex.—*The average of* 8 *numbers is* 221·25; *that of the first* 5 *is* 273·6, *that of the last* 4 *is* 111·75. *Find the fifth number.*

The sum of all the 8 numbers is $8 \times 221{\cdot}25 = 1770$,

,, the first 5 ,, $5 \times 273{\cdot}6 = 1368$,

∴ ,, ,, last 3 ,, $= 402$, by subtraction.

But ,, ,, ,, 4 ,, $4 \times 111{\cdot}75 = 447$.

Therefore the number fourth from the last, *i.e.* the fifth number, is 45, by subtraction.

EXAMPLES XCVIII.

1. Calculate to 2 decimal places the averages, for completed innings, for the following cricketers (August 13th, 1900) :—

	Total.	Innings.	Not out.
R. E. Foster	1563	26	2
Abel	1972	35	3
Hayward	2270	42	4
J. R. Mason	1337	26	2

2. Calculate to 2 decimal places the average number of runs per wicket for the following bowlers (August 13th, 1900) :—

	Runs.	Wkts.		Runs.	Wkts.
Rhodes	2466	197	Haigh	1878	128
Mold	1153	75	C. J. Kortright	1095	62

3. Calculate the mean of the following numbers :—

(i.) 3, 5, 7, 9, 11, 13 ; (ii.) 1·2, 3·9, 4·5, $5\frac{1}{3}$, 6·$\dot{3}$, 7.

4. Calculate the averages for—

(i.) 2, 4, 6, 8, 10, 12 ; (ii.) $1\frac{1}{2}$, $2\frac{1}{3}$, $3\frac{1}{4}$, 4, 5·5, $6\frac{1}{6}$.

The following (Questions 5-10) were, for the schools named, the athletic records up to 1898, in minutes (′) and seconds (″) or feet (′) and inches (″). Calculate in each group the average :—

5. *Mile Race*—Berkhamstead, 5′ $14\frac{4}{5}$″; Bradfield, 4′ 54″; Charterhouse, 4′ 30″; Dulwich, 4′ $56\frac{1}{5}$″; Leys, 4′ $46\frac{4}{5}$″; Sedbergh, 4′ $39\frac{1}{5}$″.

6. *High Jump*—Dulwich, 5′ 8″; Leys, 5′ $6\frac{1}{4}$″; Westminster, 5′ $7\frac{1}{2}$″; Rugby, 5′ 5″; Winchester, 5′ $5\frac{1}{2}$″; Eton, 5′ 6″; Sedbergh, 5′ $3\frac{1}{8}$″.

7. *Quarter Mile*—Bradfield, 51″; Epsom, 52″; Eton, 52″; Haileybury, $53\frac{4}{5}$″; University College, $53\frac{2}{5}$″; St. Paul's, $54\frac{1}{3}$″; Glenalmond, $54\frac{2}{5}$″.

8. *Long Jump*—Framlingham, 23′ $6\frac{1}{2}$″; Cranleigh, 17′ $10\frac{1}{2}$″; Weymouth, 21′ 11″; Giggleswick, 19′ 2″; Rossall, 20′ 7″; Dulwich, 21′ 2″; Brighton, 20′ $1\frac{1}{2}$″.

9. *Cricket Ball*—Fettes, 122 yds.; Leys, 103 yds. 1′ 4″; Rugby, 102 yds. 0′ 2″; Glenalmond, 111 yds.; St. Paul's, 107 yds.; Dulwich 100 yds. 2′ 6″.

10. *Putting the Weight*—Rugby, 39′ 7″; Rossall, 32′ 7″; Leys, 30′ 6″; Winchester, 31′ 11″; Felsted, 35′ 2″; Bradfield, 28′ 7″.

11. Eight of the prefects at Kingswood School were measured in July, 1900, with the following results :—

Ages (in years and months)—18.9 ; 17.11 ; 17.3 ; 17.11 ; 17.7 ; 16.11 ; 18.2 ; 18.1.

Weights (in stones and pounds)—16.3 ; 10.3 ; 12.0 ; 9.0 ; 9.8 ; 10.3 ; 10.12 ; 9.6.

Heights (in feet and inches)—$6.1\frac{1}{4}$; $5.7\frac{1}{4}$; $6.1\frac{3}{8}$; $5.3\frac{1}{4}$; $5.2\frac{5}{8}$; $5.7\frac{1}{8}$; $5.8\frac{1}{8}$; $5.6\frac{3}{4}$.

Lung capacity (in cubic inches)—321 ; 240 ; 310 ; 183 ; 184 ; 218 ; 285 ; 238.

Find the average of each group.

12. Eight of the prefects at Kingswood School were measured in June, 1892, with the following results:—

Ages (in years and months)—18.7; 18.4; 18.11; 18.5; 18.7; 17.7; 18.4; 16.10.

Weights (in stones and pounds)—11.3; $9.4\frac{1}{2}$; $8.3\frac{1}{2}$; $8.8\frac{1}{2}$; $11.0\frac{1}{2}$; $8.12\frac{1}{2}$; $8.6\frac{3}{4}$; $6.10\frac{1}{4}$.

Heights (in feet and inches)—5.8; $5.4\frac{3}{4}$; 5.2; 5.7; 5.9; $5.6\frac{3}{4}$; $5.4\frac{5}{8}$; $5.5\frac{1}{4}$.

Lung capacity (in cubic inches)—336; 204; 206; 232; 276; 248; 154; 180.

Find the average of each group.

13. The "Cornishman" leaves Paddington at 10.35 a.m. and reaches Bristol at 12.50. The distance is $118\frac{1}{2}$ mls. It stops 10 minutes at Bristol. Supposing its average speed to be the same after Bristol as before, at what time is it due at Exeter, which is $75\frac{1}{2}$ mls. from Bristol?

14. If from Paddington to Didcot (53 mls.) a train averages 50 mls. per hour, while from Paddington to Bath (107 mls.) it averages 52 mls. per hour, find the average speed from Didcot to Bath, supposing that there is no stop between Paddington and Bath.

15. The average interest produced by Consols from 1805 to 1809 (both dates inclusive) was £4 15s. 7d. per cent., from 1805 to 1818 was £4 13s. 4d. Calculate the average from 1810 to 1818.

16. The average price of Consols from 1850 to 1859 (both inclusive) was 94, the average price in 1850 was $96\frac{5}{8}$, and the average price during 1860 and 1861 was $92\frac{15}{16}$. Find the average price from 1851 to 1861, correct to the nearest sixteenth.

17. In buying a school-book, I obtain 4 copies from a local bookseller at 3s. $4\frac{1}{2}$d. each, also one slightly damaged at 2s. 6d., and 5 second-hand at 2s. each; in addition, I get 3 dozen copies from a wholesale house at 3s. each. What have I paid on the average per copy? (Answer to nearest penny.)

18. Of three numbers the first is double the second, the second is double the third, and the average of the three is 77. What are the numbers?

CHAPTER XXII.

PRACTICE.

170. When the smaller of two numbers or quantities is contained an exact number of times in the larger, it (the smaller) is called an **aliquot part** or **sub-multiple** of the larger. The number expressing this aliquot part is a fraction, having unity for its numerator and an integer for its denominator.

Thus 2s. $= \frac{1}{10}$ of £1; 8d. $= \frac{1}{3}$ of 2s.; 5 cwts. $= \frac{1}{4}$ of a ton.

Practice is a method in which we consider a whole as made up of successive portions, each of which is an *aliquot part* of some preceding portion, and use our knowledge to find multiples of this whole.

Thus, £1 2s. 8d. may be separated into

£1 + 2s. + 8d., *i.e.* £1 + $\frac{1}{10}$ of £1 + $\frac{1}{3}$ of 2s.

So, 1 ton 15 cwts. 2 qrs. = 1 ton + 10 cwts. + 5 cwts. + 2 qrs.

= 1 ton + $\frac{1}{2}$ a ton + $\frac{1}{2}$ of 10 cwts. + $\frac{1}{10}$ of 5 cwts.

The classes of problems in which Practice may be used are as follows:—(1) Multiply £1 2s. 8d. by 153; (2) Find the cost of 9237 things at £3 17s. 4½d. each. These are examples of *Simple Practice.*

In all such questions, if the pence and halfpence form together a number divisible by 3, it will probably be equally short, and will often be very much shorter, to decimalise the sum of money and perform a simple multiplication. This method will be exhibited below. It is not recommended unless the pence and halfpence together are divisible by 3, for the decimal of £1 will not otherwise be terminating. Thus it might be applied to the second example quoted, because 4½d. *is* divisible by 3, but is not recommended for the first.

We may also use the method for such a problem as (3) Find the cost of 3 tons 17 cwts. 2 qrs. at £4 5s. 6d. per ton. Such applications are called *Compound Practice* (§ 172).

Simple Practice.

171. It will be noticed that this is really an alternative method for Compound Multiplication.

Ex. 1.—*Multiply* £1 2s. 8d. *by* 153.

This can be done as already explained in § 68, or we can multiply the £1, the 2s., and the 8d. separately by 153, and add the three results. Thus:—

		£	s.	d.
	£1 × 153 =	153	0	0
Since 2s. is £$\frac{1}{10}$, therefore 2s. × 153 is £$\frac{1}{10}$ × 153 =		15	6	0
Since 8d. is $\frac{1}{3}$ of 2s., therefore 8d. × 153 is $\frac{1}{3}$ of 2s. × 153 =		5	2	0
Therefore	£1 2s. 8d. × 153 =	173	8	0

N.B.—This is, of course, also the answer to the following:—Find the cost of 153 articles at £1 2s. 8d. each. The working may be arranged as in Ex. 2.

Ex 2.—*Find the cost of* 9237 *things at* £3 17s. 4½d. *each.*

	£	s.	d.			£	s.	d.	
Method (i.)—	9237	0	0	=	cost of 9237 things at	1	0	0	each.
			3						
	27711	0	0	=	,, ,,	3	0	0	,,
10s. = £½	4618	10	0	=	,, ,,	0	10	0	,,
5s. = ½ of 10s.	2309	5	0	=	,, ,,	0	5	0	,,
2s. = ⅕ of 10s.	923	14	0	=	,, ,,	0	2	0	,,
4d. = ⅙ of 2s.	153	19	0	=	,, ,,	0	0	4	,,
½d. = ⅛ of 4d.	19	4	10½	=	,, ,,	0	0	0½	,,
	£35735	12	10½	=	,, ,,	3	17	4½	,,

Here we first multiply the top line by 3 to get the cost at £3 each The 17s. 4½d. is separated into 10s. + 5s. + 2s. + 4d. + ½d. The 2s. is not an aliquot part of the immediately preceding portion 5s., but is ⅕ of 10s. To get the cost at 2s. each we must therefore divide the cost at 10s. each by 5.

We might have separated the 17s. 4½d. into

10s. + 6s. 8d. + 8d. + ½d.,
or into 10s. + 5s. + 2s. + 3d. + 1½d.,
or into 10s. + 4s. + 3s. 4d. + ½d.

Considerable practice and some ingenuity are needed to discover the best means of making the separation.

Method (ii.).—As previously stated, we might have decimalised £3 17s. 4½d., and then used ordinary multiplication. If the decimalisation is performed exactly (Rule II., § 158), as in the solution marked

(A), the work is longer than in Method (i.). If it is approximate (Rule III., § 158), as in (B), the answer is inexact. Explain why.

£		£	
3·86875		3·869	
9237		9237	
34818·75		34821	
773·750	(A).	773·8	(B), *Inexact*
116·0625		116·07	
27·08125		27·083	
£35735·64375 = £35735 12s. $10\frac{1}{2}$d.		£35737·953 = £35737 19s. $0\frac{3}{4}$d.	

Ex. 3.—*Find the cost of* $786\frac{3}{4}$ *cwts. at* £2 14s. 7d. *each.*

If 1 cwt. costs £1, $\frac{3}{4}$ cwt. costs £$\frac{3}{4}$ = 15s.

	£	s.	d.				£	s.	d.	
	786	15	0	= cost of $786\frac{3}{4}$ cwts. at			1	0	0	per cwt.
			2							
	1573	10	0	=	,,	,,	2	0	0	,, ,,
10s. = £$\frac{1}{2}$	393	7	6	=	,,	,,	0	10	0	,, ,,
3s. 4d. = $\frac{1}{3}$ of 10s.	131	2	6	=	,,	,,	0	3	4	,, ,,
1s. 3d. = $\frac{1}{8}$ of 10s.	49	3	$5\frac{1}{4}$	=	,,	,,	0	1	3	,, ,,
	£2147	3	$5\frac{1}{4}$	=	,,	,,	£2	14	7	,, ,,

Here we might first have found the cost of 786 cwts. at £2 14s. 7d. per cwt., and then have added in the last line the cost of $\frac{3}{4}$ cwt. at £2 14s. 7d. per cwt., found by the method of Ex. 1, § 156. This method should be adopted whenever, as in Ex. 4, the value of the given fraction of an article cannot be expressed in any coin.

Ex. 4.—*Find the cost of* $871\frac{4}{7}$ *lbs. at* 8s. 4d. *per lb.*

	£	s.	d.									s. d.
	871	0	0	= cost of 871 lbs. at £1 each.							7)	8 4
5s. = £$\frac{1}{4}$	217	15	0	=	,,	871	,,	5s.	,,			1 $2\frac{2}{7}$
3s. 4d. = £$\frac{1}{6}$	145	3	4	=	,,	871	,,	3s. 4d.	,,			
		4	9·14	=	,,	$\frac{4}{7}$	,,	8s. 4d.	,,			4 s. d.
	£363	3	1·14	=	,,	$871\frac{4}{7}$	,,	8s. 4d.	,,			4 $9\frac{1}{7}$ = 4 9·14

To the nearest penny the answer is £363 3s. 1d.

The thick line is drawn below the top line to indicate that it must not be added in with the rest. In such questions, decimals of 1d. *to two places* should be used rather than vulgar fractions.

Ex. 5.—*What is the value of* 8173 *cwts. at* £5 19s. *per cwt.?*

£5 19s. is just 1s. less than £6; it is therefore more convenient to find the value of 8173 cwts. at £6 each and *subtract* the value at 1s. each.

	£	s.	d.					£	s.	d.	
	8173	0	0	= value of 8173 cwts. at				1	0	0	each
			6								
	49038	0	0	=	,,	,,	,,	6	0	0	,,
1s. = £$\frac{1}{20}$	408	13	0	=	,,	,,	,,	0	1	0	,,
	£48629	7	0	=	,,	,,	,,	£5	19	0	,,

Ex. 6.—*Multiply* 4 *tons* 3 *cwts.* 2 *qrs. by* 739.

	tons	cwts.	qrs.		tons	cwts.	qrs.	
	739	0	0	=	1	0	0	× 739
			4					
	2956	0	0	=	4	0	0	× 739
2 cwts. = $\frac{1}{10}$ ton	73	18	0	=	0	2	0	× 739
1 cwt. = $\frac{1}{2}$ of 2 cwts.	36	19	0	=	0	1	0	× 739
2 qrs. = $\frac{1}{2}$ of 1 cwt.	18	9	2	=	0	0	2	× 739
	3085	6	2	=	4	3	2	× 739

Ex. 7.—*Find the dividend*† *on* £11469 13s. 4d. *at* 9s. $4\frac{1}{2}$d. *in the pound.*

	£	s.	d.
	11469	13	4
5s. = £$\frac{1}{4}$	2867	8	4
4s. = £$\frac{1}{5}$	2293	18	8
3d. = $\frac{1}{20}$ of 5s.	143	7	5
$1\frac{1}{2}$d. = $\frac{1}{2}$ of 3d.	71	13	$8\frac{1}{2}$
	£5376	8	$1\frac{1}{2}$

When the meaning of the process has been mastered, the only figures which need be given in the working are as shown in this example. The explanation of the various lines need *not* be given.

EXAMPLES XCIX.

1. Multiply 8s., 11s., 10s. 6d., 16s. 8d. by 352.

2. Multiply 7s., 15s., 11s. 3d., 17s. 6d. by 401.

3. Multiply 1s. $11\frac{1}{2}$d., 2s. 8d., 4s. $6\frac{1}{2}$d., 7s. 3d. by 500.

4. Multiply 2s. 1d., 3s. $6\frac{1}{2}$d., 5s. 4d., 8s. $7\frac{1}{2}$d. by 600.

Using *one* aliquot part only, in Exs. 5–8, find cost of—

5. 625 things at £1 10s. each, 478 at £2 5s. each, 350 at £3 2s. 6d. each.

6. 721 at £1 4s. each, 256 at £1 1s. 3d. each, 453 at £4 6s. 8d. each.

7. 459 at £1 2s. each, 873 at £2 3s. 4d. each, 900 at £2 1s. each.

8. 149 at £3 2s. 6d. each, 923 at £2 5s. each, 378 at £1 6s. 8d. each.

9. Find price of 117 things at 16s. 6d. each, at 17s. $4\frac{1}{2}$d. each, at 19s. 11d. each.

10. Find price of 97 things at 14s. 6d. each, at 18s. 4d. each, at 19s. $11\frac{3}{4}$d. each.

Find cost of—

11. 856 things at £1 14s. $3\frac{1}{2}$d. each.

12. 2420 lbs. of tea at 2s. $1\frac{1}{2}$d. per pound.

13. 1297 trucks of coal at £4 17s. $11\frac{1}{2}$d. per truck.

14. 4935 books at 11s. 7d. each.

15. 48138 articles at £2 3s. 6d. each.

† For the meaning of this phrase, see § 192.

16. 758 pairs of boots at 14s. $7\frac{1}{2}$d. per pair.
17. 764 things at £2 17s. $6\frac{1}{2}$d. each.
18. 2157 things at £2 7s. $4\frac{1}{4}$d. each.
19. 3621 things at £2 16s. 8d. each.
20. 412 things at £2 14s. $10\frac{1}{2}$d. each.
21. $7163\frac{4}{5}$ tons at £2 9s. $8\frac{1}{4}$d. per ton.
22. 108 tons at £5 14s. 1d. per ton.
23. 21840 articles at 18s. $10\frac{1}{2}$d. each.
24. 50 dozen articles at £1 7s. $8\frac{1}{2}$d. each article.
25. 7242 articles at 4s. $7\frac{1}{2}$d. per dozen.
26. 3725 articles at 7s. 9d. per score.
27. 1296 lbs. at 17s. $9\frac{1}{2}$d. per lb.
28. 3725 things at 13s. $7\frac{1}{2}$d. per packet of 10.
29. 345 shares at £40 16s. 9d. each.
30. 1020 articles at £5 3s. $7\frac{1}{2}$d. each.
31. 2874 sacks of potatoes at 8s. $4\frac{3}{4}$d. per sack.
32. 79 dozen lbs. at 10s. $7\frac{1}{2}$d. per lb.
33. 170 mls. of wire at £14 16s. 8d. per mile
34. 1056 things at £1 4s. 7d. each.
35. 318 things at £3 11s. $2\frac{3}{4}$d. each.
36. 921 things at 13s. 6d. per dozen.
37. 12 dozen articles at 16s. $1\frac{1}{2}$d. each article.
38. 333 lbs. at 18s. $7\frac{1}{2}$d. per lb.
39. 417 articles at 13s. $10\frac{1}{2}$d. each.
40. 5470 articles at 11s. $4\frac{3}{4}$d. each.
41. $1350\frac{3}{4}$ articles at 13s. 9d. each.
42. $4872\frac{7}{8}$ articles at 18s. 2d. each.
43. 25993 articles at £2 5s. 7d. each.
44. 1027 articles at £14 8s. 5d. each.
45. 743 articles at 13s. $10\frac{3}{4}$d. each.
46. 342 articles at 16s. $10\frac{1}{2}$d. per dozen.
47. 965 articles at £1 13s. 2d. each.
48. 87642 articles at 16s. 3d. per dozen.
49. 1215 articles at £4 11s. $9\frac{3}{4}$d. each.
50. 563 acs. at £2 12s. $6\frac{1}{2}$d. per acre.

In Ex. 51–55 reckon the fraction of a penny in your answer as a whole penny. Find cost of—

51. $7394\frac{3}{4}$ articles at £12 8s. $8\frac{1}{4}$d. each.
52. $429\frac{6}{7}$ articles at 18s. $6\frac{1}{2}$d. each.

53. $7264\frac{2}{3}$ articles at £4 16s. 8d. each.

54. $297\frac{4}{5}$ articles at £1 14s. 6d. each.

55. $8416\frac{7}{9}$ articles at £3 18s. $11\frac{3}{4}$d. each.

In Exs. 56–60 the answer is given exactly, but see Note to Ex. 4 above—

56. $321\frac{4}{7}$ lbs. at 18s. 4d. per lb.

57. $1716\frac{4}{11}$ yds. at £1 3s. 9d. per yard.

58. $477\frac{1}{2}$ things at £2 15s. $10\frac{3}{4}$d. each.

59. $1263\frac{5}{6}$ things at £3 14s. 5d. each.

60. $8001\frac{2}{5}$ things at £1 13s. $0\frac{1}{2}$d. each.

61. Find the dividend on £6285 15s. at 13s. 9d. in the £.

62. Find the dividend on £3570 10s. at 16s. $10\frac{1}{2}$d. in the £.

63. Find the dividend on £8756 15s. at 16s. 5d. in the £.

64. Find the dividend on £396 15s. at 18s. 11d. in the £.

65. A bankrupt's debts (liabilities) are £6235 10s., and he pays his creditors 5s. 6d. in the £. Find the amount of his assets.

66. A man owes £6285 15s., but can pay only 16s. 9d. in the £. How much do his creditors lose altogether?

67. Find the dividend on a sum of £1890 at 15s. 9d. in the £.

68. Multiply 3 tons 17 cwts. 2 qrs. by 371.

69. Multiply 3 lbs. 4 oz. 12 dwts. by 97.

70. Find the weight of 2348 packages, when each weighs 5 cwts. 2 qrs. 6 lbs.

71. Find the total length of 1500 paces, each pace being 2 ft. 6 ins. long. (Answer in yards.)

72. Find the size of 50 farms, averaging 212 acs. 3 rs. 20 pls. each.

Compound Practice.

172. The method may be seen from the following example:—

Ex.—*Find the cost of* 3 *tons* 17 *cwts.* 2 *qrs. at* £4 5s. 6d. *per ton.*

		£	s.	d.			tons	cwts.	qrs.
		4	5	6	=	price of 1 ton.			
				3					
		12	16	6	=	price of	3	0	0
10 cwts.	$= \frac{1}{2}$ ton	2	2	9	=	,,		10	0
5 cwts.	$= \frac{1}{2}$ of 10 cwts.	1	1	$4\frac{1}{2}$	=	,,		5	0
2 cwts. 2 qrs.	$= \frac{1}{2}$ of 5 cwts.	0	10	$8\frac{1}{4}$	=	,,		2	2
		£16	11	$3\frac{3}{4}$	=	,,	3	17	2

EXAMPLES C.

Find the cost of—

1. 4 cwts. 2 qrs. 11 lbs. at £48 13s. per cwt.

2. 3 tons 17 cwts. 49 lbs. 12 oz. of tapioca at £65 6s. 8d. per ton.

3. 3 lbs. 7 oz. 10 dwts. at £3 17s. 10d. per oz.

4. 7 oz. 12 dwts. 12 grs. at £2 8s. per lb.

5. 47 mls. 3 furls. 5 yds. at £38 2s. 8d. per mile.

6. 2 furls. 7 pls. $1\frac{1}{2}$ yds. at £29 6s. 8d. per mile.

7. 518 acs. 3 rs. 20 pls. at £236 15s. per acre.

8. 15 acs. 3 rs. 17 pls. at £15 13s. 4d. per acre.

9. 4 bushs. 3 pks. 1 gall. 2 qts. at £3 17s. 4d. per quarter.

10. Find cost of warming a building for 11 days 17 hours 28 minutes, if the cost is £4 10s. per day.

11. Find cost of excavating 12 c.yds. 19 c.ft. 1440 c.ins. of granite at 16s. $10\frac{1}{2}$d. per cubic yard.

12. If goods cost £9 6s. 8d. per ton, what is the price of 19 cwts. 3 qrs. 27 lbs.?

13. What is the rent of 7 acs. 3 rs. 15 pls. at £1 10s. 8d. per acre?

14. If gold be worth £3 17s. 6d. per ounce, what is the value of a lump weighing 2 oz. 3 dwts. 16 grs.?

15. What will be the cost of replacing a cistern to weigh 8 cwts. 2 qrs. 14 lbs. at the rate of £2 0s. 6d. per cwt. if the plumber allows £1 11s. 6d. per cwt. for the lead of the old one, which weighs 6 cwts. 1 qr. 10 lbs.?

16. What is the charge for fencing 2 mls. 3 furls. 170 yds. 1 ft. 6 ins. of road at £671 per mile?

17. Find total wages for 73 weeks 30 hours at £1 15s. per week of 48 hours.

18. Find rent of a farm of 899 acs. 2 rs. 30 pls. at £1 16s. 4d. an acre.

19. What is the cost of 2 tons 11 cwts. 60 lbs. of sugar at £1 10s. 11d. per hundredweight?

20. Find value of a gold cup weighing 4 lbs. 2 oz. 15 dwts. 7 grs. at £6 10s. per ounce.

173. A method akin to Practice can often be used to perform complicated multiplications. Thus, suppose we have to multiply by 3·14159. This is not far from $3\frac{1}{7} = 3{\cdot}14286$ (nearly), in fact $3{\cdot}14159 = 3{\cdot}14286 - {\cdot}00127$, but ·127 is near $\frac{1}{8}$; therefore ·00127 is near $\frac{1}{800}$. Hence we may put $3{\cdot}14159 = 3\frac{1}{7} - \frac{1}{800}$ (nearly), and this statement is *true to five figures.*

Ex.—*Multiply* 17·325 *by* 3·1416 *correctly to five significant figures.*

$$\begin{array}{rl} 17{\cdot}325 & \\ 3 & \\ \hline 51{\cdot}975 & = 3 \times 17{\cdot}325 \\ 2{\cdot}475 & = \tfrac{1}{7} \times 17{\cdot}325 \\ \hline 54{\cdot}450 & \\ {\cdot}022 & = \tfrac{1}{800} \times 17{\cdot}325 \\ \hline 54{\cdot}428 & = (3\tfrac{1}{7} - \tfrac{1}{800}) \times 17{\cdot}325 \end{array}$$

The student should verify the following results, all of which are most convenient† :—

	Correct value to six places.		Approx. value.		Decimal value of approx.
$\sqrt[3]{2}$	$= 1{\cdot}25992$	$=$	$1 + \frac{1}{4} + \frac{1}{100}$	$=$	$1{\cdot}26000$;
π	$= 3{\cdot}141592$	$=$	$3 + \frac{1}{8} + \frac{1}{60}$	$=$	$3{\cdot}14166$;
		$=$	$3 + \frac{1}{7} - \frac{1}{800}$	$=$	$3{\cdot}14160$;
$\frac{1}{\pi}$	$= {\cdot}318310$	$=$	$\frac{1}{3} - (\frac{1}{100} + \frac{1}{200})$	$=$	${\cdot}318333$;
$\frac{\pi}{4}$	$= {\cdot}785398$	$= {\cdot}7 + {\cdot}07 +$ twice $({\cdot}007) +$ twice $({\cdot}0007)$		$=$	${\cdot}7854$.

Invoices.

174. If I go into a shop to make a series of purchases, an **invoice,** or list of my purchases, will be made by the assistant in the form given below, and when the goods are sent to my house, the invoice is generally sent along with them that I may see that all which I ordered have been sent. If I pay for the goods at the time, the invoice is **receipted,** *i.e.* the cashier will put a penny stamp on it if the value of the purchase is £2 or over, and write his name and the date across the stamp; but if the value of the purchase is under £2, he will write his name and the date on the invoice without putting the penny stamp. If I do not pay for the goods at the time, the tradesman will, at the end of the week, month, or quarter, send in an **account,** which may be a repetition of all the invoices, in which case it is called a **detailed account,** or may be a **statement** of the amount of the invoices, with their dates and total.

The making up of the invoices should, as far as possible, be done mentally. In the prices common with drapers, some ingenuity is needed. Thus 36 yds. at 1s. $11\frac{3}{4}$d. per yard is worked as 36×2s. $- 36 \times \frac{1}{4}$d. (since 1s. $11\frac{3}{4}$d. is $\frac{1}{4}$d. less than 2s.), and can be easily reduced mentally to £3 11s. 3d.

† Mostly due to Prof. A. Lodge. See Langley, *Computation.* See also § 197.

The method of Ex. 4, p. 153, is most useful. Thus the value in pence of $15\frac{3}{4}$ articles @ 1s. $8\frac{1}{4}$d. is worked as follows:—

$15\frac{3}{4} \times 20\frac{1}{4} = (16 - \frac{1}{4})(20 + \frac{1}{4}) = 320 + 4 - 5$ (to the nearest farthing) $= 319$

Accordingly the value is £1. 6s. 7d.

[This method was suggested by H. F. Davis, *Educ. Times*, lvii., p. 41, 1892.]

Specimen of Invoice (receipted).

Mrs. Smith, Zion Villa.

Bought of JOHN R. HUNTLEY,

PROVISION MERCHANT,

TELEPHONE NO. 160. 1 MILSOM STREET. CLOSED AT 2 ON THURSDAYS.

			£	s.	d.
6 lbs. 2 oz.	Bacon ...	... @ 9d.		4	$7\frac{1}{4}$†
1 lb. 2 ,,	Cheese ...	... @ 8d.			9
2 lbs.	Butter ...	... @ 1s. 2d.		2	4
$1\frac{1}{2}$,,	Sausages	... @ 8d.		1	0
				8	$8\frac{1}{4}$

Received the above,
J. R. HUNTLEY, 17 *Aug.*, 1901.

EXAMPLES CI.

Make out invoices for the following goods, in proper form, with ruled lines:—

1. 12 yds. sheeting at 1s. $2\frac{1}{2}$d.; 6 yds. towelling at $10\frac{1}{4}$d.; 1 doz. pillow-cases at 1s. $1\frac{1}{2}$d. each; 3 toilet-covers at 1s. 3d.; $\frac{1}{2}$ doz. dusters at $4\frac{1}{2}$d. each; $\frac{1}{2}$ doz. tea-cloths at 1s. 2d. each.

2. 3 pairs gloves at 2s. 11d. per pair; 3 pairs ditto at 1s. $6\frac{3}{4}$d.; 5 silk ties at 1s. $0\frac{1}{2}$d.; 3 ditto at 2s. $6\frac{1}{4}$d.; 6 collars at $5\frac{1}{4}$d.; 6 pairs cuffs at $7\frac{1}{2}$d. per pair.

3. 35 yds. sheeting at 1s. $5\frac{3}{4}$d.; 19 yds. towelling at $10\frac{3}{4}$d.; 6 doz. dusters at $4\frac{3}{4}$d. each; 16 tea-cloths at 1s. $3\frac{1}{4}$d.; 23 dish-cloths at $2\frac{1}{2}$d.; 5 bed-quilts at 11s. 9d. each.

4. 37 books at 2s. 9d.; 29 at $4\frac{1}{2}$d.; 46 at 1s. $10\frac{1}{2}$d.; 3 doz. penny pencils; 47 quires of paper at $2\frac{1}{4}$d. a quire; 17 quires blotting paper at $7\frac{1}{4}$d. a quire.

† See Note on Question 5 below. This rule is followed in the answers. It must be applied to every item and not merely to the total.

5. 2 lbs. 3 oz. cheese at 9d. per lb.; 3 lbs. 5 oz. bacon at $10\frac{1}{2}$d.; $4\frac{1}{2}$ lbs. ham at $11\frac{1}{2}$d.; $1\frac{1}{2}$ doz. eggs at 9d. per doz.; 3 lbs. 9 oz. lard at $7\frac{1}{2}$d.; $2\frac{1}{2}$ lbs. butter at 1s. $7\frac{1}{2}$d. (In such questions the tradesman generally reckons the nearest farthing *above* the amount when fractional.)

6. $9\frac{3}{4}$ lbs. beef at $10\frac{1}{2}$d.; 8 lbs. 13 oz. lamb at 1s. $0\frac{1}{2}$d.; 12 lbs. 10 oz. pork at $8\frac{1}{2}$d.; $2\frac{3}{4}$ lbs. suet at $8\frac{1}{4}$d.; $\frac{1}{2}$ doz. kidneys at $2\frac{1}{2}$d. each; 10 lb. 5 oz. mutton at $9\frac{1}{2}$d.

7. A person buys 48 lambs at £1 4s. 6d. each; 81 sheep at £2 3s. $4\frac{1}{2}$d. each; 14 cows at the rate of £36 15s. 9d. for every 3 of them. What is the cost of the whole?

8. Find the total amount of duties payable on the following cargo:—500 galls. of brandy at 10s. 4d. per gall.: 512 galls. of wine at 2s. 6d. per gall.; 432 galls. of wine at 3s. 3d. per gall.; 10 galls. of Eau de Cologne at 16s. 6d. per gall.

9. What is the whole cost of 5 pairs of gloves at 2s. $11\frac{3}{4}$d. per pair; 24 yds. of muslin at 1s. $9\frac{1}{2}$d. per yd.; $17\frac{1}{2}$ yds. of ribbon at $8\frac{1}{2}$d. per yd.; and 35 yds. of flannel at 1s. $7\frac{3}{4}$d. per yd.?

10. A cheque for £3 6s. is tendered in payment of a grocer's bill for 9 lbs. of tea at 2s. 11d. per lb.; $3\frac{1}{2}$ stones of sugar at $2\frac{3}{4}$d. per lb.; 5 lbs. of coffee at 1s. $6\frac{1}{2}$d. per lb.; 6 qts. of vinegar at $4\frac{3}{4}$d. per pt.; and 2 lbs. of gelatine at $3\frac{1}{2}$d. per oz. How much honey at $9\frac{1}{2}$d. per lb. can be bought with the change?

11. A cook buys 10 lbs. of sugar at $2\frac{1}{2}$d. per lb.; 18 eggs at 8 for 1s.; $1\frac{1}{2}$ lbs. of sausages at $9\frac{1}{2}$d. per lb.; 3 lbs. 6 oz. of bacon at 10d.; $1\frac{3}{4}$ lbs. of cheese at 7d.; and a joint of mutton weighing 10 lbs. 4 oz. She receives £4 1s. 3d. change out of a £5 note. What price per lb. was she charged for the mutton?

SECTION VII.

RATIO AND PROPORTION.

CHAPTER XXIII.

DEFINITIONS.

175. We have seen in the sections on Division and on Fractions how to compare one quantity with another of the same kind by finding the number of times the first quantity contains the second. This number of times has hitherto been called "quotient" (whether integral or fractional). **Ratio** is another name to express the same idea. Thus the number of times 10 lbs. contains 5 lbs. may be written as 2, or $\frac{2}{1}$, or $2 \div 1$, or by the new symbol 2 : 1 (read 2 to 1).

So the number of times 6 mls. contains 7 mls. may be written $\frac{6}{7}$ or 6 : 7. This may be otherwise expressed by saying that the ratio of 6 mls. to 7 mls. is $\frac{6}{7}$ or 6 : 7. A ratio is (just like a fraction) a number and not a quantity. The two numbers that form the ratio are called its **terms;** the numerator being called the **antecedent,** and the denominator the **consequent.**

A proper fraction (as $\frac{6}{7}$) is also called a *ratio of less inequality,* as 6 : 7.
An improper fraction (as $\frac{7}{6}$) is ,, ,, ,, *greater* ,, ,, 7 : 6.

176. Since a fraction and a ratio mean the same thing, the properties of the former apply equally to the latter.

(1) A ratio is unaltered if each term be multiplied or divided by the same number (§ 106).

6 : 7 is the same as 12 : 14; 28 : 30 is the same as 14 : 15.

(2) A ratio of less inequality (proper fraction) is increased by adding the same quantity to its two terms.

6 : 7 (*i.e.* $\frac{6}{7}$) is changed into a larger ratio 10 : 11 (*i.e.* $\frac{10}{11}$) by the addition of 4 to each term; for $\frac{6}{7} = \frac{6 \times 11}{7 \times 11} = \frac{66}{77}$, whilst $\frac{10}{11} = \frac{10 \times 7}{11 \times 7} = \frac{70}{77}$, which is larger than $\frac{66}{77}$.

For the general proof of this result, see § 124, (i.).

(3) If a series of ratios be taken, *e.g.* 3 : 4, 5 : 6, 8 : 9, 10 : 11 (arranged in ascending order), then the ratio 3 + 5 + 8 + 10 : 4 + 6 + 9 + 11 is smaller than the greatest and larger than the smallest of the given ratios.

A general proof of this for two fractions was given in § 124, (ii.).

The method applies to any number of fractions.

Ex.—*Find the ratio of* 6 *cwts.* 3 *qrs.* 9 *lbs. to* 10 *cwts.* 3 *qrs.* 20 *lbs., and express it in its most simple form* (*called "lowest terms" in the case of fractions*).

We express both terms in the same unit, viz. lbs.

$$\therefore \quad \text{required ratio} = \frac{765 \text{ lbs.}}{1224 \text{ lbs.}} = \frac{765}{1224} = \frac{5 \times 153}{8 \times 153} = \frac{5}{8} \text{ or } 5 : 8.$$

EXAMPLES CII.

Express the following ratios in Exx. 1–14 in their most simple form—

1. 16 to 8. **2.** 8 to 16. **3.** 25 : $4\frac{1}{2}$. **4.** $4\frac{1}{2}$: 25.

5. 27 : 168. **6.** $7\frac{3}{4}$: $15\frac{1}{2}$. **7.** ·6 : ·$\dot{6}$. **8.** $\frac{3}{7}$: ·12.

9. £1 to 18s. 10d. **10.** 7s. 6d. to 4s. 8d. **11.** 1 ft. to 1 ml.

12. $5\frac{1}{2}$ yds. to 3 mls. **13.** 1 ton 16 cwts. 1 lb. to 3 tons 12 cwts. 2 lbs.

14. 28 oz. to 1 cwt.

Which is the larger in the following pairs of ratios (Exx. 15–18)?—

15. 6 : 7 or 15 : 21. **16.** 8 lbs. to 17 lbs. or 2 ins. to 6 ins.

17. 3s. 6d. to 7s. or 3 to 4. **18.** 4 : 11 or 17 : 30.

19. Arrange the following ratios in descending order:—5 : 6, 16 : 19, 18 : 23. Form a new ratio whose antecedent is the sum of the three antecedents and whose consequent is the sum of the three consequents; and prove that this new ratio lies in value between the largest and the smallest of the given ratios.

20. A boy obtains 203 marks out of a total during three lessons of 300 marks, but in the fourth lesson he obtains 100 out of 100. Prove that the ratio of his final marks to the final total is greater than the corresponding ratio at the end of the third lesson.

PROPORTION.

177. When two ratios are equal to one another, the four terms are said to be in **proportion;** thus $6 : 7 = 12 : 14$, or $6 : 7 :: 12 : 14$ (read "6 is to 7 as 12 is to 14"). So also 6d. : 7d. :: 12 tons : 14 tons, *i.e.* the first and second terms must be of the same kind, and the third and fourth terms must also be of the same kind.

The numbers 6 and 14 are called the **extremes,** 7 and 12 are called the **means.**

Do not confuse this use of "mean" with that for "average."

Since $6 : 7 = 12 : 14$, or $\frac{6}{7} = \frac{12}{14}$, we have $\frac{6 \times 14}{7 \times 14} = \frac{12 \times 7}{14 \times 7}$, *i.e.* two equal fractions have equal denominators; therefore their numerators are equal, *i.e.* $6 \times 14 = 12 \times 7$; *i.e. the product of the extremes is equal to the product of the means.* The method of proof is quite general, and this result is a fundamental property of four numbers in proportion. When any three of the four terms are given, the fourth can be obtained.

Ex. 1.—*What number of lbs. bears to 68 lbs. the same ratio as 3 to 4?*

Arrange thus: Ans. lbs. : 68 lbs. :: 3 : 4;

or Ans. : 68 :: 3 : 4 (omitting the name lbs.);

$\therefore$ Ans. $\times 4 = 68 \times 3$; $\therefore$ Ans. $= \frac{68 \times 3}{4} = 51$. 51 lbs. *Ans.*

Ex. 2.—*If 51 lbs. has to a certain number of lbs. the same ratio as 3 : 4, what is that number of lbs.?*

Arrange thus: 51 lbs. : Ans. lbs. :: 3 : 4;

$\therefore$ $51 \times 4 =$ Ans. $\times 3$; $\therefore$ Ans. $= \frac{51 \times 4}{3} = 68$. 68 lbs. *Ans.*

Ex. 3.—*Find a* **fourth proportional** *to the three numbers* 18, $4\frac{1}{2}$, 100.

I.e. Find a fourth number such that $18 : 4\frac{1}{2} :: 100 :$ Ans.

We have Ans. $\times 18 = 4\frac{1}{2} \times 100$; $\therefore$ Ans. $= 25$;

therefore 25 is the required fourth proportional.

Ex. 4.—*Find a* **third proportional** *to 4 and 6.*

I.e. Find a third number such that $4 : 6 :: 6 :$ Ans.

$\therefore$ Ans. $= 9$.

The number 6 is called a **mean proportional** between 4 and 9.

Ex. 5.—*Find a mean proportional to* 16 *and* 25.

We have 16 : Ans. :: Ans. : 25.

Therefore the answer multiplied by itself $= 16 \times 25$

$$= (4 \times 4) \times (5 \times 5);$$

$$\therefore \quad \text{Ans.} = 4 \times 5 = 20,$$

i.e. 20 is the required mean proportional.

(Such questions generally require the extraction of a square root [See Chap. xxxiii.].)

EXAMPLES CIII.

1. Find the number that is to $7\frac{2}{3}$ in the ratio of £3 1s. 3d. to £4 13s. 11d.

2. £7 10s. $7\frac{1}{2}$d. is to a certain sum of money as $7\frac{1}{2}$ to $11\frac{1}{2}$. Find that sum.

3. What number bears the same ratio to 81 that $\frac{5}{6}$ of ·123 does to $\frac{5}{14}$ of ·567?

4. What sum of money bears the same ratio to £4 8s. $6\frac{1}{2}$d. as 0·12 to 0·17?

5. What must be the first term of a proportion, if the second, third, and fourth terms are respectively $\frac{3}{8}$, $\frac{14}{15}$, and $\frac{1}{20}$?

6. Find a fourth proportional to 0·15, 0·08, 0·09.

7. Find a mean proportional to 5 and 20.

8. Three pounds seven shillings is to three pounds eighteen shillings and twopence as a certain number is to 7. What is that number?

9. Find a third proportional to the numbers 7 and 19.

10. Find a fourth proportional to 7·75, 6·2, 6·2.

11. The numbers ·369, x, ·45, ·8 form a proportion. What is the value of x?

12. What sum bears the same ratio to £5 that 13s. $11\frac{1}{4}$d. does to £3 14s. 4d.?

13. Find a fourth proportional to 275, 2·75, and ·275.

14. What sum of money bears to £1 17s. 6d. the same ratio as 10 to 15?

Simple Proportion.

178. Ex. 1. —*If* 12 *lbs. of tea cost* 18s., *what is the cost of* 16 *lbs. of the same tea?*

First Method.—We may regard this as an example in proportion in which three out of four terms are given and the fourth is required.

Since the cost of 2, 3, &c., lbs. is twice, three times, &c., the cost of 1 lb., we see that the ratio between the cost of 16 lbs. and the cost of 12 lbs. is the same as the ratio between 16 and 12;

$$\therefore \quad \frac{\text{cost of 16 lbs.}}{\text{cost of 12 lbs.}} = \frac{16}{12};$$

$$\text{cost of 16 lbs.} = \text{cost of 12 lbs.} \times \frac{16}{12} = 18\text{s.} \times \frac{16}{12} = £1\ 4\text{s.} \quad \textit{Ans.}$$

Second Method.—

12 lbs. cost 18s. ;

∴ 1 lb. costs 18s. $\div 12 =$ 1s. 6d. ;

∴ 16 lbs. cost 1s. 6d. $\times 16 =$ £1 4s.

Explanation.—The cost of 1 lb. is $\frac{1}{12}$ the cost of 12 lbs., *i.e.* $\frac{1}{12}$ of 18s. or 1s. 6d. The cost of 16 lbs. is 16 times the cost of 1 lb., *i.e.* 16 times 1s. 6d., or £1 4s.

Here we change from the quantity 12 lbs. to the quantity 16 lbs. by the intermediate quantity 1 lb. (or some other convenient unit) ; hence the name **Unitary Method** This method is employed in Exx. 2 and 3, and leads gradually to the so-called **Fractional Method** employed in the sequel.

The working might have been shown thus :

12 lbs. cost 18s.,

i.e. 3 packets of 4 lbs. cost 18s. ;

∴ 1 packet of 4 lbs. costs 6s. ;

∴ 4 packets of 4 lbs. cost 24s.,

or 16 lbs. cost £1 4s.

Here we change from 12 lbs. to 16 lbs. by the intermediate quantity 4 lbs., the number 4 being the H.C.F. of 12 and 16.

Ex. 2.—*A man walks* 14 *mls. in* 4 *hours. How far will he walk in* 10 *hours at the same rate ?*

Notice what that quantity is in terms of which the answer is to be given (in this example, "miles"). State the data of the question in an abbreviated form, always putting this quantity on the right-hand side, thus :

in 4 hours he walks 14 mls. ;

,, 1 hour ,, ,, $14 \div 4$ mls. ;

∴ ,, 10 hours ,, ,, $\frac{14 \times 10}{4}$ or 35 mls.

The results of each step should not be worked out in full, but should be left to the end to save working, as frequently certain factors in the numerator and denominator cancel.

Ex. 3.—*What will be the cost of* 3 *tons* 8 *cwts. of coal, if* 17 *cwts. cost* 15s. 6d. ?

The answer is to be "money"; in the data the money named is 15s. 6d. Therefore we have

17 cwts. cost 15s. 6d. (= 186d.) ;

∴ 1 cwt. costs $(186 \div 17)$ pence ;

∴ 3 tons 8 cwts. (= 68 cwts.) cost $\frac{186 \times 68}{17}$

$= 744$d.

$=$ £3 2s.

Note that 15s. 6d. was reduced to pence ; therefore the answer is in pence, which is afterwards reduced to £ s. d.

We might have worked thus :

17 cwts. cost $15\frac{1}{2}$s. ;

∴ 1 cwt. costs $15\frac{1}{2}$s. $\div 17$;

∴ 3 tons 8 cwts. cost $\frac{31 \times 68}{2 \times 17}$ s. $=$ £3 2s.

EXAMPLES CIV.

1. If 30 yds. of silk cost £4 10s., what is the cost of 75 yds.?

2. What is the cost of 50 lbs. of tea when 13 lbs. cost £1 6s.?

3. The railway fare for 51 mls. is 8s. 6d. What is the fare for 100 mls ?

4. If 77 yds. of cloth cost £15 8s., what is the cost of 1 yd.?

5. When half-a-dozen chairs cost £2 2s., what will three dozen of the same kind cost?

6. If 13 lbs. of tea cost £1 5s., find the cost at the same rate of 3 qrs. 7 lbs. of the same tea.

7. A train travels 45 mls. in an hour. How many yards is that in 1 minute?

8. Find the price of 748 articles at 13s. 6d. a dozen.

9. If 1 ton 11 cwts. of coal cost £1 16s., what will be the cost of 4 tons 13 cwts.?

10. If 7 lbs. of butter cost 10s. 6d., how many pounds can be bought for £1 10s.?

11. When cloth is sold at 25s. for 12 yards, how many yards can be bought for £7 10s.?

12. If 37 sheep cost £64 15s., find the cost of 49 sheep.

13. What will be the railway fare for 146 mls., if the fare for 272 mls. is £2 16s. 8d.?

14. If 14 yds. of cloth cost £2 7s. 3d., find the cost of 80 yds.

15. If 76 articles cost £25 0s. 4d., how much will 46 cost?

16. Find the value of 368724 articles at 13s. 4d. per score.

17. Find the cost of 1527 articles at £3 19s. 9d. per dozen.

18. If 9s. $4\frac{1}{2}$d. be equally divided between 15 children, how much would each child get? For how many children at the same rate would £1 suffice?

179. We give some rather harder problems :—

Ex. 4.—*If* 8 *cwts.* 1 *qr.* 26 *lbs. cost* £23 5s. $1\frac{1}{4}$d., *find the cost of* 1 *cwt.*

8 cwts. 1 qr. 26 lbs. = 950 lbs.; 1 cwt. = 112 lbs.; and £23 5s. $1\frac{1}{4}$d. = $5581\frac{1}{4}$d. Answer is to be in pence.

We now have

950 lbs. cost $5581\frac{1}{4}$ pence;

$$\therefore \quad 112 \text{ lbs. cost } \frac{5581\frac{1}{4} \times 112}{950} \text{ pence}$$

$$= \frac{22325 \times 112}{4 \times 950} \text{ pence}$$

$$= 658\text{d.} = £2\ 14\text{s.}\ 10\text{d.}$$

The intermediate step of changing to unity may frequently be omitted in the working.

Ex. 5.—*If the wages of* 12 *men for* 210 *days be* £514 10s., *how much would they receive for* 180 *days' work?*

As the number of men is the same in both parts, the number 12 does not enter into the working. The answer is to be money.

For 210 days the wages are £$514\frac{1}{2}$;

for 180 ,, ,, ,, £$\frac{1029 \times 180}{2 \times 210}$

= £441.

EXAMPLES CV.

1. What will be the railway fare for 73 mls., if the fare for 136 mls. be £1 8s. 4d.?

2. If $6\frac{1}{4}$ tons of coal cost £6 15s. 5d., what will be the price of 5 tons 3 cwts.?

3. A grocer bought 2 tons 3 cwts. 3 qrs. of sugar for £61 5s. What would he give for 5 cwts. 1 qr. 16 lbs.?

4. If 27 horses can be bought for £2400, how many can be bought for £1333 6s. 8d.?

5. A man can mow 264 sq.yds. in an hour. How long will he take to mow 1 ac. 3 rs. 34 pls. $5\frac{1}{2}$ sq.yds.?

6. If 14 cwts. 8 lbs. of steel cost £54 3s. 6d., what will be the cost of 17 cwts. 16 lbs.?

7. An Indian officer, whose annual pay was estimated in rupees, lost £41 12s. 6d. in one year by a fall in the value of a rupee from 1s. $11\frac{1}{2}$d. to 1s. $10\frac{1}{4}$d. What was his salary, estimated in rupees?

8. If a cubic foot of water weighs 1000 oz., how many cubic yards of water will weigh $337\frac{1}{2}$ tons?

9. If 3 mls. 5 furls. 36 pls. 2 yds. of telegraph wire weigh 4 tons 8 cwts. 14 lbs., how many pounds does 1 yd. weigh?

10. Find the cost of 19 lbs. of tea, if 17 lbs. of the same tea cost £2 7s. $5\frac{1}{2}$d.

11. A sum of £85 8s. 4d. is to be divided for a dinner among 1000 people. If only £31 3s. $6\frac{1}{2}$d. is collected, find how many can be provided.

12. A clock loses $30\frac{1}{2}$ seconds in 11 hours 11 minutes. How many minutes will it lose in 3 days 5 hours?

13. A man walks 15 mls. in $3\frac{1}{2}$ hours. How long would he take to walk 105 mls. at the same rate?

14. Find the value of 3 cwts. 2 qrs. of meat at 6s. 4d. per stone of 8 lbs.

15. What is the value of 1 lb. Troy of silver cut from a bar weighing 400 oz. and worth £50 16s. 8d.?

16. If the cost of reaping 32084 pls. is £32 10s., what will be the cost of reaping 13574 pls.?

17. If 42 yds. of material cost £2 6s. $4\frac{1}{2}$d., how much will 35 yds. of the same material cost?

18. If 13 acs. 1 r. 15 pls. of land are valued at £1547 17s. 6d., what is the value per acre?

19. If 78 tons 10 cwts. 3 qrs. 10 lbs. 1 oz. cost £722 4s. $4\frac{1}{4}$d., find the cost of 123 tons 8 cwts. 1 qr. 23 lbs. 13 oz.

20. If 3 tons 2 qrs. cost £15 16s. 2d., what is the price of 2 tons 12 cwts. 3 qrs. 21 lbs.?

21. If 11 cwts. 1 qr. cost £9 3s. 9d., find the value of 1 cwt. 2 qrs. 12 lbs.

22. If 2 qrs. 17 lbs. of coffee cost £5 3s. 5d., what is the value of 6 cwts. 14 lbs.?

23. If $1\frac{3}{4}$ yds. of silk cost 10s. $11\frac{1}{4}$d., what will be the cost of $75\frac{5}{6}$ yds., and how many yards can be bought for £5?

24. If a train runs 49 mls. in 1 hour 10 minutes, how many seconds will it take to run 770 yds.?

25. If 4 cwts. 1 qr. 26 lbs. of tobacco can be bought for £56 9s. 6d., what quantity can be purchased for £4 19s. 9d.?

26. A man's hotel bill for 4 weeks 5 days amounted to £28 17s. 6d. How much would he have to pay at the same rate for 6 weeks 4 days?

27. If $3\frac{1}{2}$ lbs. can be bought for £1 17s. 4d., how many pounds can be bought for £106 2s. 8d.?

28. If 1 oz. 4 dwts. 9 grs. cost 15s. $6\frac{1}{4}$d., what is the price of 5 oz. 17 dwts.?

29. Find the value of 3766 lbs. at 8s. $10\frac{1}{2}$d. per stone of 14 lbs.

30. The Irish acre is to the English acre as 49 is to $30\frac{1}{4}$. How many English acres are there in a farm containing $852\frac{1}{2}$ Irish acres?

31. Find the value of 35688 articles at £1 11s. $9\frac{3}{4}$d. per dozen.

32. Find the cost of 89604 articles at 4s. 7d. a score.

33. If a field of $3\frac{3}{4}$ acs. is let for £13 15s. a year, what should be the rent of land of the same quality measuring 15 acs. 2 rs. 4 pls. 11 yds.?

34. If the fare for a journey of 93 mls. is £1 11s. 6d., how far would one travel, at the same rate per mile, for £143 6s. 6d.?

35. If 41 qrs. 2 bushs. 3 pks. of wheat cost £86 16s. $5\frac{1}{4}$d., how much wheat can be bought for £136 8s. $8\frac{1}{4}$d.?

36. If 5 lbs. 7 oz. 9 dwts. 6 grs. of silver plate cost £22 9s. 9d., what weight of it can be bought for £18 1s. $3\frac{1}{2}$d.?

37. If 12 guineas' worth of water leak out of a reservoir in a week, what value would be lost at the same rate of leakage in 2 weeks 1 day 20 hours $42\frac{1}{2}$ minutes?

38. Given that an ounce Troy of standard gold is worth £3 17s. $10\frac{1}{2}$d., how much do 623 sovereigns weigh?

39. If $\frac{1}{6}$ of a gallon cost 3s. 6d., how much should be obtained for 5s. 3d.?

40. If 1 ton 19 cwts. of lead is worth £50 2s. $7\frac{1}{2}$d., what is the value of 10 cwts. 2 qrs. of the same?

41. What is the value of an ingot of silver which weighs 576 oz., if 22 oz. of the silver are worth £2 9s. $0\frac{1}{2}$d.?

42. If it costs £1 10s. 3d. to run a train 11 mls., how much will it cost to run it 150 mls. under the same conditions?

43. If 18 lbs. 3 oz. cost 6s. $0\frac{3}{4}$d., what will be the cost of 2 cwts. 1 qr. 3 lbs.?

44. If $\frac{7}{10}$ of a ton of metal is worth £68, what is the value of $5\frac{1}{4}$ tons?

45. If 7 lbs. of tea cost £1 3s. 4d., how much will 1 lb. 7 oz. cost?

46. If 5 oz. 11 dwts. of silver cost £1 7s. 9d., what will be the value of 1 lb. 17 dwts. 12 grs.?

47. If 9 lbs. 9 oz. 10 dwts. of silver are worth £32 6s. 3d., what is the value of 3 lbs. 4 oz.?

48. If 13 jets of gas burn 104 c.ft. of gas in 2 hours, how much will 7 jets burn in the same time?

49. If 17 cwts. 3 qrs. 14 lbs. cost £6 14s. $0\frac{3}{4}$d., find the value of $\frac{1}{2}$ ton.

50. If the cost of paving 27 ft. of road be £10 16s., how much should be paved for £24?

51. If 2 tons 14 cwts. 1 qr. 27 lbs. 9 oz. cost £305 3s. $6\frac{3}{4}$d., what will 7 cwts. 3 qrs. 3 lbs. 15 oz. cost?

52. If 140 ells cost £59 4s. $10\frac{3}{4}$d., how much *more* would 193 yds. cost? (5 yds. = 4 ells.)

Inverse Proportion.

180. All the foregoing problems are examples of what is called **direct proportion,** *i.e.* when one quantity is doubled, trebled, &c., as another quantity is doubled, trebled, &c. When, however, the multiplication of one quantity by 2, 3, &c., involves the division of the other quantity by 2, 3, &c., the proportion is said to be **inverse;** for instance, if a *fixed* quantity of money is to be divided amongst a certain number of persons, then, the *more* the persons, the *less* each will receive; or, if a definite piece of work is to be done, the *greater* the number of persons, the *less* is the time they are employed on it.

Ex. 1.—*If* 18 *men can do a piece of work in* 10 *days, in how many days will* 45 *men do the same work?*

The answer is to be in days.

18 men do the work in 10 days;

$\therefore$ 1 man does ,, ,, ,, 10×18 days;

$\therefore$ 45 men do ,, ,, ,, $\frac{10 \times 18}{45}$ days

$= 4$ days.

Explanation.—One man will require 18 times as long as 18 men to do the same work; therefore we *multiply* 10 by 18.

45 men will require only $\frac{1}{45}$ of the time that 1 man requires; therefore 10×18 must be *divided* by 45.

Ex. 2.—*If* 3 *men* **or** 5 *women take* 17 *days to do a certain piece of work, how long will* 7 *men* **and** 11 *women take?*

We are told that, as regards capacity for work, 5 women = 3 men;

$$\therefore \quad 1 \text{ woman} = \tfrac{3}{5} \text{ man};$$

$$\therefore \quad 7 \text{ men} + 11 \text{ women} = 7 \text{ men} + 6\tfrac{3}{5} \text{ men} = 13\tfrac{3}{5} \text{ men}.$$

We now have 3 men take 17 days;

$\therefore$ 1 man takes 17×3 days,

$\therefore$ $13\frac{3}{5}$ men take $\frac{17 \times 3 \times 5}{68} = 3\frac{3}{4}$ days;

i.e. 7 men and 11 women take $3\frac{3}{4}$ days.

To avoid the absurdity of speaking of the fraction of a man, the following method may be adopted:—

The L.C.M. of 5 and 11 is 55; instead then of 7 men + 11 women take 5 times as many, *i.e.* 35 men + 55 women = 35 men + 33 men = 68 men.

Now 3 men take 17 days;

$\therefore$ 68 men take $\frac{17 \times 3}{68}$ or $\frac{3}{4}$ day,

i.e. 5 times (7 men + 11 women) take $\frac{3}{4}$ day;

$\therefore$ 7 men + 11 women take $\frac{3}{4} \times 5$ or $3\frac{3}{4}$ days.

EXAMPLES CVI.

1. If 125 men can do a piece of work in 120 days, in how many days will 150 men do it?

2. Fifteen hundred men in a fortress have provisions for 48 days; but five hundred men join them. How long will the provisions now last

3. If $1002\frac{7}{13}$ c.yds. of earth are carted $3\frac{5}{7}$ miles for £70, how far should 181 c.yds 648 c.ins. be carted for the same sum?

4. If 10 men and 5 women can do a piece of work in 14 days, and the work of 4 women is equal to that of 3 men, in how many days could one man and one woman together do the work?

5. If 12 men can do a piece of work in 12 days of 10 hours each, how many men will be wanted to do the same piece of work in 10 days of 8 hours each?

6. If 5 men *or* 7 women can do a piece of work in 37 days, in what time will 7 men *and* 5 women do a piece of work twice as great.

7. If 15 men can do as much work in the same time as 21 boys, how long will it take 25 men to do what 30 boys can do in 14 hours?

8. If a certain amount of work is done by 9 men, 12 women, and 13 boys, all working together, in 11 days, how long will the same work take if 18 men, 3 women, and 5 boys are together set to do it, assuming that the ratio of a man's work to a woman's work is as 5 to 3, and a woman's work to a boy's as 4 to 3?

9. If 4 men can do a piece of work in 52 days, how long would 13 men take to do the same?

10. If 32 horses consume a certain amount of corn in 36 days, for how many days would the same amount be sufficient for 96 horses?

11. If 2 sheep *or* 3 lambs consume in 7 days a certain quantity of turnips, how many days will it take 4 sheep with 3 lambs to eat the same quantity?

12. Four men = 7 boys; 3 women = 2 men and a boy. Find how long it will take 12 men, 14 women, and 21 boys together to do a piece of work, when 15 men with 7 women and 14 boys do it in 18 days.

13. A garrison of 420 have food enough to last them 35 days. After 5 days they are reinforced by 210 men, bringing no provisions with them. How much longer will the food last?

14. If 3 men *or* 5 women can do a piece of work in 11 days, how long will 5 men with 3 women take?

Compound Proportion.

181. Ex. 1.—*If* 64 *men dig a trench* 50 *yds. long*, 2 *yds. wide, and* 2 *yds. deep in* 5 *days of* 12 *hours each, in how many days of* 8 *hours each will* 80 *men dig another trench* 75 *yds. long*, 4 *yds. broad, and* 3 *yds. deep?*

The answer is to be "days." We therefore restate the problem in tabulated form, putting the given number of days on the right-hand side of the upper line, thus:—

men		yds. long	yds. wide	yds. deep		hours a day		days
64	dig a trench	50	2	2	working	12	in	5 ;
80	,, ,,	75	4	3	,,	8	,,	*Ans.*

Now the required number of days depends on five different ratios—viz.: that of (1) the men, (2) the lengths, (3) the widths, (4) the depths,

(5) the hours per day; so that we have five distinct problems in Simple Proportion, nos. (1) and (5) being *inverse*, whilst nos. (2), (3), and (4) are *direct*. We will work these five problems separately.

(1) If 64 men dig a trench in 5 days,
∴ 1 man digs same ,, ,, 5×64 days;
∴ 80 men dig ,, ,, ,, $\frac{5 \times 64}{80} = 4$ days.

(2) If 80 men dig a trench 50 yds. long in 4 days,
∴ the same men ,, ,, 75 yds. ,, $\frac{4 \times 75}{50} = 6$ days.

(3) If 80 men dig a trench 75 yds. long and 2 yds. wide in 6 days,
∴ the same men ,, ,, ,, ,, 4 yds. ,, $\frac{6 \times 4}{2} = 12$ days.

(4) If 80 men dig a trench 75 yds. long 4 yds. wide 2 yds. deep in 12 days,
∴ the same men ,, ,, ,, ,, ,, 3 ,, $\frac{12 \times 3}{2} = 18$ days.

(5) If 80 men dig a trench 75 yds. long 4 yds. wide 3 yds. deep working 12 hours per day in 18 days,
∴ the same ,, ,, ,, ,, ,, 8 ,, $\frac{18 \times 12}{8}$ or 27 days,
i.e. required number of days = 27.

The work may also be put as follows:—

Question (*a*).—Will 80 men (the number in lower line) take more *days* or less *days* than 64 men (the number in upper line) to do the *same* work? *Ans.*, *Less* days. Therefore multiply by $\frac{64}{80}$, *i.e.* $\frac{\text{smaller number}}{\text{larger number}}$.

Question (*b*).—Will a certain number of men in digging a trench 75 yds. long (the number in lower line) take more *days* or less *days* than in digging one 50 yds. long (the number in upper line)? *Ans.*, *More* days. Therefore multiply by $\frac{75}{50}$, *i.e.* $\frac{\text{larger number}}{\text{smaller number}}$.

Question (*c*).—Will a certain number of men in digging a trench 4 yds. wide (the number in lower line) take more *days* or less *days* than in digging one 2 yds. wide (the number in upper line)? *Ans.*, *More* days. Therefore multiply by $\frac{4}{2}$, *i.e.* $\frac{\text{larger number}}{\text{smaller number}}$.

Question (*d*).—Will a certain number of men in digging a trench 3 yds. deep (the number in lower line) take more *days* or less *days* than in digging one 2 yds. deep (the number in upper line)? *Ans.*, *More* days. Therefore multiply by $\frac{3}{2}$, *i.e.* $\frac{\text{larger number}}{\text{smaller number}}$.

Question (*e*).—In working 8 hours a day (the number in lower line), will the men want more *days* or less *days* than in working 12 hours a day (the number in upper line)? *Ans.*, *More* days. Therefore multiply by $\frac{12}{8}$, *i.e.* $\frac{\text{larger number}}{\text{smaller number}}$.

$$\therefore \text{ required number of days} = \overset{(a)}{\frac{5}{1}} \times \overset{(b)}{\frac{64}{80}} \times \overset{(c)}{\frac{75}{50}} \times \overset{(d)}{\frac{4}{2}} \times \overset{(e)}{\frac{3}{2}} \times \frac{12}{8},$$

which, when simplified, becomes 27 days.

Ex. 2.—*Two men with* 7 *boys can do a piece of work in* 14 *days;* 3 *men with* 8 *boys can do the same in* 11 *days. In how many days can* 8 *men with* 6 *boys do three times the amount of work?*

	men		boys		days		men		boys		day
	2	+	7	take	14;	∴	28	+	98	take	1.
Again,	3	+	8	,,	11;	∴	33	+	88	,,	1.

Therefore 28 men + 98 boys are equivalent to 33 men + 88 boys in powers of work, *i.e.*

$$28 \text{ men} + 88 \text{ boys} + 10 \text{ boys} = 5 \text{ men} + 28 \text{ men} + 88 \text{ boys};$$

$$\therefore \quad 1 \text{ man} = 2 \text{ boys};$$

$$\therefore \quad 2 \text{ men} + 7 \text{ boys} = 11 \text{ boys}, \quad 8 \text{ men} + 6 \text{ boys} = 22 \text{ boys}.$$

The problem now becomes as follows:—If 11 boys can do a piece of work in 14 days, in how many days can 22 boys do three times the amount of work?

boys	work done	days
11	1	14
22	3	*Ans.*

The answer $= \frac{14}{1} \times \frac{11}{22} \times \frac{3}{1} = 21$ days.

Ex. 3.—*If* 17 *cwts. are carried* 15 *miles for* £1 10s., *how far can* 2 *tons* 11 *cwts. be carried for* £5?

cwts.	s.	mls.
17	30	15
51	100	*Ans.*

Question (*a*).—Will 51 cwts. be carried more *miles* or less *miles* than 17 cwts. for the same money? *Ans.*, *Less* miles. Therefore multiply by $\frac{17}{51}$.

Question (*b*).—Will the same weight be carried more *miles* or less *miles* for 100s. than for 30s.? *Ans.*, *More* miles. Therefore multiply by $\frac{100}{30}$.

$$\therefore \text{ required number of miles} = \overset{(a)}{\frac{15}{1}} \times \overset{(b)}{\frac{17}{51}} \times \frac{100}{30} = 16\tfrac{2}{3} \text{ miles}.$$

EXAMPLES CVII.

Find the answers in the following examples—

1. 10 men in 5 days earn £10.
15 ,, ,, 3 ,, ,, *Ans.*

2. 85 tons for £34 are carried 48 mls.
50 ,, ,, £30 ,, ,, *Ans.*

3. 85 tons for 8 mls. cost £34.
50 ,, ,, 12 ,, ,, *Ans.*

4. 6 men working 10 hours a day do a piece of work in 18 days.
8 ,, ,, 9 ,, ,, ,, the same ,, ,, *Ans.*

5. 18 horses plough $10\frac{1}{2}$ acs. in 15 days.
64 ,, ,, 112 ,, ,, *Ans.*

6. 6 men do a piece of work in 18 days of 10 hours a day.
8 ,, ,, the same ,, ,, 15 ,, ,, *Ans.*

7. 16 horses eat 12 pks. in 10 days.
56 ,, ,, 252 ,, ,, *Ans.*

8. 35 tons for £20 are carried 10 mls.
42 ,, ,, £18 ,, ,, *Ans.*

9. For 35 mls. £4 17s. 6d. pays for 13 cwts. 2 qrs. 19 lbs.
,, 49 ,, £20 9s. 6d. ,, ,, *Ans.*

10. 16 horses eat 112 bushs. of corn in 36 days.
54 ,, ,, 63 ,, ,, ,, *Ans.*

EXAMPLES CVIII.

1. If 36 men, working 8 hours a day for 16 days, can dig a trench 72 yds. long, 18 ft. wide, and 12 ft. deep, in how many days will 32 men, working 12 hours a day, dig a trench 64 yds. long, 27 ft. wide, and 18 ft. deep?

2. When wheat is at 5s. 9d. per bushel, the 6d. loaf weighs 3·96 lbs. What is the price of $34\frac{1}{2}$ lbs. of bread, when wheat is at 18s. 4d. per bushel?

3. If 10 men, working 7 hours a day, dig a trench 147 ft. long, how many men working 8 hours a day will dig a trench 168 ft. long and of the same breadth and depth as the first in the same number of days?

4. If 4 men with 6 boys working 9 hours a day mow 69·3 acs. of corn in 10 days, in how many days will 5 men and 4 boys working 10 hours 40 minutes a day mow 67·76 acs., it being assumed that 2 men mow as much as 5 boys?

5. If 10 sheep *or* 15 lambs eat 40 bushs. of turnips in 7 days, how long will it take 6 sheep *and* 18 lambs to eat 36 bushs.?

6. If the freight on 5 tons 13 cwts. 84 lbs. of goods amounts to £6 12s. $8\frac{1}{2}$d. for 350 mls., what is the rate per ton for 100 mls.?

7. If 6 compositors in 16 days of $10\frac{1}{2}$ hours each can set in type 720 pages, each of 60 lines with 40 letters in a line, in how many days of 7 hours each will 9 compositors set 960 pages, each of 45 lines with 50 letters in each line?

8. If 7 men *or* 12 boys can hoe a field of 12 acs. in 6 days, working 10 hours a day, in how many days of 8 hours each can 14 men *and* 6 boys hoe a field of 60 acs.?

9. The gross revenue of a business in 9 weeks of 7 days each is £441. What will be the gross revenue at the same rate in $9\frac{1}{2}$ weeks of 6 days each?

10. If 120 men working 10 hours a day can make an embankment 90 yds. long, 20 ft. wide, and 8 ft. high, in 5 days, how many days will it take 500 men working 9 hours a day to make an embankment 1200 yds. long, 25 ft. wide, and 9 ft. high?

11. If the wages of 11 labourers for 15 weeks be £103 2s. 6d., in what time will the wages of 13 labourers amount to £170 12s. 6d. at the same rate per week?

12. If a beam of timber $13\frac{1}{2}$ ft. long, $9\frac{1}{2}$ ins. wide, and $7\frac{1}{2}$ ins. in thickness, weigh $320\frac{5}{8}$ lbs., what will be the weight in pounds of another beam of the same kind of timber, $15\frac{3}{4}$ ft. long, $11\frac{1}{2}$ ins. wide, and 8 ins. thick?

13. How many days of 10 hours each would 60 men take to perform a piece of work which 45 men can do in 30 days of 12 hours each?

14. If the cost of provisioning a gunboat carrying 84 men be £598 10s. when the ship is at sea for 95 days, what will it cost to provision for 33 days a ship carrying a crew of 110 men?

15. If 35 men can do a piece of work in 12 days, working 8 hours a day, in how many days could 30 men do half the work, working 7 hours a day?

16. If 24 lbs. of spun flax make 69 yds. of linen 1 yd. wide, how many yards of linen 4 ft. wide will 164 lbs. make?

17. If a family by using 6 gas-burners 5 hours a day pay £1 5s. per quarter when gas is at 5s. per 1000 c.ft., what will a family using 8 burners 3 hours per day pay per quarter when gas is at 3s. 9d. per 1000 c.ft.?

18. If 310 men can dig a trench 11 ft. broad, 6 ft. deep, and 155 yds. long in half an hour, find what length (in yards) of trench 10 ft. broad and 7 ft. deep 2000 men can dig in 6 hours.

19. If 90 men can dig a ditch 50 yds. long in $4\frac{1}{2}$ days, how many men can dig a ditch 360 ft. long in 27 days?

20. If 17 men can do a piece of work in 24 days of 10 hours each, in how many days of 8 hours each can 18 men do the same piece?

21. If 72 horses consume 16 qrs. of corn in 12 days, how many will 24 qrs. feed for 16 days?

22. If a man walks 21 mls. in 6 hours, how many minutes will he take to walk half a mile, his speed being increased in the ratio of 8 to 7?

23. If 30 men, working 8 hours a day at a uniform rate, earn £72 in 9 days, how much will 63 men earn who work for 13 hours a day at two-thirds of the same uniform rate in 10 days, the wages being proportional to the amount of work done?

24. If the carpet for a room 17 ft. 6 ins. long and 12 ft. wide costs £4 1s. 8d., what will be the cost of a carpet of the same material for a room 23 ft. 3 ins. long and 16 ft. wide?

25. If when the price of wheat is 8s. a bushel one pennyworth of bread weighs 10 oz., what should be the weight of the threepenny loaf when wheat of the same quality costs 7s. 6d. a bushel?

26. If 9 men dig a trench 27 yds. long, 15 ft. wide, and 11 ft. deep in 19 days, how many men must be employed to dig a trench 22 yds. long, 18 ft. wide, and 20 ft. deep in 16 days?

27. A piece of work can be done by 8 men and 12 boys in 10 days. Assuming that a man does twice as much in an hour as a boy, and that the men work 8 hours a day and the boys 6, how many boys will be required to assist 13 men so that the work may be done in 5 days?

28. If 2 men and 3 boys can do a piece of work in 16 days, and 5 men and 6 boys can do the same in 7 days, how long will 4 men and 8 boys take to do it?

29. If 9 men can mow 12 acs. in 4 days, working 11 hours a day, how many hours a day must 7 men work in order to mow 14 acs. in $5\frac{1}{2}$ days?

30. If 12 labourers can dig a trench measuring 30 yds. long and 4 yds. wide in 4 days of 9 hours each, in how many days of 8 hours each will 18 men dig a trench 80 yds. long and 3 yds. wide, and of the same depth as the other?

31. If 11 coins, each $\frac{1}{20}$ of an inch thick, weigh 2 oz. 17 dwts. 7 grs., find the weight of 18 similar coins, each $\frac{1}{25}$ of an inch in thickness.

32. If 8 labourers can dig a trench measuring 150 yds. long and 4 ft. deep in 4 days of 9 hours each, in how many days of 8 hours each will 12 men dig a trench 400 yds. long, 3 ft. deep, and of the same width as the former one?

33. If 2 horses plough 7 acs. in a day, how many will plough 322 acs. in $11\frac{1}{2}$ days?

34. If 12 men do a piece of work in 36 days, how many men will do twice as much work in 27 days?

35. If 16 cwts. are carried 63 mls. for £6 6s., what weight can be carried 112 mls. for £2 16s.?

36. If 35 tons can be carried $22\frac{1}{2}$ mls. for £15 12s. 6d., how far can 63 tons be carried for £27 10s.?

37. If $\frac{4}{7}$ of a field can be dug by 5 men in 8 days, how long would 3 men take to dig the remainder?

38. If a crew of 15 men require 2550 galls. of water for a voyage of 68 days, for how many days would a supply of 3600 galls. be sufficient for a crew of 18 men?

39. If 25 men do a certain amount of work in 14 days, working 8 hours a day, in how many days will 7 men perform $\frac{2}{5}$ of the work, working 10 hours a day?

40. If 8 men can reap 16 acres in 3 days, working 10 hours a day, in how many days can 3 men reap 64 acres, working $10\frac{2}{3}$ hours a day?

41. If 8 men *or* 17 boys do a piece of work in 13 days, how many days will it take 4 men *and* 24 boys to do a piece of work 5 times as great?

42. In 11 days 250 men working 9 hours a day complete 528 yds. of an embankment which is to be $1\frac{1}{2}$ mls. long. How many additional men must be put on in order that, if all now work 10 hours a day, the embankment may be finished in 33 days more?

43. When the price of wheat was 45s. 2d. per quarter, and the yield 4 qrs. per acre, the produce of a certain piece of land was worth £153 11s. 4d. What must the yield be per acre, if the produce of the same land is worth £157 5s., when the price of wheat is 30s. 10d. per quarter?

44. Five men are employed 7 hours a day on a certain work. After they have been engaged 4 days, it is found that they have done one-twelfth of the whole. Six additional men are then employed, and all are to work $7\frac{1}{2}$ hours a day When will the work be completed, supposing that they all work equally fast?

45. If 124 men in $5\frac{1}{2}$ days of 11 hours each dig a trench $232\frac{1}{2}$ yds. long, 3 ft. 8 ins. wide, and 2 ft. 4 ins. deep, in how many days of 9 hours each will 36 men dig a trench $337\frac{1}{2}$ yds. long, 5 ft. $7\frac{1}{5}$ ins. wide, and 3 ft. 6 ins. deep, supposing that in the same time each of the second set of men does $\frac{1}{6}$ more than each of the first set?

46. If 35 men, working 9 hours a day, build 17 ft. of a wall in 18 days, how many men, working $7\frac{1}{2}$ hours a day for 21 days, will be required to build 119 ft. of the same wall?

47. If 7 men and 2 women earn £51 in 8 weeks, and 4 men with 2 women earn £46 10s. in 12 weeks, what are the weekly wages of a man and of a woman?

48. An engine while driving machinery burns coal at the rate of 1 ton 12 cwts. 2 qrs. in 8 hours 40 minutes. When the machinery is not in motion, the consumption of coal is only $\frac{4}{11}$ of this rate. How much coal will the engine burn in 1584 hours, during $\frac{1}{9}$ of which time the machinery is at rest?

49. One man, 3 women, and 4 boys can do a piece of work in 96 hours; 2 men and 8 boys can do it in 80 hours, and 2 men with 3 women can do it in 120 hours. In how many hours can it be done by 5 men and 12 boys?

50. If 9 men have done $\frac{3}{5}$ of a piece of work in 14 days, how many extra men must be engaged in order that the rest of the work may be finished in 4 days?

51. A farmer engaged a number of men and boys to reap corn; the men were to receive 4s. a day each and the boys 2s. 6d. each. If the work of 3 boys were equal to that of 2 men, and 4 men with 5 boys could reap 22 acs. in 4 days, what sum should 6 men with 7 boys receive for reaping 48 acs.?

52. Four men, 3 women, and 2 boys can reap 7 acs. of corn in 2 days; 2 men and 6 women can reap 11 acs. in 4 days; 3 men and 5 boys can reap 5 acs. in 2 days, the men working 10 hours, the women 8 hours, and the boys 6 hours a day. If a man receive 8d. an hour, a woman 6d., and a boy 4d., what would 6 men, 5 women, and 4 boys earn in reaping 33 acs., each working the same number of hours a day as before?

53. A contractor employs some men to do a piece of work which can be done by 16 men in 14 days. At the end of 5 days 7 of the men employed stop work, and 3 days afterwards half the remainder stop; the rest finish the work in 5 days. Determine the number of men originally engaged.

Proportional Division.

182. We group together in the remainder of the chapter some important applications of the principles underlying Proportion.

Ex. 1.—*Divide* £1 2s. *between* A, B, C *in the proportion of the numbers* 2, 4, 5.

Suppose the whole number of pence (viz. 264) to be arranged in piles, each containing $2+4+5$ or 11d.

There will be $264 \div 11$ or 24 such piles. From each pile A is to take 2d., B 4d., and C 5d.

$\therefore$ A's share is $2\text{d.} \times 24 = 4\text{s.}$;
B's ,, $4\text{d.} \times 24 = 8\text{s.}$;
C's ,, $5\text{d.} \times 24 = 10\text{s.}$

Hence the practical work of such problems may be shown thus:—Add together the numbers 2, 4, 5, *i.e.* 11. Form fractions with 11 as the denominator of each, and 2, 4, 5 as their respective numerators.

$$\therefore \quad \text{A's share} = \frac{2}{2+4+5} \text{ of £1 2s.} = \tfrac{2}{11} \text{ of 22s.} = 4\text{s.};$$

$$\text{B's} \quad ,, \quad = \frac{4}{2+4+5} \text{ of £1 2s.} = \tfrac{4}{11} \text{ of 22s.} = 8\text{s.};$$

$$\text{C's} \quad ,, \quad = \frac{5}{2+4+5} \text{ of £1 2s.} = \tfrac{5}{11} \text{ of 22s.} = 10\text{s}$$

The results, when added together, should come to £1 2s.

Ex. 2.—*Divide the number* 195 *into three parts proportional to* $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$.

As in Ex. 1, 1st part $= \dfrac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}$ of $195 = \dfrac{\frac{1}{2}}{\frac{13}{12}}$ of $195 = \frac{6}{13}$ of $195 = 90$.

So 2nd ,, $= \dfrac{\frac{1}{3}}{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}} = \frac{4}{13}$ of $195 = 60$;

3rd ,, $= \dfrac{\frac{1}{4}}{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}} = \frac{3}{13}$ of $195 = 45$.

The total of the three parts is seen to be 195.

Or thus:—The ratios of the three given fractions are unaltered if we multiply each by the same number. Multiply each by the L.C.M. of the denominators 2, 3, 4, viz. 12. Therefore $\frac{1}{2} : \frac{1}{3} : \frac{1}{4} :: 6 : 4 : 3$. We must now divide 195 into three parts proportional to the integers 6, 4, 3

1st part $= \frac{6}{13}$ of $195 = 90$;
2nd ,, $= \frac{4}{13}$ of $195 = 60$;
3rd ,, $= \frac{3}{13}$ of $195 = 45$.

Ex. 3.—*Divide* 840 *into two parts, one of which is* $\frac{10}{11}$ *of the other.*

If one part be denoted by 1, then the other part is denoted by $\frac{10}{11}$;
or, if ,, ,, ,, ,, 11, ,, ,, ,, ,, ,, ,, 10;
therefore divide 840 into two parts in the ratio of 10 to 11.

As in Ex. 1, 1st part $= \frac{10}{21}$ of $840 = 400$;
and 2nd ,, $= \frac{11}{21}$ of $840 = 440$.

(Test the answer by seeing whether 400 is $\frac{10}{11}$ of 440.)

Ex. 4.—*A certain sum of money is divided between* A, B, C, *so that for each* 2s. 6d. A *has* B *has* 2s. *and* C 1s. 6d. *If* A's *share is* £1 5s., *find the sum of money.*

The ratios of the three shares are as the numbers of sixpences in 2s. 6d., 2s., and 1s. 6d. (sixpence being the largest sum of money common to all), *i.e.* as the numbers 5, 4, 3.

Therefore A's share is $\frac{5}{12}$ of the whole. But A's share is £1 5s.;
$\therefore$ $\frac{5}{12}$ of the whole $= 25$s.; $\therefore$ whole $= 25\text{s.} \times \frac{12}{5} = 60\text{s.} = £3$.

EXAMPLES CIX.

1. Divide a guinea into two parts proportional to 3 and 4.

2. A guinea is to be divided between two persons so that one may receive half-a-crown for every shilling which the other receives. How much does each get?

3. Divide £11 5s. into equal numbers of half-crowns and florins.

4. If 20 galls. of water be mixed with 80 galls. of spirit, what fraction of the whole is water?

5. The sum of £1 7s. is made up of sixpences and half-crowns, an equal number of each. How many coins are there altogether?

6. Divide 650 in the ratio of 4 : 9.

7. The sum of £780 is divided between two persons in the ratio of $\frac{1}{4} : \frac{1}{9}$. How much does each get?

8. Gunpowder is composed of nitre, sulphur, and charcoal in the ratio of 33, 5, 7. How many lbs. of each are there in 270 lbs. of powder?

9. Divide 256 into parts proportional to 4·5, 3·7, and $\cdot\dot{3}$.

10. Divide £140 into sovereigns, half-sovereigns, half-crowns, florins, and sixpences, an equal number of each.

11. Divide 840 into two parts, one of which is $\frac{3}{4}$ of the other.

12. How many lbs. of gunpowder can be made from 66 lbs. of nitre, with the necessary quantities of sulphur and charcoal. (See Ex. 8.)

13. Divide 5 tons 11 cwts. into two parts, one of which is $\frac{17}{20}$ of the other.

14. If A receives 2s. 6d. for each 2s. which B receives, how will a sum of £5 12s. 6d. be divided between them?

15. Divide 204 into three parts proportional to the numbers 7, 8, 9.

16. Divide £2220 amongst A, B, C in the proportion of ·7, ·28, ·056.

17. It was intended that 19s. 6d. should be divided between three boys, A, B, C in the proportion of 2, 3, 4; but, by mistake, the distribution was made in the proportion $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$. How much does each gain or lose by the error?

18. Divide £847 19s. 3d. among four people in the ratio of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$.

19. A sum of £256 1s. 3d. was divided between A, B, C so that their shares were respectively proportional to 4, 5, 8. What was A's share?

20. A certain sum of money was divided between A, B, C so that for every sovereign which A had B received 15s. and C half-a-guinea. C's share was £4 14s. 6d. What was the whole sum divided, and what were the shares of A and of B?

21. A certain quantity of tea is put into five boxes, containing quantities proportional to the numbers 10, 12, 14, 20, 21. The last box contains 102 lbs. What is the whole quantity of tea? (Ans. in cwts., &c.)

183. Ex. 5.—*The sum of £32 3s. 6d. is made up of half-sovereigns, half-crowns, and shillings, the numbers used of each coin being in the same ratio as the number of sixpences in the coin. How many of each kind were used?*

The numbers used of half-sovereigns, half-crowns, and shillings are as 20, 5, 2. Suppose we put into one bag 20 half-sovereigns, 5 half-crowns, and 2 shillings, *i.e.* £10 + 12s. 6d. + 2s., or £10 14s. 6d.

The number of bags, each containing £10 14s. 6d., that are required for £32 3s. 6d. is the number of times £32 3s. 6d. contains £10 14s. 6d., *i.e.* 3 times.

But each bag contains 20 half-sovereigns, 5 half-crowns, and 2 shillings; therefore the required numbers of coins are, respectively, 60, 15, 6.

(Test by seeing whether all these coins amount to £32 3s. 6d.)

Ex. 6.—*If £5 be divided among 6 men, 12 women, and 17 boys, so that 2 men may receive as much as 5 boys and 2 women as much as 3 boys, how much will each man, woman, and boy receive?*

Express all in terms of a boy's share.

6 men's shares = 15 boys' shares;
12 women's ,, = 18 ,, ,,
17 boys' ,, = 17 ,, ,,

Therefore total number of boys' shares is 50.

∴ share of 1 boy is £5 ÷ 50 or 2s.;
∴ ,, 1 woman is 3s.;
and ,, 1 man ,, 5s.

(Test by seeing whether the shares of 6 men, 12 women, and 17 boys together amount to £5.)

Ex. 7.—*In an examination there are five subjects, and all have the same maximum. A candidate's marks are in the proportion of 3 : 4 : 5 : 6 : 7, and his aggregate is three-fifths of full marks. In how many subjects did he make more than half marks?*

Suppose the maximum for each subject to be 100 (or any other convenient number). Therefore the maximum for five subjects is 500. Therefore the boy gets $\frac{3}{5}$ of 500, or 300 marks; and these are divided amongst the five subjects in the proportion of 3, 4, 5, 6, 7. Therefore in the various subjects he gets $\frac{3}{25}$, $\frac{4}{25}$, $\frac{5}{25}$, $\frac{6}{25}$, and $\frac{7}{25}$ of 300, *i.e.* 36, 48, 60, 72, 84 marks. But in each subject the maximum is 100. Therefore he gets over half marks in the last *three* subjects.

Ex. 8.—*Divide £300 between* A, B, C *so that* A *may have £20 more than* B *and* B *twice as much as* C.

Subtract the £20 from £300, *i.e.* £280. Now divide £280, so that A's share is equal to B's, and B's is twice C's.

I.e. divide £280 in the ratio of 2, 2, 1.

Therefore A's share of the £280 = $\frac{2}{5}$ of £280 = £112. So the shares of B and of C are £112 and £56 respectively. Now add on to A's share the £20 originally subtracted, and we have the required shares to be £132, £112, £56, respectively.

Ex. 9.—*A gentleman divided £4 18s. among 150 school children, giving the girls one shilling each and the boys sixpence. How many boys and girls were there?*

£4 18s. = 196 sixpences. Now each girl got one shilling; therefore the total number of sixpences for all the girls is twice the number of girls. Hence we have

twice the number of girls + the number of boys = 196.

Also the number of girls + the number of boys = 150.

Therefore, by subtraction, we get that

the number of girls = 46;
∴ the number of boys = 150 − 46 = 104.

Ex. 10.—A, B, *and* C *rent land;* A *puts* 500 *sheep on it for* 4 *months,* B 600 *sheep for* 3 *months, and* C 1200 *for* 2 *months. If the whole rent is* £279, *how much should each pay?*

Each should pay a sum depending partly on the number of sheep he puts on the land, and partly on the time he keeps them there. As the three times are different, we reduce all first to one common time—viz. *one* month.

Now 500 sheep for 4 months may be reckoned as 2000 sheep for 1 month.
So 600 ,, ,, 3 ,, ,, ,, ,, 1800 ,, ,, ,,
and 1200 ,, ,, 2 ,, ,, ,, ,, 2400 ,, ,, ,,

We must then divide £279 into three parts proportional to 2000, 1800, 2400, *i.e.* proportional to 10, 9, 12 (dividing each by the H.C.F., 200).

Therefore A, B, C should pay respectively $\frac{10}{31}$, $\frac{9}{31}$, $\frac{12}{31}$ of £279, *i.e.* £90, £81, £108.

Ex. 11.—*A bag contains* 196 *coins, consisting of sovereigns, half-crowns, and sixpences; the total value of the coins of each kind is the same. How many coins are there of each kind, and what is the whole sum of money in the bag?*

As sixpence is the largest sum of money common to a sovereign, a half-crown, and a sixpence, we may say that

no. of sovs. × 40 = no. of half-crowns × 5 = no. of sixpences × 1...(A).

Now, as one sovereign is greater in value than one half-crown, which itself is greater than one sixpence, we see that, to make up the *same* sum of money from each of these coins separately, the *number* of sovereigns would be less than the *number* of half-crowns, which again would be less than the *number* of sixpences. Hence, from (A), we may state that

the no. of sovs. : the no. of half-crowns : the no. of sixpences $:: \frac{1}{40} : \frac{1}{5} : \frac{1}{1}$
$:: 1 : 8 : 40.$

Now divide 196 in the proportion of 1 : 8 : 40.

Therefore the numbers of sovereigns, of half-crowns, and of sixpences are respectively $\frac{1}{49}$, $\frac{8}{49}$, $\frac{40}{49}$ of 196.

Therefore the bag contains 4 sovereigns, 32 half-crowns, and 160 sixpences, making a total of £12.

EXAMPLES CX. (HARDER.)

1. A gets 2s. 6d. for every shilling which B gets, and B gets four times as much as C; between them they get £4 15s. 7½d. What does each person get?

2. A, B, and C are employed together and work 10, 8, and 7 hours a day respectively. If the total amount paid be £1 19s. 7d., how much ought each to receive?

3. A's income is $\frac{3}{4}$ of B's income; A's expenditure is $\frac{1}{2}$ of B's income; B's expenditure is equal to the whole of A's income. Show that each has the same sum left.

4. A boy took up five papers in an examination. His marks for these papers were in the proportion of 6 : 7 : 8 : 9 : 10. Full marks were the same in each paper, and the boy got three-fifths of the total full marks. Find in how many papers he got more than half-marks.

5. If 13 men working 6 days, 8 women working 5 days, and 5 boys working 4 days, earn between them £13 15s., and the wages per day of a man, a woman, and a boy are as 5 : 3 : 2, what sum is each of them entitled to?

6. Seven men, 8 women, and 11 children divide £71 0s. 1½d. among them in such a manner that each woman receives three times as much as a child, and each man receives as much as a woman and child together. How much did each man receive?

7. Divide £208 between A, B, and C, so that A may have £20 more than B and C together, and C £20 more than half of B's share.

8. Three people, A, B, C, start on a tour with £12 each, and at the end A has £1 11s. 6d., B has twice as much, and C has £4 12s. How are they to settle accounts so that the expenses of the tour may be shared equally?

9. A, B, and C rent a pasture for a year. A puts in 500 sheep for the first half of the year, B 750 for the second half, and C 400 all the year round; the feed is considered to be worth twice as much in the first as in the other half of the year. Find (1) what fractional part of the rent each person ought to pay, and (2) the amount to be paid by each when C's share is £21 more than A's.

10. A, B, and C play cricket. A's runs are to B's, and B's runs are to C's, as 3 : 2. They get altogether 342 runs. How many did each get?

11. Divide £523 10s. among three persons, A, B, and C, so that A may get one-third of what B receives, and C may get as much as A and B together.

12. Divide £383 among four persons so that the first may have thrice as much as the second, the third twice as much as the fourth, and the first and second together £95 more than the third and fourth together?

13. For every penny which A gets B gets 2½d.; and for every £1 which B gets C gets 21s. If amongst them they get £97 17s. 11½d., what is each man's share?

14. Standard gold consists of pure gold and copper in the ratio of 11 to 1, and from 40 lbs. troy of this compound metal are coined 1869 sovereigns. How many ounces of pure gold are there in 100 sovereigns?

15. Divide £40 17s. 5d. among three persons, A, B, C, so that A may have £2 18s. 7d. more than B, and B £1 18s. 2d. more than C.

16. The sum of £25 14s. 6d. is to be divided among 7 boys, 6 women, and 5 men. If a woman has twice as much as a boy, and a man three times as much as a woman, what will each receive?

17. A, B, and C divide amongst themselves 719 marbles, A taking 17 to every 16 of B's, whilst C takes 14 to every 15 which B takes. What number does each get?

18. A sum of £38 consisted of a certain number of half-crowns, twice as many florins, and three times as many shillings. How many were there of each?

19. An alloy is made of zinc, tin, and lead. The quantity of zinc is $\frac{3}{4}$ that of tin, and the quantity of tin is $\frac{4}{5}$ that of lead. How much of each metal will there be in 4 lbs. of the alloy?

20. Divide 3 guineas among 5 persons so that two of them may each receive 2s. 4d. more than each of the other three.

21. The sum of £26 2s. 6d. is divided between 4 persons, A getting 3s. where B gets 2s., B getting 3s. where C gets 4s., and C getting 7s. where D gets 6s. How much does each get?

22. If £15 is divided between 5 men and 3 women so that a man's share is to a woman's as 7 : 5, what will be the share of one man and of one woman?

23.* A casket contains diamonds and pearls. The diamonds are worth as many times the pearls as the pearls are worth the casket, and the values of the diamonds and the casket are £3411 9s. and one guinea respectively. Find the value of the pearls.

24. Three graziers, A, B, and C, rent a piece of pasture land for a month. A puts in 27 cattle for 21 days, B 19 for 24 days, and C 23 for 25 days. If, at the end of a month, the rent and other charges amount to £33 5s. 10d., how much ought to be paid by each?

25. I have between 150 and 200 coins, some of which are sixpences, some florins, and the rest half-crowns. The total value of the coins of each kind is the same. How many have I of each?

26. Between two railway stations, the first, second, and third class fares were fixed at first in the ratio of 8 : 6 : 3, but afterwards the first class fare was reduced by 2d. in the shilling, and the second class by 1d. in the shilling. In a year the numbers of first, second, and third class passengers were respectively as 9 : 12 : 26, and the money taken at the booking offices was £1088. How much was paid by first class passengers?

27. A bag contains £26 5s. in half-crowns, florins, and shillings. There are three times as many florins and four times as many shillings as half-crowns. Find how many coins of each kind the bag contains.

28. Divide £106 4s. 3d. between A, B, and C so that A shall get £5 for every £2 that B gets, and for every £3 that B has C shall get £4.

29. Three men are employed on some work, working respectively 8, 9, 10 hours a day and receiving the same daily wages. After 3 days, each works 1 hour a day more and the work is finished in 3 more days. If the total sum paid for wages be £2 7s. 6¼d., how much of it should each receive?

30. If £9 15s. be divided amongst 80 children, giving each girl 2s. 6d. and each boy 2s., how many boys are there?

31. An alloy is made of zinc, tin, and lead. The quantity of zinc is $\frac{3}{4}$ that of tin, and the quantity of tin is $\frac{4}{5}$ that of lead. How many pounds of each metal will there be in 1 cwt. of the alloy?

32. Two settlers in New Zealand own adjoining farms of 3000 and 5000 acs. respectively. They unite their farms, taking at the same time an additional partner, who pays them £8000 on the understanding that a third share of the land shall in future belong to each. How is the £8000 to be divided between the original owners?

33. A mixture weighing 1320 oz. consists of three liquids in the proportion by volume of $7 : \frac{1}{4} : \frac{3}{4}$. The weights of equal volumes of the liquids are respectively as $1 : 1{\cdot}445 : 1{\cdot}185$. What weight of the second liquid is contained in the mixture?

34. The weight of a gold 10-franc piece is 49·78 grs.; the weight of a silver 5-franc piece is 385·8 grs., of which the equivalent in pure silver is ·782 oz. troy. If the relative value of coined gold and silver be 19·29 : 1, find the intrinsic value of a gold 10-franc piece, the intrinsic value of pure silver being 4s. 2d. per oz. troy.

35. A, B, C start on a journey, having £9, £7 10s., £5 10s., respectively, with them. At the end it is found that A has 10s., B £5, and C £1 15s. For expenses common to B and C only B has paid £1 10s.; for private expenses A has paid £1, B 10s., and C £1; all other payments have been made to defray expenses in common. How should they divide the money which remains over?

36. An alloy containing 7 times as much gold as silver is worth 3 times as much as one that contains 3 times as much silver as gold. In what ratio must these alloys be mixed to form one worth £2 10s. an oz., gold being worth £3 17s. 11d. an oz.?

37. Standard gold consists of 11 parts of pure gold to 1 part of alloy. If 1 oz. troy of standard gold is worth £3 17s. $10\frac{1}{2}$d., how many grains (to the nearest integer) of pure gold are contained in a sovereign?

PARTNERSHIP.

184. Ex. 1.—*Two men, with £700 and £400 capital, respectively, become partners in business and make a profit of £220. How should they divide it?*

The share of each is proportional to his capital; therefore divide £220 into two parts proportional to 700 and 400, *i.e.* to 7 and 4.

Therefore their respective shares are $\frac{7}{11}$ and $\frac{4}{11}$ of £220, *i.e.* £140 and £80.

Ex. 2.—A, B, C *enter into partnership*, A *putting in* £2000 *for the whole year*, B *putting in* £3000 *at first and increasing it to* £4000 *at the end of* 4 *months, whilst* C *puts in* £4000 *at first, but withdraws* £1000 *at the end of* 9 *months. How should they, at the end of a year, divide a profit of* £8475?

The times during which the various capitals are employed are different; we must therefore, for the purpose of comparison, reduce them all to one common standard, viz. *one* month (Calendar).

A has £2000 employed for 12 months. This is equivalent to £24000 for 1 month.

B has £3000 employed for 4 months, and then £4000 for 8 months. This is equivalent to £12000 + £32000 or £44000 for 1 month.

C has £4000 employed for 9 months, and then £3000 for 3 months. This is equivalent to £36000 + £9000 or £45000 for 1 month.

The profits must be divided in the proportion of 24000, 44000, 45000, *i.e.* 24, 44, 45.

$$\therefore \quad \text{A's share is } \tfrac{24}{113} \text{ of } £8475 = 24 \times 75 = £1800.$$

Similarly, the shares of B and of C are respectively £3300 and £3375.

Ex. 3.—A, B, C *enter into partnership, with* £1500, £2500, £3000 *capital, respectively.* A, *as manager, is to receive one-tenth of the total profits, and the remainder is to be divided amongst the three in proportion to their capitals. If* A's *share is* £369, *what are the shares of* B *and of* C?

A first receives $\frac{1}{10}$ of the profits; therefore $\frac{9}{10}$ of the profits remain for division amongst the three in the ratio of 1500, 2500, 3000, *i.e.* 3, 5, 6.

$\therefore$ A's full share $= \frac{1}{10}$ of profits $+ \frac{3}{14}$ of $\frac{9}{10}$ of profits $= \frac{41}{140}$ of profits.

$\therefore \quad \frac{41}{140}$ of profits $= £369$; $\quad \therefore$ profits $= £1260$.

$\therefore$ B's share $= \frac{5}{14}$ of $\frac{9}{10}$ of £1260 = £405

and C's share $= \frac{6}{14}$ of $\frac{9}{10}$ of £1260 = £486.

EXAMPLES CXI.

1. Two partners make a profit of £182; one puts £300, the other £400, into the business. How should they divide the profit?

2. Two troops of cavalry rent a field in common, for which they are to pay £23 10s. One troop puts into the field 27 horses for 29 days, and the other 19 horses for 33 days. How much should each pay?

3. Three partners, A, B, C, invest £4730, £3680, £2840, respectively, in their business. What is the share of each in a profit of £595 6s. 3d.?

4. Three men enter into partnership, putting into the business £300, £400, £500, respectively. How much should each receive out of every £100 of profits?

5. A, B, C start in business together; A puts in £5000, B £6000, and C £9000. At the end of 3 months C leaves, and at the end of 7 months more B leaves, both taking out their capital with them, but the profits are not divided till the end of the year, when C receives £50. How much will A and B then receive?

6. Three partners, A, B, C, own a business; B's capital is twice A's, and C's capital is twice B's. Their profits, which are one-tenth of the whole capital, amount to £2331. What capital has each partner in the business?

7. Three persons, A, B, C, enter into partnership; A contributes £840, B £1200, C £1560. A acts as manager, for which he receives one-tenth of the profits, and the remainder of the profits is divided between all three partners in proportion to their contributions. What fractional part of the whole profits does each receive?

8. A and B enter into partnership on the following terms:—(1) B is to put in half as much capital again as A; (2) A, who is the working partner, is to receive out of the profits £8 for each £100 of his capital and B is to receive £4 for each £100 of his capital; (3) the rest of the profits are to be divided between them in the ratio of their respective capitals. At the end of the first year, A receives in all £2456 and B £3084. What capital did each put into the business?

9. A and B entered into partnership. A put in at first £2000, and at the end of 8 months £1000 more; B put in at first £750, and at the end of 4 months £3000 more, but he took out £1300 at the end of 3 months more. At the year's end they had gained £1635. What should each receive?

10. A and B enter into partnership, their capitals being as 5 : 6. At the end of 2 months they withdraw respectively $\frac{1}{3}$ and $\frac{1}{4}$ of their capital, and 4 months afterwards B increases his capital by the addition of $\frac{1}{3}$ of his original capital. How are they to divide the profits amounting at the end of the year to £379 2s. 6d.?

11. Three men, A, B, C, go into business, A contributing £5000, B £3000, and C £2500, of the capital on the understanding that, after allowing one-eighth of the profits to C as manager, the remainder should be divided amongst them all in proportion to the amount of capital contributed by each. At the end of a year C receives £280. What are the total profits of the business, and how much do A and B receive respectively?

12. A has a capital of £666 13s. 4d. in a business for 12 months. B has £500 in the same business for 7 months, which he raises to £766 13s. 4d. for the remaining 5 months. C has £800 in the same business for the first 3 months, and only £300 for the last 9 months. If B's share of the profits is £53 0s. 10d. more than C's, what should A's share be?

SECTION VIII.

COMMERCIAL ARITHMETIC.

CHAPTER XXIV.

PERCENTAGES, PROFIT AND LOSS, RATES AND TAXES.

185. It has been shown in § 108 that, if we wish to find the relative magnitude of a given set of fractions, we must reduce them all to a new set with a common denominator, and then compare their numerators. The object of this process is to express them all in terms of one common standard.

Again, suppose that in three arithmetic examinations a boy obtains 120, 200, and 125 marks respectively out of a possible total of 150, 500, and 250. His success in the three papers may be represented by the fractions $\frac{120}{150}$, $\frac{200}{500}$, $\frac{125}{250}$, or by the fractions $\frac{4}{5}$, $\frac{2}{5}$, $\frac{1}{2}$, or (reducing them to a common denominator 100) by the fractions $\frac{80}{100}$, $\frac{40}{100}$, $\frac{50}{100}$, *i.e.* if the maximum in each paper had been 100 marks, he would have obtained 80, 40, 50 marks respectively. We express this by saying that his percentage of marks is 80, 40, 50. The term "per cent." (Latin *centum*, a hundred) means "for each hundred," is abbreviated as p.c., and is represented by the symbol %. Thus 3% of a ton means three parts out of every 100 parts into which the ton may be supposed divided.

Any fraction or ratio may be expressed as a percentage—*e.g.*

$$\frac{6}{7} = \frac{\frac{6}{7}}{1} = \frac{\frac{6}{7} \times 100}{100} = \frac{85\frac{5}{7}}{100},$$

i.e. $85\frac{5}{7}$ parts in every 100 parts, or $85\frac{5}{7}$ per cent., or $85\frac{5}{7}$ p.c., or $85\frac{5}{7}$%.

$$\cdot 25 = \frac{25}{100} = 25\%.$$

So any percentage can be expressed as a fraction—*e.g.* $65\% = \frac{65}{100} = \cdot 65$ or $\frac{13}{20}$.

186. The term "per cent." is of very common use in commercial life (see chapters on "Interest," "Discount," &c.). A rent or debt collector charges for his services a certain percentage on the money which he collects. This charge is called his *commission*.

The increase or decrease of the population of a town may be expressed as a percentage on the *original* population. (It is also sometimes reckoned as so much per 1,000.) The results of a chemical analysis are generally expressed in a similar way. Rates and taxes are most frequently expressed as so much in the £.

Ex. 1.—*What is the value of* $6\frac{1}{2}$°/₀ *of* £300?

$$6\tfrac{1}{2}\%\text{ of }£300 = \frac{6\frac{1}{2}}{100}\text{ of }£300 = £300 \times \frac{6\frac{1}{2}}{100} = £3 \times 6\tfrac{1}{2} = £19\ 10\text{s.}$$

Ex. 2.—*How much* °/₀ *is* 4s. 6d. *of* £3?

We first express 4s. 6d. as a fraction of £3, and then reduce this fraction to a percentage.

$$4\text{s. }6\text{d.} = \frac{4\frac{1}{2}}{60}\text{ of }£3 = \frac{3}{40}\text{ of }£3;\ \text{and}\ \frac{3}{40} = \frac{\frac{3}{40} \times 100}{100} = \frac{7\frac{1}{2}}{100}.$$

Ans. $7\frac{1}{2}$°/₀

Ex. 3.—*What is the sum of money of which* 35% *is* 2s. 4d.?

$$35\% = \tfrac{35}{100} = \tfrac{7}{20}\ \therefore\ \tfrac{7}{20}\text{ of required sum} = 28\text{d.}$$

$$\therefore\ \text{Required sum} = 28\text{d.} \div \tfrac{7}{20} = 28 \times \tfrac{20}{7}\text{d.} = 6\text{s. }8\text{d.}$$

Ex. 4.—*If the population of a town at the beginning of a year was* 765,000, *and the birth rate was* 53·2, *while the death rate was* 31·2, *per* 1,000 *of the population, find the net increase in the population at the end of the year.*

The original population was 765 thousand;

$\therefore$ No. of births was $53{\cdot}2 \times 765 = 40698$,

No. of deaths was $31{\cdot}2 \times 765 = 23868$;

$\therefore$ Net increase in population $= 16830$;

or shorter thus:—

Net increase of births over deaths per $1{,}000 = 53{\cdot}2 - 31{\cdot}2 = 22$; therefore total increase on 765 thousand $= 22 \times 765 = 16830$.

Ex. 5.—*A's annual income is reduced from £750 to £600, while B's is increased from £600 to £750. Find the percentage of decrease in A's income and of increase in B's.*

On A's *original* income of £750 there is a decrease of £150; therefore if A originally had an income of £100 there would be a decrease of £150 × $\frac{100}{750}$, or 20%.

On B's *original* income of £600 there is an increase of £150; therefore if B originally had an income of £100 there would be an increase of £25, or 25%.

Ex. 6.—*A boy spends 30% of his pocket money, and has 10s. 6d. left. How much had he at first?*

He spends 30% of the whole; therefore he has 70% of the whole left; therefore 70% of the whole = 10s. 6d.; therefore, as in Ex. 3, his original pocket money = 10s. 6d. × $\frac{100}{70}$ = 15s.

EXAMPLES CXII.

1. What percentage is 16s. 8d. of £1?

2. Find the value of $33\frac{1}{3}$% of £7 2s. 6d.

3. Reduce £4 6s. 2d. to the percentage of £17 4s. 8d.

4. Calculate 30% of £882 5s.

5. Express as a percentage 15 cwts. 2 qrs. 17 lbs. ÷ 19 tons 11 cwts. 1 qr. 5 lbs.

6. Find the value of $12\frac{1}{2}$% of 3 cwts. 1 lb. 8 oz.

7. What is the number, 75% of which is 1,320?

8. How much remains out of £5, after 63% of it has been spent?

9. Find $2\frac{1}{4}$% of £1 13s. 4d.

10. In an examination the maximum number of marks was 3,600. Three candidates, A, B, C, obtain respectively 2,420 marks, 70% of the whole, and two-thirds of the whole. Arrange them in order of merit.

11. Out of a class of 32 boys, 4 were absent; 25% of the remainder failed in their work. How many were successful?

12. In a town of 60,000 inhabitants the death rate was 10·3 per 1,000. How many deaths were there?

13. Find, correct to the nearest penny—

(1) $9\frac{1}{2}$% on £473 8s. 5d. (2) 11·75% on £763 8s.

14. Calculate—

(1) $4\frac{2}{3}$% of 3 cwts. 1 qr. 11 lbs. (2) $9\cdot\dot{3}$% of 1 ac. 3 rs. 20 pls.

(3) 80% of 2 mls. (*Ans.* in ml. and yds.) (4) $20\frac{3}{4}$% of £3 6s. 8d.

15. A book is published at 10s. What price is paid by a customer if the bookseller allows a deduction of 2d. in the shilling?

16. Find the commission at the rate of $2\frac{3}{4}$% on (1) £250; (2) £675; (3) £1,518 15s.

17. An agent charges a commission of 3% for collecting debts to the amount of £1,050. What balance does he pay after deducting for his services?

18. How much should I pay for goods costing £15 if the tradesman allows me a reduction of 5% for ready money?

19. What percentage is 33 qts. of 88 galls.?

20. Find the value of 15% of £96 17s. 6d.

21. A merchant buys goods to the amount of £686 13s. 4d. If from this sum he is allowed a deduction of $4\frac{3}{4}$%, what does he actually pay?

22. Find the value of 17% of £7 3s. 9d.

23. If $3\frac{3}{4}$% on a certain sum of money amounts to £46 17s. 6d., what is that sum?

24. The population of a certain town has risen from 35,000 to 48,055. What is the increase per cent.?

25. How many quarts are there in $3\frac{3}{4}$% of 20 galls.?

26. How much is 9% of £660?

27. A cask containing 64 galls. lost $6\frac{1}{4}$% through leakage. How much remained in the cask?

28. If a man's income is £640, and he spends $87\frac{1}{2}$% of it, how much has he left?

29. The population of a certain town was 6,864 in 1881, and in 1891 it was 7,762. What was the increase per cent? Also what was the increase per 1,000? (Both answers to two decimal places.)

30. How much % of £162 10s. is £11 15s. $7\frac{1}{2}$d.?

31. How much is $\frac{3}{4}$% of £266 13s. 4d.?

32. The rain injures $6\frac{1}{2}$% of a truck-load of corn weighing 4 tons 9 cwts. 1 qr. 4 lbs. What is the weight of the damaged grain?

33. (1) If the population of a town increase from 6,705 to 7,152, find the percentage of increase. (2) If the population had decreased from 7,152 to 6,705, what would have been the percentage of decrease?

34. Find the duty on goods worth £372 10s. at $7\frac{1}{2}$%.

35. A man's income is £750. How much does he pay as income tax at the rate of 6d. in each £? Express this rate as a percentage.

36. On what sum does a rate of $4\frac{1}{4}$% produce £54 3s. 9d.?

37. After spending 25% of his money, a man has £24 left. How much had he at first?

38. Out of an annual income of £350, a man spends £280. What fractional part of his income does he save? What is the equivalent percentage?

39. A boy spends 20% of his pocket money on oranges, one quarter of it on apples, and 5% in paying fines. What fractional part of the whole has he left?

187. Ex. 7.—*A bankrupt's liabilities* (i.e. *his debts*) *are* £7,625 10s., *and his assets* (i.e. *money available for paying debts*) *are* £953 3s. 9d. *How much can he pay in the* £?

Instead of paying in full £7,625½, he pays only £953 3s. 9d.

Therefore, instead of paying in full £1, he pays only £953 3s. 9d. ÷ 7625½; or 2s. 6d.—*i.e.* he can pay 2s. 6d. in the £.

Ex. 8.—*A reduction of* 5 °/₀ *in the price of tea enables one to purchase* 2 *lbs. more for* £3 3s. 4d. *What is* (1) *the reduced price per lb.*; (2) *the original price per lb.?*

Owing to the reduction of price, there is a saving of 5 °/₀ on £3 3s. 4d., or $\frac{1}{20}$ of 760d.—*i.e.* 38d. With this 38d. one can purchase 2lbs. at the reduced price; therefore reduced price per lb. is 1s. 7d.

Again, 95 °/₀ of original price per lb. = 1s. 7d.; therefore original price per lb. = 1s. 8d.

Ex. 9.—*A costermonger has a certain number of oranges, of which* 2 °/₀ *are bad. He sells* 95 °/₀ *of the remainder, and then has* 49 *oranges left. How many had he originally?*

Suppose he had 100 oranges at first, then 2 of these would be bad, and, rejecting these, he would have 98 left. He sells 95 °/₀ of these; therefore he would have 5 °/₀ of 98 left, or $4\frac{9}{10}$.

Therefore $4\frac{9}{10}$ are left out of a supposed original 100; therefore 49 are left out of an original 1,000; therefore 1,000 oranges is the answer.

The fraction $\frac{9}{10}$ does not really affect the *argument*, but it may be avoided thus: 5 °/₀ of the remainder, after rejecting the bad ones, amounts to 49; therefore the remainder, after rejecting the bad ones, amounts to 980.

But the remainder consisted of 98 °/₀ of the whole: therefore 98 °/₀ of the whole amounts to 980; therefore the whole = 1,000.

EXAMPLES CXIII. (Harder.)

1. Of 5,460 candidates who presented themselves at a certain examination 35 °/₀ failed to pass, whilst 15 °/₀ passed with honours. Find the number of candidates who passed the examination, but failed to obtain honours.

2. If 27 °/₀ of a certain number of sovereigns is equivalent to 5,400 shillings, what is the number of sovereigns?

3. The population of a certain town has risen 23½ °/₀ in the last ten years, and is now 17,784. What was the population ten years ago?

4. Twenty-four per cent of a sample of milk is found to consist of added water, how much added water is this in a mixture of 12½ galls.?

5. A man is charged 32s. per annum to insure his house for £800 What vulgar fraction represents the rate of premium per cent.?

6. The annual expenditure of a school is £1,500. Of this $36\frac{1}{2}$% is paid by the Government, £202 10s. by subscriptions, and the remainder by school fees. What percentage of expenditure is paid by fees?

7. A general, having lost $\frac{2}{7}$ of his men in battle and 6% of the remainder by sickness, found he had 95,880 men left, how many had he at first?

8. In a forest 5% of the trees are blown over in a gale, and after 3% of those remaining are cut down there still stand 55.290 trees. How many trees were there in the forest before the gale?

9. The population of a city is a million; at the end of each year it is found that the increase is $1\frac{1}{2}$% of what it was at the beginning of that year; find the population at the end of three years.

10. Express $\frac{1}{10}$ as a percentage of $\frac{5}{7}$.

11. Out of a mixture of 75 galls. of spirits and 5 galls. of water, what percentage is spirit?

12. Of the children attending a Board school 55% are boys, and the remainder, 153 in number, are girls. How many children are there in the school?

13. A casting which weighs 4 cwts. 1 qr. 24 lbs. is composed of 92% copper and the remainder tin; how many lbs. of tin does it contain?

14. A man in 1895 spends $\frac{10}{11}$ of his income. In 1896 his income was $17\frac{1}{2}$% more than in 1895. He spends $\frac{11}{12}$ of it and saves £7 15s. more than in the previous year. What were his incomes in 1895 and 1896? (Ans. to nearest penny.)

15. At the commencement of a year a certain town contained twice as many females as males. At the end of the year the females had increased 8%, and the total population had increased 3%. At what rate per cent. had the males increased or decreased?

16. A man has a yearly income of £600. He saves one-sixth of it, pays 15% of the remainder in rent and taxes, and spends the rest. How much does he spend, excluding rent and taxes?

17. The maximum number of marks for an examination paper is 300. A boy obtains correct answers to ten out of the twenty-four questions set, the average marks for each of the ten being 8. Through bad arrangement of his work, &c., he loses 10% of his marks. What percentage of marks does he obtain on the whole paper?

18. A reduction of 10% in the price of sugar enables one to purchase 8 lbs. more for 15s. What is the reduced price per lb.?

19. If in 1871, when 702,201 tons of sugar were consumed in the United Kingdom, the consumption was 49·93 lbs. per head, and the corresponding quantities for 1881 were 989,208 tons and 62·77 lbs. per head, find to the nearest whole number the increase per cent. in the population during the interval.

20. A farm of 200 acs. is rated at 2s. 6d. an acre. After the rent has been reduced by 4 %, and the rates have been increased by £5, it is found that the rates amount to $12\frac{1}{2}$ % of the rent. Determine the original rent of the farm.

21. The population of a certain town increases every year by $7\frac{1}{2}$ %. If in April, 1892, the population was 704,000, what will it be in April, 1895?

22. A bankrupt's debts are £5,317 5s., and his assets are £4,076 11s. 2d. How much can he pay in the £ after defraying the expenses connected with the bankruptcy, which amount to 10 % of the debts?

23. 360 persons voted on a motion; but, on the motion being reconsidered, the opponents were increased by $37\frac{1}{2}$ %. The Bill was then rejected by a majority twice as great as that by which it was formerly passed. How many voted for and against the Bill in each case?

24. A man whose income is £650 pays taxes to the amount of £40 12s. 6d. How much is this in the £, and what percentage is it of his income?

25. It is estimated that the number of bread-eaters in the world increases at the rate of 12 % per decade (period of 10 years). At present (1898) the population of the world is 516 millions. Supposing the area available for wheat cultivation is 263 million acs., and that an acre produces on an average 12·7 bushels per annum, find in what decade the wheat supply will be insufficient to meet the demand if each bread-eater requires $4\frac{1}{3}$ bushels of wheat per annum.

26. A reduction of 20 % in the price of beef would enable a purchaser to obtain 6 lbs. more for a sovereign. What is the reduced price?

27. A reduction of 20 % in the price of apples would enable a purchaser to get 120 more for a sovereign. Find the first price of one apple.

28.* A boy measures a room with what he takes to be a foot-rule, and computes its area to be 720 sq. ft. But it is found that his rule is $8\frac{1}{3}$ % short of a true foot. Find the correct area in square feet.

29. I deduct 10 % from a sum of money, and from the remainder I further deduct $33\frac{1}{3}$ % of itself, and find I have £45 left. What was the original sum of money?

30. A reduction of 10 % in the price of coal would enable a purchaser to obtain for the sum of £13 10s. exactly two tons more than at the higher price. Find the price per ton before reduction.

31. An official in India receives a salary of 300 rupees a month, each rupee being worth 1s. 8d. If the rupee fall in value 20 %, find in English money the corresponding fall in his annual income.

32. A clock loses $\frac{1}{2}$ % on true time during one week and gains a $\frac{1}{4}$ % on true time during the next week. If it is set right at 12 o'clock on Saturday morning, what time will it indicate at the end of a true fortnight?

33. A reduction of 10 °/₀ in the price of iron would enable a purchaser to obtain 1 cwt. more for £1. What is the price per cwt. before reduction?

34. The price of coal falls 4 °/₀. How many tons may I now buy for the same money which was sufficient to buy 24 tons at the higher price?

35. A obtains $33\frac{1}{3}$ °/₀ of the marks in a paper for which the maximum was 300. B is ahead of A by 40 °/ of A's marks, while C is ahead of B by two-ninths of his own marks. How many marks does C get?

36. Genuine milk contains 88·75 °/₀ by bulk of water, 2·75 °/₀ of fat, and 8·5 °/₀ of non-fatty solids. A purchaser buys 7 galls. of milk at 3d. a quart. The milk, on being analyzed, is found to contain 90·84 °/₀ of water, 2·24 °/₀ of fat, the residue being non-fatty solids. Find whether anything besides water has been added to it, and find the sum of which the purchaser has been defrauded.

Percentages: Profit and Loss.

188. When a manufacturer enters upon his business he hopes to sell his goods at a higher price than their cost of production. He sells them "wholesale," that is, in large quantities, to the retail tradesman, who also hopes to sell them at a higher price than that which he gave for them (including any expenses incurred in the carrying on of his business). In both cases they are said to be selling at a profit. If, however, the goods are sold at a lower price than that which they cost, they are said to be sold at a loss.

Thus, suppose a grocer buys from a wholesale dealer tea which costs the grocer 2s. a lb. and sells the same to a customer at 2s. 6d. a lb. he makes a profit of 6d. on each lb. he sells. The 2s. is called the *cost price* (C.P.) or the *outlay*, and the 2s. 6d. is called the *sale* or *selling price* (S.P.). The 6d. is his *actual profit* per lb.

Suppose he also buys sugar at the rate of $1\frac{1}{2}$d. a lb. and sells it at 2d. a lb., his actual profit is $\frac{1}{2}$d. on each lb. sold.

To obtain, in the first case, the 6d. profit, he lays out 2s., but, in the second case, to obtain $\frac{1}{2}$d. profit he lays out $1\frac{1}{2}$d. To compare his profits it is not enough merely to compare the 6d. with the $\frac{1}{2}$d., but he must take into account also what it has cost him to get these. He therefore expresses his actual profit as a percentage of profit on his outlay or on the cost price.

In the first case, on an outlay of 2s., *i.e.* 4 sixpences, he gets a profit of 1 sixpence. Therefore on a supposed outlay of 100 sixpences he gets a profit of 25 sixpences, *i.e.* his profit is 25 °/₀ on his outlay.

So, in the second case, his profit is $33\frac{1}{3}$°/₀ on his outlay.

Therefore an equal outlay in both cases produces more profit in the second than in the first case.

We thus have

Actual profit = Sale Price − Cost Price,
or Sale Price = Cost Price + Actual Profit,
or Cost Price = Sale Price − Actual Profit,

with similar statements when goods are sold at a loss.

When the actual profit and the C.P. are known, the profit per cent. can be found by the method of proportion, but care must be taken to calculate it as a percentage on the *C.P. or outlay*, unless otherwise directed.† The relative profit or loss may sometimes also be expressed as so much in the shilling or pound, *e.g.* 2d. in each shilling.

189. Ex. 1.—*An article costs* £18 15s. *and is sold at a profit of* 12°/₀ *on the C.P. Find the S.P.*

Actual profit = 12°/₀ on the C.P. = $\frac{12}{100}$ of £18 15s. = £2 5s.

But S.P. = C.P. + actual profit; ∴ S.P. = £18 15s. + £2 5s. = £21 *Ans.*

If the article is sold at a *loss* of 12°/₀ on C.P., then S.P. = £18 15s. − £2 5s. = £16 10s.

Ex. 2.—*An article is sold for* £21 *at a profit of* 12°/₀ *on the C.P. Find the C.P.*

C.P. + actual profit = S.P.;

∴ C.P. + $\frac{12}{100}$ of C.P. = £21;

∴ $\frac{112}{100}$ of C.P. = £21; ∴ C.P. = £21 ÷ $\frac{112}{100}$ = £18 15s. *Ans.*

A common mistake in such examples is to find 12°/₀ on £21, and subtract the result from £21. The profit is, however, not 12°/₀ on £21, but 12°/₀ on an unknown C.P.

Or thus—Assume £100 to be the C.P., then the actual profit would be £12; therefore S.P. would be £112. We now have a problem in proportion.

The S.P. is £112 when the C.P. is £100;

$$\therefore \quad \text{S.P. is £21 when the C.P. is } \frac{£100 \times 21}{112} = £18\ 15\text{s.}$$

This method of assuming the C.P. to be £100 (or 100s. or 100d., &c.), and then forming a problem in proportion should be carefully noted. It is similar to the method used in algebra in solving problems, where we let an unknown quantity (x) denote what we wish to find and then form an equation.

If the article is sold at a loss of 12°/₀ on the C.P. and the S.P. is £16 10s., find the C.P. We should subtract £12 from £100, the supposed C.P. Then S.P. is £88, when C.P. is £100; therefore S.P. is £$16\frac{1}{2}$, when C.P. is £$16\frac{1}{2} \times \frac{100}{88}$ = £18 15s.

† In some cases, however, business men do estimate their profit or loss per cent. on the sale price.

Ex. 3.—1 *cwt. of tea is bought for* £11 4s. *At what price per lb. must it be retailed so as to make a profit of* $16\frac{2}{3}$ °/₀ *?*

Actual profit = $16\frac{2}{3}$ °/₀ on £11 4s. = $\frac{1}{6}$ of £11 4s. = £1 17s. 4d. ;

∴ 112lbs. are sold for £11 4s. + £1 17s. 4d. = £13 1s. 4d. ;

∴ 1lb. is sold for £13 1s. 4d. ÷ 112 = 2s. 4d. *Ans.*

We might have begun by finding the C.P. of 1 lb., *i.e.* 2s., and to this adding $16\frac{2}{3}$ °/₀ of itself, or 4d., making a total of 2s. 4d. ; but this method frequently involves unnecessary fractions.

Note that the *percentage* of profit is the same, whether we speak of a cwt. or a lb.

Ex. 4.—*A man buys eggs at* 11d. *a dozen, and sells them at the rate of* 11 *eggs for* 1s. *Find his gain per cent.*

We find the C.P. and the S.P. of one egg or of any number of eggs, provided we take the same number in each case. As L.C.M. of 12 and 11 is 132, we may (to avoid some fractions) suppose he buys 132 eggs.

The C.P. of these is 11d. × 11 = 121d. ;

The S.P. of same is 12d. × 12 = 144d. ;

∴ On outlay of 121d. he gains 23d. ;

∴ On outlay of 100d. he gains $100 \times \frac{23}{121}$d. ; or gain °/₀ is $19\frac{1}{121}$.

Ex. 5.—*By selling an article for* £25 13s. *a man loses* 5 °/₀. *At what price should he sell it to gain* 5 °/₀ *?*

As both the loss and the gain are reckoned on the C.P., we find the C.P. first.

Suppose it to be £100 ; therefore S.P., when he loses 5 °/₀, is £95 ; therefore C.P. = £25 13s. × $\frac{100}{95}$ = £27. In the second case he gains 5 °/₀ on £27.

∴ Required S.P. is £27 + $\frac{1}{20}$ of £27 = £28 7s. *Ans.*

We need not actually calculate the true C.P. Thus, still suppose £100 to be the C.P.

That, which at a *loss* of 5 °/₀ is sold for £95, should be sold for £105 to *gain* 5 °/₀.

Therefore that, which at a *loss* of 5 °/₀ is sold for £25 13s., should be sold for £25 13s. × $\frac{105}{95}$ = £28 7s. *Ans.*

Ex. 6.—*A sells an article to B at a profit of* 5 °/₀, *B sells the same to C for* £4 9s. 3d., *thereby making a profit of* 2 °/₀. *What did A pay for it ?*

Suppose A paid £100 for it ; then he would sell it to B for £105.

Now B's profit is calculated on what B paid for it (viz., £105) ;

∴ C would pay to B £105 + $\frac{2}{100}$ of £105 = £ $\frac{105 \times 102}{100}$;

∴ C would pay £$\frac{105 \times 102}{100}$ for an article for which A paid £100 ;

∴ C would pay £4 9s. 3d. for an article for which A paid

$$£4\ 9s.\ 3d. \times \frac{100 \times 100}{105 \times 102} = 1{,}071d. \times \frac{20 \times 50}{21 \times 51} = 1{,}000d.$$

$$= £4\ 3s.\ 4d.\ \textit{Ans.}$$

Or thus—As B gains 2°/₀ on what the article cost him,

∴ B actually pays $\frac{100}{102}$ of £4 9s. 3d.

So A gains 5°/₀ on his own outlay;

∴ A paid $\frac{100}{105}$ of $\frac{100}{102}$ of £4 9s. 3d. $= \frac{100 \times 100}{105 \times 102}$ of 1,071d.

$=$ £4 3s. 4d. *Ans.*

Note that B gains 2°/₀ on his own outlay and not on A's outlay.

B pays to A £4 3s. 4d. $+ \frac{5}{100}$ of £4 3s. 4d. = £4 7s. 6d., and C pays £4 7s. 6d. + 2°/₀ on £4 7s. 6d. or £4 9s. 3d.

If A had sold direct to C at a profit of 7 (*i.e.* 5 + 2)°/₀, then C would have paid £4 3s. 4d. $+ \frac{7}{100}$ of £4 3s. 4d. = £4 9s. 2d.

EXAMPLES CXIV.

1. Find the actual gain, and also the gain per cent.—

(1) When a horse which cost £20 is sold for £25.

(2) When a house is bought for £800 and sold for £900.

(3) When an article is sold for 17s. 6d. which cost 15s.

(4) When the cost price of an article is £75, and the selling price is £100.

2. Find the actual loss and the loss per cent.—

(1) If I sell for £20 a horse which cost £25.

(2) When a house which cost £900 is sold for £800.

(3) When an article costs 17s. 6d. and is sold for 15s.

(4) When the cost price of an article is £100 and the selling price is £75.

3. What is the sale price in the following cases?—

(1) Cost price £16, sold at a gain of 10°/₀.

(2) Cost price £13 6s. 8d., sold at a profit of 15°/₀.

(3) Cost price 10d., sold at at a loss of 50°/₀.

(4) Cost price £17 2s. 3d., sold at a gain of $33\frac{1}{3}$°/₀.

(5) Cost price £18 16s., sold at a gain of $\frac{1}{4}$ of the cost price.

(6) Cost price £257, sold at a loss of 2s. on every £.

4. What is the cost price when goods are sold for?—

(1) £168 with a gain of 5°/₀ on cost price.

(2) £510 at a loss of 15°/₀ on cost price.

(3) £500 at a profit of 100°/₀ on cost price.

(4) 25s. at a loss of $\frac{1}{4}$ of the cost price.

(5) £1 6s. 3d. at a gain of $16\frac{2}{3}$°/₀ on cost price.

(6) £2 10s. 6d. at a profit of 1°/₀.

5. Each of ten articles costs 1d. What is the gain per cent. if the whole are sold for 1s. 3d.?

6. A ton of sugar is bought for £16 13s. 4d., and retailed at 2d. a pound. What is the actual gain, and also the gain per cent.?

7. Goods are bought at £16 per cwt.; trade expenses amount to 10s. per cwt. At what price per ton must they be sold to gain $33\frac{1}{3}$°/₀ on the whole outlay?

8. A man buys goods which he sells again for £11 18s. $0\frac{1}{2}$d., making a profit of 16°/₀. What is the buying price?

9. A man buys eggs at the rate of 10 for a shilling, and sells 32 for 4s. What is his gain per cent.?

10. A ton of goods which cost £28 is sold retail at 4d. a pound. Find the total gain and the gain per cent.

11. A merchant gains 4°/₀ by selling goods for £16 18s. What is the cost price? How much per cent. would he have gained by selling them for £16 11s. 6d.?

12. If 123 yds. of cloth are bought for £8 4s. and sold at 1s. 6d. per yard, how much profit per cent. on the cost is made?

13. A man sells goods for £85, thus losing 9°/₀ on his outlay. At what price ought he to have sold them to make $13\frac{3}{4}$°/₀ profit?

14. If by selling a book for £2 7s. 6d. I lose one-twentieth of what it cost me, find (1) its cost price; (2) at what price I ought to sell it to gain $4\frac{1}{2}$°/₀.

15. By selling goods for £1 6s. $10\frac{1}{2}$d., a man gains $7\frac{1}{2}$°/₀ on their cost price. What should he have charged for them in order that his profit might have been 10°/₀?

16. If 2,000 articles are retailed at $7\frac{1}{2}$d. each, and a profit of £12 10s. is made on the whole, find the wholesale price of each.

17. A merchant bought 1,260 lbs. of tea, and sold it at 1s. 4d. per lb., thereby making a profit of £16 10s. Afterwards, when the wholesale price had risen $7\frac{1}{2}$°/₀, he purchased 840 lbs., and retailed it at 1s. 6d. per lb. What was his total gain on the second transaction?

18. Coal, bought at £4 6s. per truck, each truck containing 5 tons 7 cwts. 2qrs. is sold at 19s. per ton. What is the profit per cent.?

19. Find the gain or loss per cent. in the following (giving answers in vulgar fractions):—

(1) When goods are bought at 30s. a cwt. and sold at 4d. a lb.

(2) When goods are sold at 30s. a cwt. which cost 4d. a lb.

(3) When coal is bought at 14s. 7d. per 21 cwts. and sold at 16s. 8d. per ton.

(4) When toys are bought at 5 for 8d. and sold at 2d. each.

(5) When 17 tons 15 cwts. cost £120 and are sold at £8 10s. 4d. per ton.

(6) When goods are sold for £100 with a profit of £15.

20. One thousand yds. of cloth are bought at 1s. 9d. a yd.; the cost of carriage, &c., is £2 10s. At what price per yard must the cloth be sold so as to make a profit of $11\frac{1}{9}$°/₀ on the whole outlay?

21. A has a horse which cost him £100; he sells it to B at a profit of 20°/₀. B then sells it to C, making a profit of 10°/₀ on the price he paid to A. What does C pay to B?

22. An article is sold by A to B at a profit of 12°/₀. It is then sold for 5s. 3d. by B, who thereby gains $12\frac{1}{2}$°/₀ on what he paid for it. What did A pay for it?

23. If 8°/₀ is lost by selling an article for £38 5s., find the gain per cent. on selling it for £57.

24. A merchant buys 540 oranges at Seville for 6s. 8d. The cost of packing is $7\frac{1}{2}$°/₀ additional, and the cost of carriage is a penny more than 60·75 times the original cost of an orange. If the oranges are then retailed at 5 for 2d., find the profit on 100 oranges.

25. A coal merchant supplies 7 tons of coal at 20s. a ton, 30 tons at 16s. 6d. a ton, and 10 tons 2 cwts. 3 qrs. 14 lbs. at 15s. a ton. The average cost of the coal was 13s. 4d. a ton. Find (1) his actual profit; (2) his gain per cent. correct to 2 places of decimals.

26. A grazier sold 36 sheep for £74 5s., and made thereby a profit of 10°/₀. What did he give for each sheep?

27. By selling a carriage for £29 18s. 9d., a man loses one-sixth of what it cost him. How much would he have lost or gained by selling it for £36?

28. By selling a horse for £85 a man lost $7\frac{1}{2}$°/₀. Find what would have been his gain or loss per cent. if he had sold it for £96 10s.

29. An article is first sold at a profit of 10°/₀. The purchaser then sells it for 4s. 7d., and his gain is 15°/₀ of the price at which he *sells* it. How much did the article originally cost?

30. An article of commerce passes successively through the hands of three dealers, each of whom, in selling, adds 10°/₀ of the price at which he bought it as his profit. What did the first dealer pay for the article which the third dealer sells for £11 1s. 10d.?

31. A man sells a horse for £12 more than he gave for it, and realizes a profit equal to two-sevenths of its cost price. What was its cost price?

32. A tradesman who is selling off makes a reduction of 10°/₀ from the marked price. What is the marked price of an article for which he receives 15s. 9d.?

33. A tradesman sold 318 cwts. of sugar for £775 2s. 6d. and realized a profit of £39 15s. by the sale. What did the sugar cost him per cwt.? Also what percentage of profit did he make?

34. By selling goods for £240 a tradesman gains 25°/₀. How much would he have gained per cent. by selling them for £204?

35. A dealer sold a watch for £41 10s., and by so doing lost 17°/₀; at the proper price he would have cleared 20°/₀. What was the proper price of the watch?

36. If 3,000 bottles of wine are bought for £463 and then sold at the rate of 5 dozen for £11 11s. 6d., what is the whole gain on the transaction? What is the gain per cent.?

37. A grocer sells sugar at £2 per cwt., making a profit of 20%. At what price per lb. should he sell it to make a profit of 26%?

38. If a profit of 5% is made by selling tea at £12 5s. per cwt., what per cent. profit would be made by selling it at 2s. 4d. a lb.?

39. A tradesman bought a quantity of tea and sold it at 2s. per lb., thereby gaining $33\frac{1}{3}$%. If his total gain was £14, how many pounds of tea did he buy?

40. A person selling an article for £2 12s. 6d. gains 5%. What would he have gained or lost per cent. by selling it at £2 7s. 6d.?

41. If I sell for £30 16s. what cost me £27 10s., what is my gain per cent.?

42. If 15% profit on the cost of production would be made by selling 1,000 qrs. of barley for £1,078 2s. 6d., what is the cost of production per quarter?

43. If 5 tons of potatoes are bought for £18 15s. and sold at 6d. per stone, what is the total gain, and what the gain per cent.?

44. A coal-dealer sells some coal at 22s. a ton, and thereby loses a penny on every shilling of the cost price. What would he have gained on every shilling by selling his coal at 27s. a ton?

45. Sugar passes through the hands of three dealers, each of whom makes 10% profit on his outlay, and is sold by the last of the three at $2\frac{3}{4}$d. a lb. Find, to the nearest penny, the original price of the sugar per cwt.

46. A man bought 800 lbs. of tea for £93 6s. 8d., and sold it so as to gain 25%. At what price per lb. did he sell it?

47. A sells goods to B at a gain of $22\frac{1}{2}$%, and B sells them to C at a gain of $7\frac{1}{2}$%. C gave £526 15s. for them. How much did A give for them?

48. If I were to sell my horse for £72 I should lose 25%. What must I sell it for in order to gain 25%?

49. A man buys a number of articles at 6d. each, and sells them at $6\frac{1}{2}$d. each; also a number at 1s. 3d. each, which he sells at 1s. $3\frac{1}{2}$d. each. What is the ratio between his two percentages of profit?

50. If oranges are bought at 11 for 10d. and sold at 10 for 11d., what is the gain per cent.?

51. Property which cost £522 10s. is sold for £418. What is the loss per cent.?

52. A builder buys a piece of land for £476 18s., and builds a house upon it at a further cost of £1,047 2s. If he sells the house and land for £1,905, what percentage will he gain on the whole outlay?

53. Determine the cost of sugar per cwt. if 12°/₀ profit is made by selling it at $3\frac{1}{2}$d. per lb.

54. A merchant buys 54 dozen of wine for £97 4s. At what price per bottle must he sell it so as to gain $5\frac{5}{9}$°/₀?

55. If 8°/₀ is gained by selling a piece of ground for £773 11s., what would have been the gain per cent., and what would have been the actual amount gained, if it had been sold for £787 17s. 6d.?

56. A grocer buys 3 tons of sugar for £66 13s. 4d. and sells it at $2\frac{1}{2}$d. per lb. What is his profit per cent.?

57. A man sells a horse for £76 at a loss of 5°/₀. At what price should he have sold it to gain 5°/₀?

58. On a sum of £29 3s. 4d. a profit of £2 9s. 7d. is made. How much is this per cent.?

59. A greengrocer sells 10 oranges for the same money which he paid for 11. What does he gain per cent. on his outlay?

60. Marbles are sold so that the same money is obtained for 40 which was paid for 50. Find the gain per cent. on the outlay.

Mixtures.

190. The following exercises involve the principles of profit and loss in relation to mixtures :—

Ex. 7.—*If* 34 *lbs. of tea at* 2s. 3d. *a lb. be mixed with* 29 *lbs. of tea at* 1s. 6d. *a lb., at what price per lb. must the mixture be sold so as to gain* 5°/₀ *on the total outlay?*

34 lbs. at 2s. 3d. cost 918d.,

29 lbs. at 1s. 6d. cost 522d.;

∴ 63 lbs. mixture cost 1,440d.

Add on 5°/₀ profit, *i.e.* 72d.;

∴ 63 lbs. mixture are sold for 1,512d.,

Or 1 lb. mixture is sold for 24d. or 2s. *Ans.*

Ex. 8.— *A merchant mixes* 50 *galls. of spirit costing* £45 *with some water costing nothing, and sells the mixture at* 16s. 6d. *a gall. thereby making a profit of* 10°/₀ *on his outlay. How much water is there in one gallon of mixture? How much water is there in the mixture?*

The C.P. of 1 gall. of mixture + 10°/₀ on this C.P. = $16\frac{1}{2}$s;

∴ C.P. of 1 gall. of mixture is $16\frac{1}{2} \times \frac{10}{11}$s. = 15s.

Now 1 gall. of spirit costs £45 ÷ 50 = 18s; therefore as the water in the gallon of mixture costs nothing, this gallon can contain only $\frac{15}{18}$ or $\frac{5}{6}$ of a gall. of spirit; therefore 1 gall. of mixture contains $\frac{1}{6}$ of a gall. of water, or he adds 1 gall. of water to every 5 galls. of spirit; therefore he adds 10 galls. of water in all.

191. Certain questions of this type were formerly solved by a rule known by the name of *Alligation*, in consequence of the mysterious bars by which the data were linked together. This rule (which appears in Exx. cxlv. 30) solves some problems by rote which are better solved by reasoning, and provides an imperfect and unsatisfactory solution of certain other questions which really belong to the subject known in Algebra as Indeterminate Equations. We solve some problems formerly treated under this head.

Ex. 9.—*In what proportion must a grocer mix teas which cost him* 1s. 6d. *and* 2s. 3d. *a lb. respectively, so that by selling the mixture at* 2s. 6d. *a lb. he may make* 20 % *on his outlay?*

If S.P. of 2s. 6d. brings 20 % profit, the C.P. of mixture must be $\frac{100}{120} \times$ 2s. 6d., *i.e.* 2s. 1d. This is the price therefore at which he must sell in order exactly to meet his expenses.

If 1 lb. of the cheaper tea, C.P. 1s. 6d., is sold at 2s. 1d., the grocer gains 7d. on it.

If 1 lb. of the dearer tea, C.P. 2s. 3d., is sold at 2s. 1d., the grocer loses 2d. on it.

He must sell just as much of the cheaper tea as will balance his loss on the dearer. But to gain 1d. on the cheaper tea, he has to sell $\frac{1}{7}$ lb.; and to lose 1d. on the dearer, he has to sell $\frac{1}{2}$ lb. If, therefore, he mixes $\frac{1}{7}$ lb. of the cheaper and $\frac{1}{2}$ lb. of the dearer, he will meet his expenses. The mixture must therefore be in the ratio $\frac{1}{7} : \frac{1}{2}$, *i.e.* 2 : 7. The principle of such questions is *Equalize gains and losses.*

Ex. 10.—*In what proportions must teas, worth respectively* 1s. 6d., 2s., *and* 2s. 3d. *per lb., be mixed to produce a mixture worth* 1s. 10d.?

Call the three teas A, B, C. On each lb. of A sold at 1s. 10d. there is a gain of 4d., on B there is a loss of 2d. per lb., on C of 5d. However we combine the losses, we can always equalize them by the gains, *i.e.* however we mix B and C, we can always take sufficient of A to balance the loss. Thus, if we take 3 lbs. of B, and 2 of C, we get a total loss of $(3 \times 2 + 2 \times 5)$d. = 16d. To balance this, we must take 4 lbs. of A. Hence, one solution is to mix the teas in the proportion 4 : 3 : 2. Clearly there is an infinite number of such. The student should verify the following possible answers:—

$$3 : 1 : 2, \quad 5 : 5 : 2, \quad 6 : 7 : 2, \text{ \&c.}$$

The readiest way of obtaining a series of solutions to such questions is to act as in Ex. 9, and to reduce the gains and losses to unity. Thus on $\frac{1}{4}$ lb. of A there is a gain of 1d., on $\frac{1}{2}$ lb. of B a loss of 1d, on $\frac{1}{5}$ lb. of C a loss of 1d. If we lose x pence on B, y pence on C, we shall lose nothing on the whole if we gain $(x+y)$ pence on A. Hence we may take $\frac{x+y}{4}$ lbs. of A, $\frac{x}{2}$ lbs. of B, $\frac{y}{5}$ lbs. of C, *i.e.* mix the teas in the proportion $5(x+y) : 10x : 4y$, where x and y are any integers whatever. This formula includes all possible solutions.

For other problems solved by the principles underlying this subject see § 217, Ex. 17, and also Exx. cxlii. 27. cxlv. 23, 29, 30.

EXAMPLES CXV.

1. A grocer buys 1 cwt. of tea for £8 8s.; but 4 lbs. are damaged and rendered worthless. At what price per lb. must he sell the remainder so as to gain $12\frac{1}{2}$ % on the original outlay?

2. A man mixes 6 galls. of spirit at 15s. a gall. with 2 galls. at 12s. 6d. a gall., and sells the mixture at 17s. 3d. a gall. What percentage of profit does he make?

3. A grocer mixes 24 lbs. of tea at 2s. $6\frac{1}{2}$d. per lb. with 8 lbs. at 3s. $2\frac{1}{4}$d. per lb. He gains $33\frac{1}{3}$%. At what price per lb. does he sell the mixture?

4. A person buys a certain quantity of tea at 1s. 9d. per lb. and an equal quantity of tea at 2s. 3d. per lb. He mixes them, and sells the mixture at 2s. 2d. per lb. How much does he gain per cent.?

5. A spirit merchant buys 70 galls. of whisky at 17s. a gall., and 130 galls. more at 12s. a gall. He mixes them and sells the mixture at 16s. 6d. a gall. What does he gain per cent.?

6. A dealer buys coffee at £5 15s. per cwt., and chicory at £2 12s. 6d. per cwt., and mixes them in the proportion of 4 parts of chicory to 7 parts of coffee. Find at what price per cwt. he should sell the mixture so as to gain 20% on his outlay.

7. If 30 tons of coal at 20s. 6d. per ton be mixed with $22\frac{1}{2}$ tons at 24s. per ton, find the cost of 18 tons 6 cwts. of the mixture.

8. A mixture is made of 7 cwts. 3 qrs. 19 lbs. of tea costing 1s. $8\frac{1}{2}$d. per lb. with 23 cwts. 3 qrs. 1 lb. of tea costing 1s. $10\frac{1}{2}$d. per lb. If the mixture be sold at 1s. 11d. per lb. find the profit per lb., and also how much is gained upon the whole transaction.

9. A mixture consists of 15 parts of coffee, purchased at £4 4s. per cwt. and one part of chicory, purchased at £1 16s. per cwt. It is sold at 1s. 4d. per lb. What profit would be made on the sale of 2 cwts. 24 lbs.?

10. If 48 galls. of spirit at 12s. a gall., $2\frac{2}{3}$ galls. at 10s. 6d. a gall., and $19\frac{1}{2}$ galls. at 1s. 4d. a gall. be mixed with $19\frac{5}{8}$ galls. of water (costing nothing), and the mixture be sold at 7s. $10\frac{1}{2}$d. a gall., what is the gain per cent.?

11. A man buys $2\frac{1}{2}$ cwts. of tea at £8 3s. 6d. per cwt. and $1\frac{1}{2}$ cwts. at £11 7s. 6d. per cwt.; he mixes the whole together and sells the mixture at 1s. $10\frac{1}{2}$d. per lb. What is his gain per cent?

12. A man bought a number of oranges at 3 a penny, and an equal number at 2 a penny. At what price per dozen should he sell them to make a profit of 20%?

13. In mixing tea 1 lb. in every 100 lbs. is wasted. In what proportions must a dealer mix teas which cost him 1s. 9d. and 1s. 4d. per lb. respectively so as to gain 10 % by selling the mixture at 1s. 8d. per lb.?

14. There are three kinds of tea valued respectively at 1s. 6d., 2s. 1d., and 2s. 3d. per lb. They are mixed in particular proportions, and the mixture is found to be worth 1s. 11d. per lb. If 15 lbs. more of the second kind were used than of the third, and 8 lbs. more of the second kind than of the first, what was the total weight of the mixture?

Rates and Taxes.

192. It has already been pointed out (see Percentages) that certain payments are calculated at a given rate in the £, instead of at a given rate per cent.—*e.g.* rates, income tax, bankrupts' dividends. The two methods are but different ways of expressing the same thing. Thus a payment of 2s. 6d. in the £ is the same as a payment of 1 half-crown in every 8 half-crowns, or £1 in every £8, or £$12\frac{1}{2}$ in every £100, *i.e.* $12\frac{1}{2}$%. So 3s. 4d. in the £ represents the same as the fraction $\frac{1}{6}$ or the percentage $16\frac{2}{3}$.

Rates are certain payments which various local authorities are empowered by law to levy in order to defray different local expenses. They are calculated as so much in the £ on the annual (or rateable) value of a man's property.

Income Tax is a tax levied on a man's income, and is payable to the Government of the country. It also is reckoned as so much in the £. At present (1902) all incomes not exceeding £160 are entirely exempt from taxation; whilst from those which are larger than £160, but do not exceed £700, certain deductions are allowed, and the tax is levied on the remainder only.

The term *Liabilities* is applied to the sum total of a man's debts. His *Assets* consist of the money which he has in cash or which is due to him from various *Debtors*, and the money which would be obtained by the sale of his property. The payments which a bankrupt is able to make to his *Creditors* are called *Dividends*, and are reckoned at so much on each £ of his liabilities.

Premium is the annual sum (reckoned as a percentage) paid by a man to an insurance company when he insures his life or his property.

For the term *Commission* see § 186.

193. Ex. 1.—*Find the commission on* £850 10s. *at* $2\frac{1}{2}$%.

Using the unitary method, we say—

On £100 the commission is £$2\frac{1}{2}$;

$\therefore$ On £850½ the commission is £850½ $\times \frac{2\frac{1}{2}}{100}$ = £850½ $\times \frac{1}{40}$ = £21 5s. 3d.

Ex. 2.—*Find the gross value of an estate which yields £2,880 after a commission of 4°/₀ has been deducted.*

When 4°/₀ has been deducted there remains 96°/₀; $\therefore$ 96°/₀ of the gross value = £2,880; $\therefore$ Gross value = £2,880 $\div \frac{96}{100}$ = £3,000.

Or thus—£96 remains out of a supposed gross value of £100;

$\therefore$ £2,880 remains out of a gross value of £2,880 $\times \frac{100}{96}$ = £3,000.

N.B.—By *gross* value we mean the full value before any deductions are made. The remainder is called the *net* value.

Ex. 3.—*What is the whole rate at* 4s. 6½d. *in the £ on property of which the rateable value is £850?*

On each £ the rate is 4s. 6½d.; therefore on £850 the rate is 4s. 6½d. × 850 = £193 0s. 5d. This result can be obtained by ordinary multiplication or by the practice method. (See § 171, Ex. 7.)

Ex. 4.—*A man's income is £550. What remains after paying an income tax of* 5d. *in the £ on the whole?*

The tax amounts to 5d × 550 = £11 9s. 2d.; therefore his net income = £538 10s. 10d.

It is sometimes better, in such problems, to note that 5d. in the £ is the same as $\frac{1}{48}$; therefore, after paying the tax on £1, he has £$\frac{47}{48}$ left, &c.

Ex. 5.—*A cargo is valued at £2,945. For what must the owner insure it so that in case of loss he may obtain sufficient to cover the value of the cargo and the premium, which is at the rate of 5°/₀?*

If he insures for £100, then in case of loss he would recover £5 premium and also £95 as the value of goods. Therefore he must insure £95 as if it were £100, for which he would then pay a premium of £5;

$\therefore$ £2,945 must be insured as if it were £2,945 $\times \frac{100}{95}$ = £3,100.

Ex. 6.—*How much in the £ can a bankrupt pay when his liabilities are £9,875, whilst his assets are only £6,994 15s. 10d.?*

When he ought to pay £9,875 he can pay only £6,994 15s. 10d.;

$\therefore$ When he ought to pay £1 he can pay only £6,994 15s. 10d. $\div$ 9875 = 14s. 2d. *Ans.* 14s. 2d. in the £.

EXAMPLES CXVI.

1. Find the commission on—

(1) £250 at 3%
(2) £366 13s. 4d. at 1½%.
(3) £1,883 6s. 8d. at ⅛%.
(4) £476 15s. at 2d. in the £.
(5) £873 10s. 4d. at 2½d. on each *complete* £.
(6) £480 17s. 6d. at 5%.

2. An agent collects rents the gross value of which is £353 15s. On this he charges for his services a commission of 4%. What is the net amount he pays to the landlord?

3. Find the gross value of an estate which yields £12,749 after the payment of a 5 % commission to an agent.

4. The rent of a house is £40 a year, and the tenant pays yearly rates which amount to 3s. 5d. in the £. What is the total amount paid annually by the tenant for rent and rates?

5. A house is rented at £70 a year; but the rates are calculated on six-sevenths of the rent. If these rates consist of two half-yearly payments of 2s. 4½d. for each £, at which the house is assessed, find the total cost to the tenant for rent and rates.

6. A rate of 2s. 7½d. in the £ is levied in a parish of which the rateable value is £27,000. What is the total amount thus raised?

7. An estate is assessed at £15,450. Find the rates on it at 4s. 3¼d. in the £.

8. If a rate of 1s. 6½d. in the £ produces £728 8s. 9d., what is the value at which the property is assessed?

9. The property in a parish is assessed at £87,000. What is the rate in the £ which will produce £20,118 15s.?

10. What is the net income left out of £450 after paying an income tax of 6d. in the £?

11. After paying income tax at the rate of 8d. in the £ a man's income is reduced to £275 10s. What was his gross income?

12. A man's income is £950. He pays £57 10s. for life insurance, on which he claims exemption from income tax. How much does he pay as income tax at 7d. in the £?

13. An increase in the income tax of 1d. in the £ reduces a man's net income by £1 17s. 6d. What is his gross income?

14. An income of £580 was charged income tax at 8d. in the £ on the excess over £120. What was the net income?

15. A man pays an income tax of £10, which is at the rate of 8d. in the £ on the excess of his gross income over £150. What is his gross income?

16. A man's income is £750 a year. He pays income tax at 8d. in the £ on the whole. His house is rented at £80 a year, but assessed at four-fifths of the rent. On this assessment he pays two rates of 1s. 6½d. and 1s. 7¾d. respectively. What is his net income after deducting rent, rates, and tax?

17. A man whose annual income is £620 pays income tax at the rate of 8d. in the £, but claims that the premiums which he pays for life insurance should be untaxed. If the amount of income tax paid is £18 14s. 10d., what is the amount of the premiums?

18. A man pays 5½ % of his gross income in life insurances; on what remains he pays 8d. in the £ in income tax; and, again, on what then remains he pays 3⅓ % in subscriptions. If his net income is £1,103 16s. 3d., find his gross income.

19. A man allows his agent 5% on his gross income of £16,292 10s. for collecting it. He spends one-seventh of what he has then left in insuring his life, and this portion is exempt from income tax. The income tax being 8d. in the £, find its amount.

20. Income tax was levied in the year 1884 upon incomes less than £400 at the rate of 5d. in the £ on the excess of the income above £120. A certain man's income and the income tax on it were both of the same amount as they were in the year 1874, in which year the tax was reckoned at the rate of 3d. in the £ on his whole income. What is his income?

21. In 1893 a man paid income tax at the rate of 7d. in the £ on the portion of his income exceeding £120; in 1894 he paid at the rate of 8d. in the £ on the portion of his income exceeding £160. If his income was the same in each year, and he paid 5s. less in 1894 than in 1893, what was his income?

22. An estate has been let for £480 a year, less landlord's taxes amounting to 15d. in the £. If the rent is reduced to £450 a year, and taxes rise to 16d. in the £, by how much per cent. will the landlord's net receipts be reduced?

23. If on a gross income of £562 10s. you pay taxes amounting to £11 6s. 8d. per £100, as well as income tax, and then find your net income to be £487 0s. $7\frac{1}{2}$d., at what rate in the £ is the income tax?

24. A man's income is reduced to £522 after he has paid an income tax of 8d. in the £. Find what would have been his net income if the income tax had been 7d. in the £.

25. The income tax is raised from 6d. to 10d. in the £, but the revenue is increased by 10% only. Find the decrease per cent. in the amount of income taxed.

26. During the first half of a certain financial year the income tax was fixed at 5d. in the £, but during the second half it was increased to 8d. in the £. A gentleman whose income is at the rate of £1,000 a year more in the second than in the first half year, finds that he pays for the year £35 8s. 4d. more income tax than he would have paid if his income and the income tax had remained unchanged throughout the year. Find his income for the whole financial year, and the whole amount of income tax which he pays.

27. A man insures his life for £500. What annual premium does he pay at £1 18s. 3d.%?

28. Find the total premium on a policy of £8,250 at $2\frac{3}{4}$%.

29. The owner of a ship's cargo pays a premium which, at the rate of $7\frac{3}{8}$% amounts to £711 13s. 9d. What is the value of the cargo?

30. A cargo is valued at £6,971 10s. What must it be insured for so that in case of loss the owner may recover the value of the cargo and the premium, which is at the rate of $4\frac{1}{2}$%?

31. A man insures his life for £700 at the rate of £1 19s. 7d. per £100. After a few years he insures for an additional £500, but at a higher rate. If the premiums on the £700 and on the £500 be the same, what is the rate in the latter case?

32. A man insures property at 3 % for an amount sufficient, in case of loss, to repay the value of the property and the premium. If the latter amount to £492, what is the value of the property, and for how much does he insure it?

33. The assets of a bankrupt are £8,473 15s., whilst his debts amount to £50,842 10s. How much in the £ can he pay?

34. The realized value of a bankrupt's estate is £876 13s. 6d.; but his debts absorb the whole of this amount and £292 4s. 6d. more. What dividend in the £ is he able to pay?

35. A bankrupt owes three creditors £840, £480, £600 respectively. How much will each receive, if his assets amount to £840?

36. Find the dividend on £957 10s. at 5s. 6d. in the £; also (to the nearest penny) on £18,490 18s. at 14s. 10d. in the £.

37. A creditor receives £350, which is at the rate of 8s. 4d. in the £ on the full amount due to him. What is this amount?

38. A bankrupt pays successive dividends of 2s. 6d., 3s. 4d., and 6d. in the £ respectively on the full value of his debts, which amount to £8,525. How much does he pay altogether, and how much does a creditor receive to whom he owes £482?

39. The expenses of bankruptcy proceedings are 2 % of the liabilities, which amount to £25,450. If these expenses are paid first and in full, how much in the £ can the bankrupt pay to his creditors if his total assets amount to £10,000? (Answer to nearest farthing.)

40. A creditor is owed £480 12s. 6d. He receives a first dividend at the rate of 10s. 6d. in the £, and a further dividend on the remainder of 3s. 4d. in the £. How much does he receive altogether? How much in the £ is this on the whole original debt?

41. Three creditors are owed respectively £721 12s., £450, £621 5s. 4d. The first receives £451. How much in the £ is the dividend, and how much do the other two creditors receive respectively?

42. A creditor is owed £850. How much does he lose owing to the debtor being able to pay only 16s. 8d. in the £?

43. A tradesman is owed £321 10s., £420 12s. 6d., £250, £175 18s., on which he receives respectively 4s., 3s 4d., 16s., and 20s. in the £. His stock is valued at £679 14s. 9d. If his own liabilities amount to £1,690, how much in the £ can he pay?

CHAPTER XXV.

INTEREST, SIMPLE AND COMPOUND.

194. *Interest* is the payment made for the use of borrowed money. The sum borrowed is called the *Principal*. The interest is reckoned as a percentage of the principal. The element of time enters into its calculation, since the longer the time during which the principal is lent, the greater should be the sum paid for its use. The time is generally expressed in years. When the interest is actually paid at the end of each year (or each interval of time), the principal is said to be lent at *Simple Interest* (S.I.); but if, instead of being thus paid as it becomes due, it is retained by the borrower, and added on to the principal, the money is said to be lent at *Compound Interest* (C.I.). (See § 199.)

Thus, if £500 is lent for three years at 4% per annum (S.I.), the borrower should pay *each year* £4 for every £100 lent—*i.e.* £20 each year; therefore for the three years he should pay £60.

The interest is thus proportional to both the number of years and to the rate per cent.—*i.e.* at a given rate the interest for 2, 3, 4, &c., years would be twice, thrice, four times, &c., the interest for one year; or, for a given number of years, the interest at 2, 3, 4, &c., per cent. would be twice, thrice, four times, &c., the interest at 1%.

If the principal and the interest be added together, the total is called the *Amount*.

Ex. 1.—*Find* S.I. *on* £450 *for* 3 *years at* 5% *per annum.*

The interest on £100 for 1 year is £5.

∴ The interest on £100 for 3 years is £5 × 3; therefore the interest on £1 for 3 years is £5 × 3 ÷ 100; therefore the interest on £450 for 3 years is £450 × 5 × 3 ÷ 100 = £67 10s.

(The amount would be £517 10s.)

From a study of this example it will be seen that the S.I. on any sum of money for a given number of years, and at a

given rate per cent., is found as follows:—*Multiply the principal by the number of years and by the rate per cent., and divide the result by* 100.

This may be conveniently remembered in the following symbolical form:— s.I. $= \frac{P \times n \times r}{100}$, where $P =$ principal, $n =$ number of years, $r =$ rate per cent.

The beginner is advised to work a few examples by the unitary method, as shown in Ex. 1 above, until the reason for the rule is clear.

Ex. 2.—*Find* s.I. *on* £323 17s. 6d. *for* 4 *years at* 5% *per annum.*

First method—

£	s.	d.	
323	17	6	
		4	Multiply by the number of years.
1295	10	0	
		5	Multiply by the rate per cent.
64·77	10	0	Divide by 100.
15·5ø s.			(See p. 85.)
6·ø d.			*Ans.* £64 15s. 6d.

Second method—

$$\text{s.I.} = \frac{£323\ 17\text{s. } 6\text{d.} \times 5 \times 4}{100} = \frac{£323\ 17\text{s. } 6\text{d.}}{5} = £64\ 15\text{s. } 6\text{d.}$$

EXAMPLES CXVII.

Find the simple interest on—

1. £500 for 2 years at 3%.
2. £850 for 4 years at 3%.
3. £133 6s. 8d. for 3 years at 2%.
4. £250 for 2 years at 4%.
5. £700 for 3 years at $2\frac{1}{2}$%.
6. £360 for 2 years at 5%.
7. £340 12s. 6d. for 3 years at 4%.
8. £480 17s. 6d. for 2 years at $3\frac{1}{3}$%.
9. £488 16s. 8d. for 5 years at $3\frac{1}{2}$%.
10. £487 15s. for 6 years at $8\frac{1}{3}$%.

195. If the time given be not an integral number of years, express the odd months or days as a fraction of a year, reckoning 12 months, or 365 days, in a year.

Note that 365 contains the factor 73 five times; therefore 1×73, 2×73 (= 146), 3×73 (= 219), 4×73 (= 292) days, can be written at once as $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$ of a year.

Ex. 3.—*Find the amount on* £721 13s. 6d. *for* 2 *years* 9 *months at* 3 °/₀ *per annum.*

s.I. = £721 13s. 6d. × $2\frac{3}{4}$ × 3 ÷ 100
= £721 13s. 6d. × $8\frac{1}{4}$ ÷ 100
= £59 10s. 9·165d.
∴ Amount = principal + interest
= £781 4s. 3·165d.

£	s.	d.
721	13	6
		$8\frac{1}{4}$
5773	8	0
180	8	4·5
59·53	16	4·5
10·76		
9·165		

This is the exact answer; but, for practical work, it is sufficient to get an answer correct to the nearest farthing, or, more generally, to the nearest penny, or sometimes altogether neglecting the final fraction of a penny. Bankers make no entries in their books of any amount less than a penny.

In such questions as the above, and others of a more complex nature, it is better to work in decimals. Thus—

Ex. 4.—*Find the simple interest on* £4,726 16s. 8d. *for* 2 *years* 5 *months, at* $3\frac{1}{4}$ °/₀ *per annum.* (Answer to nearest farthing.)

Principal = £4726·8$\dot{3}$.

Time × rate ÷ 100 = $\frac{29}{12} \times \frac{13}{4} \times \frac{1}{100} = \frac{377}{4800}$ = ·0785416 ;

∴ Interest = £4726·8$\dot{3}$ × ·0785416
= £371·2533
= £371 5s. 0$\frac{3}{4}$d.

In the Contracted Multiplication the multiplicand and multiplier are rearranged according to the Rule of p. 217. Four places are retained to obtain the nearest farthing (p. 233).

472·6833	
·7854	166...
330·8783	1
37·8146	6
2·3634	2
·1890	7
47	3
28	3
2	8
	3
371·2533	

Or thus—

Interest = £4726·8$\dot{3}$ × $\frac{377}{4800}$
= £4726·8$\dot{3}$ × 3·77 ÷ 48
= £371·2533
= £371 5s. 0$\frac{3}{4}$d.

	47268·333		
	·377		
	14180·499	9	
	3308·783	3	
	330·878	3	
48 { 4	17820·162		..... (A)
(12)	4455·040		
	371·2533		

It is better to leave the division by 48 till the last, as it will not then in the line (A) be necessary to secure accuracy beyond the second or third decimal place, as any further figures would not, as a rule, affect the third and fourth places in the quotient.

EXAMPLES CXVIII.

The answers to Nos. 1–20 are exact.

Find the simple interest on—

1. £46 13s. 4d. for $2\frac{3}{4}$ years at $3\frac{1}{2}$%. **2.** £126 18s. for $3\frac{1}{3}$ years at $2\frac{1}{2}$%.
3. £872 10s. for $6\frac{1}{2}$ years at $5\frac{1}{3}$%.
4. £244 8s. 4d. for $2\frac{1}{2}$ years at $3\frac{1}{2}$%.
5. £55 16s. 8d. for 2 years at $3\frac{7}{8}$%. **6.** £3695 15s. for $1\frac{2}{3}$ years at $4\frac{1}{4}$%.
7. £2,970 16s. 8d. for 3 years 73 days at $3\frac{1}{8}$%.
8. £2,500 for 5 years 6 months at $4\frac{1}{2}$%.
9. £232 13s. 4d. for 5 years at $2\frac{1}{4}$%.
10. £320 15s. for 32 years at $2\frac{1}{2}$%.
11. £975 for 2 years 4 months at 4%.
12. £206 16s. 6d. for 5 years at $3\frac{1}{3}$%.
13. £215 for $4\frac{1}{4}$ years at $3\frac{3}{4}$%. **14.** £750 for $4\frac{1}{2}$ years at $3\frac{3}{4}$%.
15. £440 for 3 years 3 months at $2\frac{1}{8}$%.
16. £980 for 5 years 9 months at $3\frac{1}{6}$%.
17. £1,725 for 3 years 3 months at $5\frac{1}{2}$%.
18. £3,725 15s. for 3 years 6 months at $3\frac{1}{3}$%.
19. £11,175 for $2\frac{1}{2}$ years at 3%.
20. £468 11s $10\frac{1}{2}$d. for 4 years at 4%.

Find to the nearest penny the—

21. Interest on £356 6s. 8d. for $4\frac{1}{2}$ years at $5\frac{1}{2}$%
22. Interest on £712 13s. 4d. for $4\frac{1}{2}$ years at $5\frac{1}{2}$%
23. Amount on £240 12s. 6d. for $8\frac{3}{4}$ years at $2\frac{1}{2}$%.
24. Amount on £257 for $5\frac{1}{2}$ years at $2\frac{3}{4}$%.
25. Interest on £976 10s. for $1\frac{1}{3}$ years at $2\frac{3}{4}$%.
26. Amount on £4,751 10s. for 1 year 6 months at $2\frac{3}{8}$%.
27. Interest on £219 18s. 9d. for 2 years 6 months at $3\frac{1}{3}$%.
28. Interest on £982 7s. 6d. for 4 years at $3\frac{3}{4}$%.
29. Interest on £325 3s. 9d. for 3 years 73 days at $3\frac{3}{4}$%.
30. Interest on £481 5s. for $2\frac{1}{4}$ years at $3\frac{1}{4}$%.
31. Interest on £1,373 6s. $2\frac{1}{2}$d. for 5 years at 4%.
32. Interest on £243 10s. for 2 years 5 months at $4\frac{3}{4}$%.
33. Interest on £451 13s. 4d. for 3 months at $2\frac{3}{4}$%.
34. Amount on £2,020 10s. for 1 year $4\frac{7}{8}$ months at 4%.
35. Amount on £2,024 18s. 8d. for 3 years 3 months at $3\frac{1}{8}$%.

196. Very frequently (see Bankers' Discount, § 206) the time for which a sum of money is lent is given in days, or the dates on which it is lent and on which it is to be repaid with interest are given. In finding the number of days from the first date to the second, note that *one only* of the days named is to be included. Thus, from March 17th to September 20th the number of days would be

$$\overset{\text{March}}{(31-17=)\,14}+\overset{\text{April}}{30}+\overset{\text{May}}{31}+\overset{\text{June}}{30}+\overset{\text{July}}{31}+\overset{\text{Aug.}}{31}+\overset{\text{Sept.}}{20}$$
$$= 187 \text{ days} = \tfrac{187}{365} \text{ of a year.}$$

So, from January 18th to March 21st, 1904 (leap year), we should have

$$\overset{\text{Jan.}}{(31-18=)\,13}+\overset{\text{Feb.}}{29}+\overset{\text{March.}}{21} = 63 \text{ days,}$$

i.e. we reckon in the extra day in February; but, in expressing 63 days as a fraction of a year, it is usual to put it as $\frac{63}{365}$, and not as $\frac{63}{366}$. Continental creditors, to their own profit, avoid all trouble by reckoning the year as 360 days.

Ex. 5.—*Find to the nearest penny the simple interest on* £863 15s. 4d. *from March* 17*th to September* 20*th* (*see above*) *at* $2\frac{3}{4}\%$ *per annum.*

Principal = £863·7$\dot{6}$.

$$\text{Time} \times \text{rate} \div 100 = \frac{187}{365} \times \frac{11}{4} \times \frac{1}{100} \quad \text{(Arrange so that denominator may become 73.)}$$

$$= \frac{187 \times 11}{365 \times 2 \times 2 \times 100} = \frac{2057}{73 \times 2000} = \frac{1{\cdot}0285}{73}.$$

∴ Int. = £863·7$\dot{6}$ × 1·0285 ÷ 73
= £888·382 ÷ 73
= £12·170 (correct to 3rd place)
= £12 3s. 5d. (correct to nearest penny).

```
      8637·67|
          ·10|285
      ---------
       863·76|7
        17·27|5
         6·91|0
          ·43|1
      ---------
73 ) 888·38|( 12·1696
      158
       12 3
        5 08
          70
           5
```

The answer is required to the nearest penny. Three decimal places of a £ are therefore needed (p. 231). The product 863·7$\dot{6}$ × 1·0285 may therefore be correct to two places only, for the limit of error in this case will be ·005 for the product, and therefore ·005 ÷ 73 (*i.e.* about ·00007) for the final answer. This error will not affect the third place. To multiply 863·7$\dot{6}$ by 1·0285 correct to two places, we follow the Rule of p. 217 and rewrite the multiplier as ·10285, and the multiplicand as 8637·$\dot{6}$, or, correct to two places, 8637·67.

197. "*The third, tenth, and tenth rule.*"

The division by 73 required in the above and similar examples may be worked by a method similar to that explained in § 173.

We have $\frac{1}{73} = \cdot 0137$ nearly $= 1\cdot 37 \div 100$.

But $1\cdot 37 = 1\frac{37}{100} = 1\frac{111}{300} = 1 + \frac{100}{300} + \frac{10}{300} + \frac{1}{300}$

$= 1 + \frac{1}{3} + (\frac{1}{10} \text{ of } \frac{1}{3}) + (\frac{1}{10} \text{ of } \frac{1}{10} \text{ of } \frac{1}{3})$.

Hence, to divide by 73, we add to the dividend one-third of itself, one-tenth of that, and again one-tenth of this tenth, and then divide the sum by 100.

Thus the last example runs:

	£888·38
$\frac{1}{3}$	296·12
$\frac{1}{10}$	29·61
$\frac{1}{10}$	2·96
	1217·07
Divide by 100	12·1707 = £12 3s. 5d. to nearest penny.

It can be shown that this method gives an answer which is too large by almost exactly $\frac{1}{4}$d. for every £10 in the answer; deduct such a number of farthings to get the correct result.

EXAMPLES CXIX.

Find to the nearest penny the simple interest on—

1. £1,368 15s. for 106 days at $4\frac{3}{4}$%.

2. £7,000 for 85 days at 4%.

3. £425 for 1 year 233 days at $3\frac{1}{2}$%.

4. £475 5s. $2\frac{1}{2}$d. for 304 days at $6\frac{1}{4}$%.

5. £19,314 11s. 8d. for 19 days at 4%.

6. £7,950 18s. 4d. for 3 years 105 days at $3\frac{1}{8}$%.

7. £369 3s. 4d. for 3 years 73 days at $9\frac{3}{8}$%.

8. £518 19s. 2d. for 170 days at $9\frac{1}{8}$%.

9. £389 6s. 8d. for 75 days at $3\frac{1}{2}$%.

10. £758 6s. 2d. for 53 days at $4\frac{1}{2}$%.

11. £1,044 18s. 9d. for 173 days at $3\frac{1}{4}$%.

12. £680 14s. 5d. for 250 days at $2\frac{4}{5}$%.

13. £741 10s. from March 16th to June 30th at 3%.

14. £2,325 from Sept. 3rd, 1900, to March 4th, 1901, at $4\frac{1}{2}$%.

15. £482 15s. for 158 days at £3 17s. 8d.%.

16. £856 from Sept. 3rd, 1895, to April 6th, 1896, at £3 13s. 4d.%.

198. We have seen that Simple Interest

$$= \frac{\text{principal} \times \text{time} \times \text{rate}}{100},$$

or, using symbols $$I = \frac{P \times n \times r}{100}.$$

There are here four quantities, P, n, r, and I; and, if any three of the four are given, we can find the fourth. It has been already shown how to find I when we are given P, n, and r. It remains to examine the three converse cases, together with those in which the *amount* is given.

Case I.—Find the principal when the interest, the time, and the rate per cent. are given.

Ex. 1.—*What is the principal the interest on which for $3\frac{1}{2}$ years at 4% per annum is £105?*

The interest on a *supposed* principal of £100 for $3\frac{1}{2}$ years at 4% is £$3\frac{1}{2} \times 4$, *i.e.* £14. Adopting the unitary method, we have £14 ($= 3\frac{1}{2} \times 4$) is the interest on £100;

$$\therefore \quad \text{£105 is the interest on £}\frac{105 \times 100}{3\frac{1}{2} \times 4}, \textit{ i.e. } \text{£750}.$$

From this example we see that the required principal $= \dfrac{\text{int.} \times 100}{\text{time} \times \text{rate}}$;

or thus— $\dfrac{P \times n \times r}{100} = I$. Multiply both sides by 100;

$$\therefore \quad P \times n \times r = I \times 100; \qquad \therefore \quad P = \frac{I \times 100}{n \times r} \text{.....(A)}.$$

Note that P and I must both be expressed in the same unit, *i.e.* both pounds or both shillings, &c.

Case II.—Find the time when the principal, interest, and rate per cent. are given.

Ex. 2.—*In what time will the interest on £750 at 4% per annum become £105?*

Suppose the time to be 1 year.

The interest on £750 for *1 year* at 4% is £$\dfrac{750 \times 1 \times 4}{100} =$ £30.

Adopting the unitary method, we have

£30 is the interest for 1 year;

$\therefore$ £105 is the interest for $\frac{105}{30}$ or $3\frac{1}{2}$ years,

i.e. required time in years $= \dfrac{\text{whole int.}}{\text{int. for 1 year}}$;

or thus— $P \times n \times r = I \times 100$; $\therefore \quad n = \dfrac{I \times 100}{P \times r}$ .....(B).

Case III.—Find the rate per cent. when the principal, interest, and time are given.

This is treated exactly as Case II. Suppose the rate to be 1 %,

then required rate % $= \dfrac{\text{whole int.}}{\text{int. at 1\%}}$;

or thus— $P \times n \times r = I \times 100$; $\therefore \quad r = \dfrac{I \times 100}{P \times n}$ .....(C).

The results (A), (B), (C) should not be learnt as three separate formulæ, but should be obtained, when necessary, from

$$I = \frac{P \times n \times r}{100}.$$

Case IV.—Given the amount, the principal, and also the rate per cent. (or time), find the time (or rate per cent.).

Since amount = principal + interest, we obtain the interest by subtracting the principal from the amount. We should then know the principal, the interest, and the rate per cent. (or time), and the problem becomes the same as in Cases II. and III.

Ex. 3.—*At what rate per cent. will £750 amount to £855 in $3\frac{1}{2}$ years?*

Interest = amount − principal = £105.

The question now becomes the following:—At what rate per cent. will the interest on £750 for $3\frac{1}{2}$ years be £105?

Then, as in Case III., required rate per cent.

$$= \frac{\text{whole int.}}{\text{int. at 1\%}} = \frac{105}{26\frac{1}{4}} = 4\%;$$

or required rate per cent. $= \dfrac{105 \times 100}{750 \times 3\frac{1}{2}} = 4\%.$

Case V.—Given the amount, and also the time and rate per cent., find the principal.

We must here suppose some convenient principal—£100 is the best, work out its amount under the circumstances, and then state a proportion question.

Ex. 4.—*What principal will amount to £855 in $3\frac{1}{2}$ years at 4% per annum?*

A supposed principal of £100 will, in the given time and at the given rate per cent., *amount* to £114. Then, by the unitary method, we have

£114 is the amount on £100 principal;

$$\therefore \quad £855 \text{ is the amount on } £\frac{100 \times 855}{114} = £750.$$

N.B.—Mistakes are frequently made in this class of problems by confusing the principal with the amount, and then finding the interest on the latter (*i.e.* £855 in the above) instead of on a supposed principal of £100.

The amount is made up of the principal with its interest, *i.e.* of a fixed quantity together with a variable quantity; it is *not*, therefore, like the interest, proportional to either the time or the rate per cent.

EXAMPLES CXX.

Find the principal in Examples 1–4 when the simple interest—

1. For 3 years at 5% is £90.

2. For $2\frac{1}{2}$ years at 4% is £68.

3. For 3 years 6 months at $2\frac{1}{2}$% is £297 10s.

4. For 2 years 73 days at 3% is £41 5s.

5. In what time will the simple interest on £450 at 4% per annum become £54?

6. If the interest on £675 for 4 years be £54, find the rate per cent. per annum.

7. If the interest on £2,263 6s. 8d. at 3% be £339 10s., find the number of years for which the principal is lent.

8. What is the rate per cent. when £134 10s. is the simple interest on £672 10s. for 4 years?

9. At what rate is £84 14s. the simple interest on £2,400 for $1\frac{3}{4}$ years?

10. In what time will £1,260 amount to £1,496 5s. at $3\frac{3}{4}$%?

11. What is the sum the simple interest on which for $6\frac{1}{2}$ years at $5\frac{1}{3}$% is £302 9s. 4d.?

12. What sum of money will amount to £436 in 3 years at 3%?

13. What sum of money will produce £158 2s. 6d. interest in $11\frac{1}{2}$ years at $5\frac{1}{2}$% simple interest?

14. If the interest on £650 for 5 *months* is £12 3s. 9d., what is the rate per cent.?

15. In what time will £250 amount to £287 10s. at 3%?

16. At what rate per cent. per annum will £12 amount to £13 in $2\frac{1}{2}$ years at simple interest?

17. Find the sum of money on which the simple interest for 9 months at 6% per annum is £3 7s. 6d.

18. What is the sum of money which, if invested for 3 years 8 months at $2\frac{1}{2}\%$ simple interest, will amount to £393 ?

19. If £39 7s. 6d. is the simple interest on £750 for $1\frac{3}{4}$ years, what is the rate per cent. ?

20. In how many years will £256 6s. 8d. amount to £352 9s. 2d. if lent at simple interest at the rate of $7\frac{1}{2}\%$ per annum?

21. At what rate per cent. will £1,560 increase in $3\frac{1}{2}$ years to £1,805 14s. ?

22. The amount of £420 put out at simple interest for 8 months is £429 16s. At what rate is the interest calculated ?

23. In what time will £3,100 amount to £3,384 3s. 4d. at $3\frac{1}{3}\%$ per annum simple interest ?

24. What sum must be lent for $4\frac{1}{2}$ years at 4% to produce £618 15s. as interest ?

25. At what rate per cent. will £440 6s. 8d. amount to £511 17s. 9d. in 5 years ?

26. At what rate per cent. will the simple interest on £980 be £178 8s. 10d. in 3 years 2 months ?

27. The simple interest on £318 6s. 8d. for 17 years is £351 15s. 2d. Determine the rate per cent.

28. In what time will £1,560 amount to £1,805 14s. at 2% ?

29. What sum of money at $2\frac{3}{4}\%$ per annum will amount in 5 years, principal and interest together, to £2,616 5s. ?

30. In what time will £1,500 amount to £1,865 12s. 6d. at $3\frac{3}{4}\%$ per annum simple interest ?

Compound Interest.

199. When the interest as it becomes due is retained by the borrower and added on to the principal, money is said to be lent at *Compound Interest.*

Ex. 1.—*Suppose* A *lends to* B £25 *to be repaid at the end of* 3 *years with compound interest at* 5% *per annum, how much should* A *receive altogether ?*

We reason as follows:—At the end of the 1st year the interest on £25 at 5% is £1 5s. Instead of paying this £1 5s. to A for the use of his money, B borrows or retains it, and then has in his possession the original £25 + £1 5s. interest, *i.e.* he *begins* the 2nd year with £26 5s. of A's money.

The interest on this for one year at 5% is £26 5s. $\times \frac{5}{100}$ = £1 6s. 3d. B again borrows or retains this interest, and therefore he *begins* the 3rd year with £26 5s. + £1 6s. 3d. or £27 11s. 3d. of A's money. The interest on this for one year at 5% is £27 11s. 3d. $\times \frac{5}{100}$ = £1 7s. $6\frac{3}{4}$d. Therefore, at the *end* of the 3rd year B owes A £27 11s. 3d. + £1 7s. $6\frac{3}{4}$d. or £28 18s. $9\frac{3}{4}$d. This, then, is the sum of money which A should receive from B. It is made up of the original £25 together with £3 18s. $9\frac{3}{4}$d. interest (*i.e.*, the sum of the three separate interests £1 5s. + £1 6s. 3d. + £1 7s. $6\frac{3}{4}$d.).

If B had paid annually to A the interest as it became due, instead of retaining it to the end, he would have paid £1 5s. each year, or £3 15s. for the three years. The extra 3s. $9\frac{3}{4}$d. is the excess of the compound interest over the simple interest.

200. From a study of the above example the following statements should be clear :—

(1) The successive interests are gradually increasing, because the sums on which they are calculated are gradually increasing. The actual increase in interest from one year to the next is equal to the interest for one year on the interest for the former of the two years.

(2) The *principal* with which B. starts the second or third year is equal to the *amount* for 1 or for 2 years respectively.

(3) The final *amount* due from B. is equal to the original principal together with the total interest; or total interest = whole amount − original principal.

(4) This total interest is the same as the sum of the separate yearly interests.

201. In the above example each year's principal and interest have, for the purpose of explanation, been worked out in full (£ s. d.); but in the practical working of examples decimals should be employed throughout, the reduction to shillings and pence taking place at the end. The work would then stand as follows :—

	£
Principal for 1st year	25
Interest for 1st year	1·25
Amount for one year or Principal for 2nd year	26·25
Interest for 2nd year	1·3125
Amount for two years or Principal for 3rd year	27·5625
Interest for 3rd year	1·378125
Amount for three years	28·940625
	= £28 18s. $9\frac{3}{4}$d.

Here we multiply each successive principal by the rate per cent., viz., 5, and divide by 100, as a mental process, without writing down the 5, the division by 100 being done by putting down the right-hand figure of the product two places towards the right. By so doing we secure the decimal points being vertically below one another all the way down, and each figure in its right place, units under units, and so on.

Note that multiplying by 5 and dividing by 100 is the same as at once dividing by 20, the quotient being written down so as to get all decimal points in the same vertical line.

Ex. 2.—*Find to the nearest penny the amount on* £1,856 16s. 8d. *for* 4 *years at* 4% *per annum.*

	£
Principal for 1st year	1856·833**3**33
Interest for 1st year	74·27333
Amount for one year or Principal for 2nd year	1931·106**6**66
Interest for 2nd year	77·24426
	2008·350**9**2
Interest for 3rd year	80·33403
	2088·684**9**5
Interest for 4th year	83·54739
	2172·23234

In order to obtain an answer to the nearest penny, we should secure a result correct to 3 decimal places; therefore use 5 places in the working. Begin each multiplication by 4 at the figures in thick type (as a figure in the 3rd place will, after division by 100, be placed in the 5th place), carrying on anything from the 4th figure.

∴ required amount = £2,172 4s. 8d.

The total interest is obtained by subtracting £1,856 16s. 8d. from £2,172 4s. 8d.; therefore interest = £315 8s., which result we should also get by adding up the four separate interests.

The same number of decimals in the working will give a result correct to the nearest farthing.

202. It will be observed in the above example that in the calculation of the amount for 1, 2, 3 years (or the principal for the 2nd, 3rd, 4th years respectively) there are three operations involved:—(1) Multiplying the principal by the rate per cent., (2) dividing by 100, (3) adding the result to the principal, It is sometimes convenient to reduce these three operations to one, thus:—

Multiplying any principal by 5 and then dividing by 100 is the same as multiplying by ·05. The third step of adding the result to the principal gives the same result as multiplying the original principal by 1·05,

e.g. (£25 × 5 ÷ 100) + £25 = £25 × ·05 + £25 = £25 × 1·05 = £26·25,

i.e. amount for 1 year = £25 × 1·05.

So amount for 2 years = £25 × 1·05 × 1·05 = £25 × $(1·05)^2$;

,, 3 ,, = £25 × $(1·05)^3$

= £25 × 1·157625

= £28·940625

= £28 18s. 9¾d.

This method is not usually shorter than the one already given It is, however, useful in working problems converse to the above (see § 203).

The general result may be put as follows:—

$$\text{Amount (not interest)} = \text{principal} \times \left(1 + \frac{\text{rate}\%}{100}\right)^n,$$

where n denotes the number of years (or periods of time).

The interest is then found by subtracting the original principal from the last amount.

The student who is acquainted with progressions will note that the successive amounts form a geometrical progression.

Ex. 3.—*What is the compound interest on* £783 15*s.* 6*d. for* 2 *years* 9 *months at* 4% *per annum?*

The third period of time is only three-fourths of a year, and the interest for this time is obtained by multiplying the principal for the third year by $\frac{3}{4} \times \frac{4}{100} = \frac{3}{100}$, *i.e.* we may continue the working for the third period, regarding it as a full year but at the reduced rate of 3%.

783·775	
31·351	Multiply by 4 and divide by 100.
815·126	
32·60504	Ditto.
847·73104	
25·43193	Multiply by 3 and divide by 100.
873·16297	
89·38797	Subtract the original principal.

∴ required interest = £89 7s. 9d.;

or thus— Amount = £783·775 × $(1{\cdot}04)^2$ × (1·03).

Ex. 4.—*Find the amount on* £1,856 16s. 8d. *for* 2 *years at* 8% *per annum, interest being due each half-year.*

As the interest for half a year is half the interest for a whole year, this example is equivalent to the following:—

Find the amount on £1,856 16s. 8d. for 4 periods of time (half-years) at 4% per period (half-year).

The working would then be exactly the same as in Ex. 2. So, if the interest were due each *quarter*, we should find the amount on £1,856 16s. 8d. for 8 periods of time (quarters) at 2% per period.

Sometimes special devices may save labour in calculation. For instance:—

Ex. 5.—*Find the amount on* £1,666 13s. 4d. *for* 3 *years at* 5%.

£1,666 13s. 4d. = £$\frac{5000}{3}$. Find the amount on £5,000, and divide the result by 3.

Or

$$\text{amount on £}\frac{5000}{3} = \text{£}\frac{5000}{3}\left(1 + \frac{5}{100}\right)^3 = \text{£}\frac{5000}{3} \times \frac{21}{20} \times \frac{21}{20} \times \frac{21}{20}$$

$$= \text{£}\frac{5 \times 21 \times 21 \times 7}{8} = \text{£1,929 7s. 6d.}$$

EXAMPLES CXXI.

(Answer to the nearest penny, except in those marked otherwise.)

Find the amount at compound interest on—

1. £2,150 for 3 years at 5% (exact).

2. £1,000 for 3 years at 5% (exact).

3. £25,000 for 6 years at 10% (exact).

4. £4,500 for 2 years at $4\frac{1}{4}$% (exact.)

5. £3,255 4s. 2d. for $1\frac{1}{2}$ years at 8% per annum, interest due half-yearly (exact).

6. £520 16s. 8d. for $1\frac{1}{2}$ years at 4% per annum, interest due half-yearly (exact).

7. £4,765 for 3 years at $3\frac{3}{4}$% per annum.

8. £3,600 for $3\frac{1}{2}$ years at $2\frac{1}{2}$% per annum.

9. £425 for 3 years at 4% per annum.

10. £100 for 1 year at 5% per annum, interest being payable quarterly.

11. £1,650 for 2 years at $2\frac{3}{4}$% per annum.

12.* £1 for 20 years at 5% per annum.

Find the compound interest on—

13. £1,425 for 3 years at $4\frac{1}{2}$% per annum (neglecting fraction of a penny in the answer).

14. £1250 for 3 years at 5% per annum (exact).

15. £5,432 11s. for 3 years at $4\frac{1}{2}$% per annum.

16. £8,457 for $2\frac{1}{4}$ years at $1\frac{1}{4}$% per annum.

17. £50,000 for 3 years at $2\frac{3}{4}$% per annum.

18. £1,800 for 3 years at 5% per annum (exact).

19. £1,000 for 1 year at 4% per annum, interest due each quarter.

20. £125 for 3 years at 5% per annum.

21. £1,250 for 4 years at 4% per annum.

22. £2,342 11s. 4d. for 4 years at $2\frac{1}{2}$% per annum.

23. £2,000 for $1\frac{1}{2}$ years at 2% per *half-year*.

24. £250 for 3 months at 1% per month.

Converse cases :—

203. *Case I.—Given the amount, time, and rate, to find the principal.*

As in simple interest, we assume some convenient principal (£1 is the simplest), and find the amount on it for the given time and rate, and form an example in proportion. This practically means using the formula

$$\text{Amount} = \text{principal} \times \left(1 + \frac{\text{rate } \%}{100}\right)^n.$$

Ex. 6.—*Find the principal (to the nearest penny) which in 4 years at 4% will amount to £2,172 4s. 8d. (Converse to Ex. 2.)*

The amount on a supposed principal of £1 is $£(1{\cdot}04)^4 = £1{\cdot}16985856$.

$$\therefore \quad £2172{\cdot}2\dot{3} = \text{required principal} \times (1{\cdot}04)^4,$$

or $$\text{required principal} = £2172{\cdot}2\dot{3} \div (1{\cdot}04)^4.$$

We can perform four separate divisions by 1·04, or one division by 1·16985856, employing contracted division in the latter case. The result is £1856·833, *i.e.* £1,856 16s. 8d. *Ans.*

If the interest (not the amount), time, and rate be given, and the principal be required, we first find the interest on £1, and form an example in proportion.

Case II.—Given the amount (or interest), the principal, and the rate, find the time.

Ex. 7.—*In how many years will £1,600 amount to exactly £1,723 0s. 6d. at $2\frac{1}{2}$% ?*

$$£1{,}723\tfrac{1}{40} = £1{,}600\left(1 + \frac{2\frac{1}{2}}{100}\right)^n = 1600\left(\frac{41}{40}\right)^n;$$

$$\therefore \quad \left(\frac{41}{40}\right)^n = \frac{1723\frac{1}{40}}{1600} = \frac{68921}{64000}.$$

Find by trial how many times $\frac{41}{40}$ must be multiplied by itself so as to produce $\frac{68921}{64000}$. Three times is suggested by the denominator, and, on trial, we find this to be the right number of times for the numerator; therefore answer = 3 years.

If it is known that the number of years (or periods of time) is integral, the above method would always give that number. The method in the next example could be used when the number of periods is either integral or fractional.

Ex. 8.—*In how many years will* £783 15s. 6d. *produce an interest of* £89 7s. 9d. *at* 4% *per annum?* (Compare Ex. 3.)

Amount = £783 15s. 6d. + £89 7s. 9d. = £873 3s. 3d. = £873·1625.

783·775
31·351
815 126
32·60504
847·73104

When we add in the interest for the third year (which is at least £32·6) we should get an amount larger than £873 3s. 3d. Hence the required number of years lies between 2 and 3. We now find the fractional part of a year for which the interest on £847·73104 is £25·43146 (*i.e.* £873·1625 − £847·73104).

This fraction of a year

$$= \frac{25{\cdot}43146 \times 100}{847{\cdot}73104 \times 4} = \frac{2543{\cdot}146}{3390{\cdot}92416} = \frac{2543}{3390} \text{ nearly,}$$

$$= {\cdot}75 \text{ or } \tfrac{3}{4} \text{ almost exactly;}$$

$\therefore$ Required time = $2\frac{3}{4}$ years.

Case III.—Given the amount, the principal, and the time, to find the rate per cent.

Such examples would generally involve the use of logarithms, but in easy cases the rate can be found by extracting a square root, a cube root, &c. (See Chap. xxxiii.)

Ex. 9.—*Find the rate per cent. at which* £8,000 *will amount to* £9,261 *exactly in* 3 *years.*

Using the formula

$$\text{amount} = \text{principal} \times \left(1 + \frac{\text{rate }\%}{100}\right)^n,$$

we have

$$9261 = 8000\left(1 \times \frac{\text{rate }\%}{100}\right)^3;$$

$$\therefore \quad \left(1 + \frac{\text{rate }\%}{100}\right) = \sqrt[3]{\frac{9261}{8000}} = \frac{21}{20} = 1 + \frac{1}{20};$$

$$\therefore \quad \frac{\text{rate }\%}{100} = \frac{1}{20} \text{ or rate } \% = 5.$$

N.B.—From this example we have—

Amount on £1 for 3 years at 5% = $(£\frac{21}{20})^3$;

and amount on £1 for 6 years at 5% = $£(\frac{21}{20})^6$,

which is found by *squaring* $(\frac{21}{20})^3$, and not by doubling it.

So the amount for 9 years at 5% is found by cubing $(\frac{21}{20})^3$, and not by trebling it.

EXAMPLES CXXII.

1. Of what principal is £154 7s. the amount for 3 years at 5% compound interest? (Exact answer.)

2. What sum invested for 3 years at 5% per cent. per annum will amount to £424 9s. 3d.? (Exact answer.)

3. Find the principal to the nearest penny the compound interest on which for 2 years at 4% is £51 0s. 7d.

4. When the interest is due quarterly, what is the principal which will amount to £78 2s. 6d. in 1 year at 4% per annum? (Answer to nearest penny.)

5. In how many years will the compound interest at 5% per annum on £1,250 be £197 0s. 7½d.?

6. The compound interest on £8,600 for between 2 and 3 years at 3½% is £773 15s. 1d. Find, as nearly as possible, the exact time.

7. £1,000 is lent for a period of time between 3 and 4 years at the rate of 4% per annum for the 3 complete years, whilst the rate for the remaining portion of a year is 5% per annum. If the final amount be £1,167 0s. 11d., find the exact time to the nearest month.

8. In how many years exactly will the compound interest at 4% per annum on £390,625 be £66,351?

9. If £50 amounts exactly to £76 0s. 10½d. in 3 years at compound interest, find the rate per cent.

10. In 4 years' time £97,656 5s. will amount to £114,244. Find the rate per cent.

11. Determine at what rate per cent. £3,750 will amount to £4,341 1s. 10½d. (exactly) in 3 years at compound interest, it being given that the ratio of the amount to the principal is the cube of the ratio of the amount of £1 at the same rate in 1 year to £1.

12. Find (correct to the nearest penny) the compound interest on £327 12s. 9d. for 1¼ years at 4%, the interest being due quarterly.

13. Find the difference between the simple and compound interest on £500 for 3 years at 5%.

14. A company obtain a loan from their bankers of £2,150 at 5%; they do the same a year afterwards, and again a year after that. What, exactly, will their whole debt be at the end of the third year?

15. Find the principal which will amount at compound interest in 2 years to £739 7s. 6d. and in 3 years to £768 19s.

16. A sum of money is borrowed and paid back in two equal annual instalments of £882, allowing 5% compound interest. What was the sum borrowed?

17. Find the difference between the simple and compound interest on £2,343 15s. for 2 years at 4%.

18. A boy puts out £30 on his sixteenth birthday to accumulate at $2\frac{1}{2}\%$ compound interest (interest being added yearly). What will be the amount on his twentieth birthday? (Correct to the nearest penny.)

19. The compound interest on a sum of money invested for 3 years at 4% is £390 4s. What would be the compound interest on the same sum for the same time at 5%? (Exact answer.)

20. A man lends £114 11s. 8d. for a year at 3%. How much should he receive at the end of the year? If he should lend the whole of the sum received at the end of this year for the same period at the same rate per cent., to how much would he be entitled at the close of this second year?

21. The rent of a house is £200 per annum, due by half-yearly payments on January 1st and July 1st, and no payment has been made since the beginning of last year. What sum will be due at the beginning of next year, interest being added to arrears, at the beginning of each half year, at the rate of 5% per annum? (Answer to the nearest penny.)

22. A sum of money put out at compound interest amounts in 2 years to £1,760 8s. 4d. and in 3 years to £1,830 16s. 8d. Find the rate of interest and the original sum.

23. Find the compound interest on £2,960 for 2 years at $3\frac{3}{4}\%$ per annum. How much greater is it than the simple interest for the same time and rate?

24. On what sum of money would the compound interest for 3 years at 5% per annum exceed the simple interest by £38 2s. 6d.?

25. Find, to the nearest penny, the difference between the simple and compound interest on £7,545 10s. for 2 years at 4% per annum, interest being payable half-yearly.

26. The difference between the compound and simple interest on a sum put out for 2 years at $4\frac{1}{2}\%$ was £1 13s. 9d. Find the sum.

27. The sum of £405 11s. 4d. is lent upon condition that the borrower shall owe to the lender, as interest, one forty-sixth part of the sum borrowed for each half-year that he retains the money; and, if the money be not returned at the end of the first or any subsequent half-year, the interest due, instead of being paid, is to be added to the amount lent. How much is due from the borrower to the lender at the end of a year and a half?

28. A sum of £5,000 is lent on the condition that, if it is not repaid within a year, compound interest will be charged. The option is given of 5% interest throughout, or $2\frac{1}{2}\%$ for the first year, 5% for the second year, $7\frac{1}{2}\%$ for the third year, and so on, the interest increasing by $2\frac{1}{2}\%$ every year. Show that, if the loan is repaid in 3 years, the latter system will be in favour of the borrower, and find the exact difference on the two systems if it be not paid off till the end of the fourth year.

29. If £100 be put out at compound interest at 6% per annum for 4 years, find to the nearest whole number of pounds the sum to which it will amount when the interest is payable (1) yearly, (2) monthly, being given that $(1{\cdot}005)^{24} = 1{\cdot}1272$.

30. A's capital exceeds B's by $\frac{41}{400}$ of B's capital. B puts his out at compound interest for 3 years at 5%. At what rate per cent. simple interest must A put his out in order that, at the end of the 3 years, the two amounts may be the same?

31. A sum of £1,640 is borrowed, to be paid back in 2 years by two equal annual payments, allowing 5% compound interest. Find the annual payment.

32. A certain sum is put out to compound interest for 4 years at 5% per annum. The interest for the third year exceeds that of the second year by £26 5s. Find the total interest for the 4 years, and also the original sum.

33. The amount on £200 for 3 years at 5% is £231 10s. 6d. exactly. Without finding the separate interests for the fourth, fifth, and sixth years, calculate to the nearest penny the amount for 6 years at the same rate.

34. What will be the amount of £1 accumulating at compound interest for 3 years, the rates for the successive years being 2%, 3%, 4%? Deduce at once the amount for 6 years, the rates for the fourth, fifth, and sixth years being the same respectively as for the first, second, and third years.

35. Find the amount (to the nearest pound) on £1,000 for 9 years at 3% per annum, given $(1{\cdot}03)^3 = 1{\cdot}0927$.

36. If the population of a town containing at present 60,000 inhabitants increases annually at the rate of 3%, what will be the total increase at the end of 3 years? (Answer to the nearest integer.)

37. Two districts of equal population are found from annual statistics the one to increase and the other to decrease annually at the rate of 24 per 1,000. Compare the populations of the two districts at the end of 3 years. (Answer in the form of $a : 1$, where a is calculated to two decimal places.)

38. A £5 Bank of England note, after being in circulation for 10 years, is presented at the Bank for cash payment. If compound interest be reckoned at $2\frac{1}{2}$% per annum, what profit has the Bank made? (Answer to nearest penny.)

39. An annual payment of £100 due at the *end* of each year remains unpaid for 3 years. If compound interest be reckoned at 5% per annum, what sum will be due at the end of the third year? If the payment is due at the *beginning* of each year, what sum will then be due at the end of the third year?

CHAPTER XXVI.

DISCOUNT AND PRESENT WORTH.

204. *Discount* is the general term applied to the reduction made in a "Bill" in consideration of an immediate or cash payment.

*Cash Discount.**—A retail tradesman will sometimes allow a customer who pays cash a certain reduction of, say, 5°/₀ from his bill or account. Such reduction is known as Cash Discount, and is calculated just like simple interest, except that the element of time does not affect it.

Thus, suppose a customer buys goods to the value of £25, and is allowed 5°/₀ off for cash. The discount would be $\frac{5}{100}$ of £25 = £1 5s.; he would therefore pay in cash £23 15s.

True Discount.—Again, suppose that A promises to pay B £204 at the end of 8 months, but that after 2 months have passed A proposes to B a settlement of the debt at once by a cash payment. What would be a fair arrangement, interest being at the rate of 4°/₀ per annum? A proposes to pay his debt 6 months before it is really due; he would therefore be entitled to some reduction, and should pay to B such a sum of money as would, if put out to interest by B for these 6 months at 4°/₀ per annum, amount to £204.

Now £100 will amount to £102 in 6 months at 4°/₀.

∴ £102 due in 6 months will be paid for by £100 cash.

∴ £204 due in 6 months will be paid for by £200 cash.

∴ A reduction of £4 is made in consideration of cash payment.

This £4 is called the True (mathematical or theoretical) Discount on the £204; and the £200 is called the True Present Worth (or Value), abbreviated P.W.

From this example it will be seen that the problem of finding the true present value (or true discount) on a given sum of money reduces to that of finding the principal (or interest)

* A discount allowed by a wholesale firm to the retail dealer, often as high as 30-50%, is known as *Trade Discount.*

when we are given the amount, the time, and the rate per cent., and this has been already considered in the chapter on Interest (§ 198, v.). Hence a large number of the problems in this subject are simply repetitions under different names of those already dealt with. The sum due corresponds to what has previously been called the amount; the present worth (or value) corresponds to the principal, and the discount to the interest on that principal.

Remember carefully—

(1) The sum due = true present value + true discount.

(2) The true discount = the interest on the true present value,

both of which statements are true, whether the interest be simple or compound.

205. Ex. 1.—*Find the true present value (to the nearest penny) of* £450 *due in* 3 *months, interest being at the rate of* 3% *per annum.*

This is the same as the following:—Find that principal (= true present value) which in 3 months at 3% will amount to £450 (= the sum due).

An assumed principal of £100 will in 3 months at 3% amount to £$100\frac{3}{4}$, *i.e.* £$100\frac{3}{4}$ due in 3 months can be paid at once by a cash payment of £100.

∴ of £$100\frac{3}{4}$ (the supposed sum due) the true present value is £100;

∴ of £450 (the actual sum due) the true present value is

$$£450 \times 100 \div 100\tfrac{3}{4};$$

∴ required true present value = £450 × 100 × 4 ÷ 403 = £446 13s. (to the nearest penny).

The true discount = sum due − true present value = £3 7s. (to the nearest penny). If the discount is asked for, it is not necessary to find the P.W. first; for, if £$100\frac{3}{4}$ due in 3 months is paid at once by £100, there is a reduction of £$\frac{3}{4}$.

∴ on £$100\frac{3}{4}$† (the supposed sum due) the discount is £$\frac{3}{4}$;

∴ on £450 (the actual sum due) the discount is £$450 \times \frac{3}{4} \div 100\frac{3}{4}$

$$= £\tfrac{450}{1} \times \tfrac{3}{4} \times \tfrac{4}{403} = £3\ 7s. \text{ (to the nearest penny).}$$

These results mean that a debt of £450 due in 3 months' time can be paid at once by £446 13s., a reduction of £3 7s. being made. Further, this £3 7s. is the interest (to the nearest penny) on the £446 13s. for 3 months at 3%.

† A very common mistake is made by putting £100 instead of £$100\frac{3}{4}$.

Ex. 2.—*If* £15 15s. *is the true discount on* £365 15s., *find when the latter sum is due, interest being at the rate of* 6% *per annum.*

This is the same as the following:—Given the amount on a certain principal to be £365 15s., the interest on the same principal to be £15 15s., and the rate 6%, find the time. (See § 198, case iv.)

True P.W. = sum due − true discount = £350.

But true discount = interest on true P.W.;

$\therefore$ £$15\frac{3}{4}$ = £350 × time × 6 ÷ 100;

$\therefore$ Time = $\frac{3}{4}$ year = 9 months. *Ans.*

A similar method is adopted when the time is given and the rate per cent. is required.

Ex. 3.—*What must a man invest now so that he may receive* £9,261 *at the end of* 3 *years, compound interest being* 5% *per annum?*

Here we have to find the true P.W. of £9,261 due in 3 years' time, or the principal which in 3 years will amount to £9,261.

By § 202 we have amount = principal × $\left(1+\frac{\text{rate}}{100}\right)^n$;

$\therefore$ required principal or present value

= £9,261 ÷ $(1{\cdot}05)^3$

= £8,000. (Compare Ex. 9, § 203.)

Ex. 4.—*A man is entitled to receive* £200 *annually for* 3 *years, the payments being due at the end of* 1, 2, *and* 3 *years from the present. What immediate payment would be a fair equivalent, compound interest being* 4% *per annum? (Answer to the nearest pound.)*

Required P.W. = sum of the present values of £200 due at the end of 1, 2, 3 years respectively

= £200 ÷ $(1{\cdot}04)$ + £200 ÷ $(1{\cdot}04)^2$ + £200 ÷ $(1{\cdot}04)^3$

= £200 × (·9615 + ·9246 + ·8890)

= £200 × 2·7751

= £555 to the nearest pound.

EXAMPLES CXXIII.

1. What sum of money will in $4\frac{1}{2}$ years at $4\frac{1}{2}$% per annum, simple interest, amount to £481?

2. If £50 10s. is the amount on a certain sum for 3 months at 4% per annum, what is the interest on the same sum for the same time and rate?

At simple interest find both the true present value and the true discount in Exx. 3 to 8—

3. On £381 10s. due at the end of 3 years at 4 %.

4. On £328 13s. 5d. due 3 months hence at 4 %.

5. On £135 due in 3 months time at 5 %.

6. On £962 due in $4\frac{1}{2}$ years at $4\frac{1}{2}$ %.

7. On £54 3s. 4d. due at the end of $2\frac{1}{2}$ years at $3\frac{1}{3}$ %.

8. On £155 12s. 6d. due in 18 months at $2\frac{1}{2}$ %.

9. Find the present worth of £6,433 3s. 4d. due 3 months hence, the rate of interest being $3\frac{1}{3}$ % per annum.

10. What is the true discount on £51 18s. due $2\frac{1}{2}$ years hence at $3\frac{1}{4}$ % per annum?

11. What is the cash payment equivalent to £1,919 5s. $11\frac{1}{4}$d. due in 2 years 7 months, simple interest being reckoned at 3 % per annum?

12. If a debt of £76 15s. due in 8 months time be discharged by a cash payment, what reduction should be made, interest being at $3\frac{1}{2}$ % per annum?

13. Find the true present value on £84 10s. due in 2 years, compound interest being at the rate of 4 % per annum.

14. Find the discount on £86 16s. $5\frac{1}{4}$d. due in 3 years, compound interest being reckoned at 5 % per annum.

15. Find the present worth of £1,616 2s. 8d. due 3 years 9 months hence, simple interest being 4 % per annum.

16. The true discount on a certain sum due $2\frac{1}{2}$ years hence at 3 % simple interest is £17 14s. 6d. What is the sum?

17. If the true discount on £2,696 13s. 4d. due 3 months hence is £30, what is the rate per cent. simple interest?

18. What is the true discount on £451 10s. due 2 years hence, reckoning simple interest at $3\frac{3}{4}$ %?

19. Find the true discount on a bill of £483 11s. due 146 days hence, at $4\frac{1}{2}$ % interest; and show that this discount is equal to the simple interest on the present worth of the bill, at the same rate and for the same time.

20. If interest is at the rate of 4 % per annum, what reduction is made when a bill of £412 is paid 9 months before it is due?

21. Allowing interest at $3\frac{1}{8}$ % per annum, what sum of money will now discharge a debt of £903 11s. 3d. which becomes due 11 months hence?

22. What is the present value of £14,830 7s. 3d. due in $3\frac{1}{2}$ years, simple interest being reckoned at $4\frac{1}{4}$ % per annum?

23. If £766 13s. 4d. is the true discount on £4,600 due in $2\frac{1}{2}$ years, what is the rate per cent. at simple interest?

24. The present worth of a certain sum due in 3 years at 5% per annum, compound interest, is £966 13s. 4d. What is the discount?

25. What discount at 5% per annum, simple interest, must be allowed for the present payment of £2,200 due 2 years hence?

26. Find the amount of a bill due in 18 months, if the true discount on it, reckoning simple interest at 8% per annum, is £37 10s.

27. What is the present value of £151 10s. due in 80 days, the rate of interest being $4\frac{9}{16}$% per year of 365 days?

28. If a debt of £1,013 2s. 6d. payable after a certain interval of time can be discharged by a cash payment of £1,000, what is that interval, simple interest being 3% per annum?

29. Find the present value of £236 5s. $1\frac{1}{2}$d. due 7 months hence when the rate of interest is $3\frac{1}{3}$% per annum.

30. What is the present value of £276 10s. 5d. due 219 days hence, interest being $3\frac{1}{2}$% per annum?

31. If the present value of a bill of £3,820 6s. 3d. due 5 months hence is £3,750, what is the present value of a bill of £3,876 11s. 3d. due 9 months hence at the same rate of interest?

32. A person holding a bill for £6,433 3s. 4d. disposes of it through an agent at the reduced rate of £100 for £$106\frac{1}{3}$ of the nominal value, and the agent charges a commission of $\frac{1}{6}$% on the purchase money, what does the person receive?

33. A bicycle manufacturer prints in his lists a price which would give him a profit of 60% on the cost of manufacture, and then allows for "cash down" a discount of 22% off the list price. What profit per cent. does he actually make?

34. A dealer bought a horse for £110, and sold it the same day for £121 15s., allowing the buyer 5 months' credit. Money being worth $3\frac{1}{2}$% per annum, what was his gain per cent.?

35. What is the least sum (to the nearest sovereign) which a man must invest at the beginning of a year so that at the end of each of the next 4 half-years he may receive £50, compound interest being 4% per annum?

36. Find to the nearest sovereign how much a man must invest now so that in 12 years time his child may receive £1,000, compound interest being 3% per annum.

37. A debt of £2,030, with compound interest at 3%, is paid off by two equal annual payments. Find the amount of each payment.

206. Banker's Discount.—In commerce, when a man buys goods (say from a wholesale dealer) he frequently does not pay "cash down" for them, but gives what is called a *Bill of*

Exchange, which is practically an agreement to pay for the goods after a certain time. (A promissory note, which differs in form, is sometimes used for the same purpose.) The following is a form of a bill of exchange, which would be drawn on "stamped paper"† :—

BILL OF EXCHANGE. £1,010. *Due June 7th, 1900.*		*London,* *January 4th, 1900.*
Five months after *order the sum of One* *Pounds for value*	*Accepted, payable at the City Bank, London.* THOMAS SMITH.	*date pay to myself or* *Thousand and Ten* *received.*
To Mr. THOS. SMITH. *Bristol.*		HENRY FISON.

Here Thos. Smith has bought from H. Fison goods to the value of £1,010 (the "*face value*" of the bill), but is not prepared to pay for them at once. He thinks that he can do so in, say, five months' time. H. Fison therefore draws up a bill of exchange, as above, and forwards it to T. Smith for his "acceptance." If the latter agrees to it, he signs his name across the bill with the information that the money can be obtained from the City Bank on the date agreed upon. T. Smith is said to "accept" the bill, which he returns to H. Fison. It is said to be drawn "at five months."

The date on which the bill is "drawn" is January 4th. Five months (always calendar months) from that date would be June 4th. But H. Fison cannot legally claim the money till three days after this, called "Three Days of Grace." This is a survival from times when communication between places was not so easy as at present. The date on which the bill

† *I.e.* paper bearing an Inland Revenue stamp. For the bill stated the stamp would be 11s.

"matures" (*i.e.* when the money becomes legally due) is June 7th. H. Fison cannot therefore claim the money on the bill till June 7th, when he will present it to the City Bank, and receive from the bank £1,010 of T. Smith's money. The transaction is then closed.

But another course (and one very commonly taken) is open to H. Fison. Any time between January 4th and June 7th he may take the Bill to a banker (or bill-broker) and ask him to *discount* it. If the banker is willing, he keeps the bill, and, in exchange for it, pays over to H. Fison a certain amount of cash. On the date when the bill "matures" (June 7th) the banker will present the bill to the City Bank, and receive £1,010 of T. Smith's money. The transaction is then closed.

There arises now the question as to how much the banker (or broker) who "discounts" the bill pays over to H. Fison. If the banker pays him the *true* present value of the £1,010, he would make no profit for himself,† and a part of the regular business of a banker consists in thus discounting bills. It is the custom, therefore, for the banker to pay over in cash the face value of the bill *less the simple interest* on it for the number of days the bill has yet "to run," *i.e.* the number of days from the date on which the banker discounts it up to June 7th.

Banker's discount is therefore the same as the simple interest on the bill for the unexpired time.

It is sometimes known as *Practical or Commercial Discount.*

207. The detailed working would then be as in the following example:—

Ex. 5.—*H. Fison draws a bill whose face value is £1,010 on January 4th at 5 months, and offers it to a banker to be discounted on March 26th. If the banker's rate of discount is 5% per annum, what will he pay for the bill?*

The bill is drawn on January 4th and is therefore legally due on June 7th (including 3 days of grace). It is discounted by the banker on March 26th; therefore the bill has yet "to run" from March 26th to June 7th, *i.e.* 5 (March) + 30 (April) + 31 (May) + 7 (June) (one only of the two extreme dates is included, § 196) = 73 days.

The banker's discount = the simple interest on £1,010 for 73 days at 5% = £10 2s.

† Except that of securing an investment for his money at a rate fixed by an association of bankers, which rate is always considerably in advance of that yielded by a perfectly safe investment.

Therefore the banker discounts (or buys) the bill for £1,010 less £10 2s., *i.e.* £999 18s. He keeps the bill, and on June 7th presents it at the bank named on it and receives £1,010.

The *true* present value of £1,010 for 73 days at 5 % can be shown to be £1,000, and the *true* discount to be £10; therefore, if the banker pays for the bill only £999 18s. instead of £1,000, he makes a profit of 2s.

This is expressed by saying that banker's discount is greater than true discount by 2s.

Ex. 6.—*Find the banker's discount on a bill for* £734 15s. *drawn Jan.* 17*th*, 1896, *at* 3 *months, and discounted on Feb.* 5*th, at* $2\frac{1}{2}$ % *per annum.*

Bill is legally due April 20th,† and is discounted February 5th

No. of days from February 5th to April 20th is

$$(29-5=)\ 24+31+20=75,$$

i.e. $\frac{75}{365}$ (not $\frac{75}{366}$; see § 196) of a year.

$$\therefore \text{ required banker's discount} = £734{\cdot}75 \times \frac{75}{365} \times \frac{5}{2 \times 100}$$

$$= £3{\cdot}774 = £3\ 15\text{s. } 6\text{d. (to the nearest penny).}$$

In working examples the 3 days of grace should be added on when the form of the question so admits it, *i.e.* when the actual dates are given; but otherwise they are supposed to be already included. (See Ex. 7 below.)

Note also that one month from January 31st (the last day) will lead to Feb. 28th (the last day); so two months from July 31st will lead to September 30th, the word "month" always meaning "calendar month."

Ex. 7.—*What is the face value* (see § 206) *of a* 3 *months' bill when the banker's discount at* 2 % *per annum is* £4 6s. 8d. *?*

(Here the 3 days of grace are supposed included already in the 3 months.)

We have to find the principal the interest on which for 3 months at 2 % is £4 6s. 8d.

Principal $\times \frac{1}{4} \times \frac{2}{100} = £4\frac{1}{3}$; $\therefore$ principal $= £4\frac{1}{3} \times 200 =$ £866 13s. 4d.

Therefore face value of the bill is £866 13s. 4d.

Ex. 8.—*A banker discounts a* 2 *months' bill of* £1,000 *at* 3 % *discount. What rate of interest per annum is he charging ?*

Banker's discount is readily found to be £5.

$\therefore$ he advances £995 in payment of the bill, and in 2 months will receive £1,000.

$\therefore$ £5 is the interest on £995 for 2 months at a certain rate per cent.

$$\therefore \quad £5 = £\frac{995 \times 1 \times \text{rate}}{6 \times 100}.$$

$$\therefore \quad \text{required rate} = \frac{600}{199} = 3\frac{3}{199}\ \%. \quad \textit{Ans.}$$

† Be careful not to say *March* 20th.

EXAMPLES CXXIV.

(Answers, when not exact, to the nearest penny.)

1. Find the banker's discount on a bill of £855 paid 27 days before legally due at 3 % per annum.

2. How much will a broker give for a bill of £1,000 which he discounts at $2\frac{1}{2}$ % 86 days before it is legally due?

3. Find the banker's discount on a bill for £2,240 drawn June 8th at 3 months and discounted June 30th at 4 %.

4. A bill for £6,132 10s. drawn on March 14th and payable 9 months after date was discounted by a banker on May 12th at $3\frac{1}{4}$ %. How much did the banker give for the bill?

5. How much would a banker give for a bill for £4,100 due 6 months hence and discounted at 5 % per annum?

6. Find the amount of a bill due in 18 months if the banker's discount on it at 8 % per annum is £37 10s.

7. A bill of £649 is dated June 23rd at 6 months, and is discounted on July 8th at $3\frac{1}{4}$ %. What does the banker charge, and what is the *true* discount?

8. A bill of £1,865 is discounted at 5 % 3 months before due. What will the owner of the bill receive?

9. Find the true discount, allowing 3 days of grace, on a bill for £603 drawn October 4th at 4 months and discounted November 26th at $2\frac{1}{2}$ %.

10. Calculate the exact true present worth of a bill for £554 17s. $11\frac{3}{4}$d. drawn April 15th at 6 months, and paid August 6th at $3\frac{1}{2}$ %, the usual days of grace being allowed.

11. What is the commercial discount on a bill whose face value is £700, and which is discounted 93 days before it is legally due at $3\frac{3}{4}$ %?

12. What is the exact difference between the true and commercial discount on a bill of £2,515 drawn October 8th at 4 months and discounted November 30th at 3 %?

13. What is the difference in value (true) of a bill for £1,212 due in 70 days and one for £1,228 due in 65 days, the rate of discount being 5 %, and the usual 3 days of grace being allowed?

14. At what rate per cent. would the banker's discount on £2,696 13s. 4d. due 3 months hence be £30 6s. 9d.?

15. What is the true discount on £1,025 due 2 years 8 months hence, simple interest being $3\frac{1}{2}$ % per annum?

16. Find the true discount on a bill for £97 17s. 8d. fully (or legally) due on November 7th and discounted on September 13th at $3\frac{3}{4}$ % per annum.

17. The true discount upon £220 due at the end of a certain number of years is £44. Find the commercial discount upon the same sum of £220 for the same number of years and at the same rate. Determine also the number of years and the rate per cent., given that these are indicated by whole numbers greater than unity.

18. A person holding a bill due 30 days hence discounts it *without* the usual 3 days of grace at 4 % per annum (true discount), and invests the proceeds at 5 %, thereby gaining 11s. 3d. in the thirty days. Find the amount for which the bill was drawn.

19. Find the difference between the interest on £266 13s. 4d. for 3 months at $4\frac{1}{2}$ % and the true discount on £83 due 15 months hence at 3 %.

20. What rate of interest per annum does a man get for his money when, in discounting a bill due in 10 months, he deducts as discount 4 % of the total amount of the bill?

21. The debts of a bankrupt amount to £1,067 5s. 3d. and his assets consist of property worth £458 7s. 8d. and an undiscounted bill of £256 10s. due 4 months hence, simple interest being reckoned at 4 %. How much in the £ will he pay? (Reckon true present value of the bill.)

22. A discount house takes money on deposit and uses the money in discounting bills. It pays 4 per cent. per annum interest on the deposits and makes $4\frac{1}{4}$ % upon them. If the deposits average £10,000,000, what annual profit is made?

23. A bill for £264 6s. 9d. is discounted, and the true discount charged amounts to £1 16s. 9d. If the rate of interest be $3\frac{1}{2}$%, find how many days the bill has still to run.

24. The debts of a bankrupt amount to £2,134 10s. 6d. His assets consist of property worth £916 15s. 4d. and of a bill for £513 due 4 months hence, simple interest being at 4 % per annum. How much in the £ can he pay his creditors? (Reckon true present value of the bill.)

25. What is the difference between banker's and true discount on a bill of £816 due in 6 months at 4 %?

26. Goods are bought and paid for by a 3 months' bill of £120; they are sold again the same day for a 6 months' bill of £150; if commercial discount be 3 % per annum, find the present gain or loss.

27. A banker discounts a 3 months bill of £1,000 at 4 % per annum. What does he pay for the bill? At what rate per cent. of interest per annum is he advancing money?

28. A tradesman marks some goods for £2 10s., allowing 12 months credit. If his rate of discount be 3 %, what ready money would he take for the same goods?

29. A bill of £850 drawn on May 27th for 4 months was discounted on July 19th for £843 4s. What was the rate of banker's discount?

30. A bill of £1,460 was drawn on September 11th for 3 months and discounted at 4 % for £1,454 8s. On what date was it discounted?

208. The following miscellaneous examples are a little more difficult. Their solution depends mainly on the following statements, which have been already mentioned, but are again collected here:—

(1) True discount on a bill = interest on the true present value of the bill.

(2) Banker's discount on a bill = interest on the bill.

From these it follows that

banker's discount − true discount = interest on (bill − true P.W.)
= interest on true discount;

or

banker's discount = true discount + interest on true discount
= amount on true discount.

209. Ex. 9.—*If £12 is the true discount on £412 for 9 months, for how long is £12 the true discount on £612 at the same rate?*

£12 is the true discount on £412. ∴ £400 is the true P.W. of £412.

∴ £12 is the interest on £400 for 9 months at a certain rate.

∴ $£\frac{400 \times 9 \times \text{rate}}{12 \times 100} = £12.$ ∴ The rate is 4 %.

Again, £12 is the true discount on £612.

∴ £12 is the interest on £600 for the unknown time at 4 %.

∴ $£\frac{600 \times \text{no. of years} \times 4}{100} = £12.$

∴ Required time = ½ year = 6 months. *Ans.*

Ex. 10.—*If £7 19s. is the true discount on £272 19s. for a certain time, what is the true discount on the same sum for double that time, the rate being the same in both?*

£7 19s. is the true discount on £272 19s.

∴ £7 19s. is the interest on £265.

∴ $£\frac{265 \times \text{no. of years} \times \text{rate}}{100} = £7{\cdot}95.$

∴ no. of years × rate = 3. ∴ twice no. of years × rate = 6.

∴ on £106 the true discount is £6.

∴ on £272 19s. the true discount is $£\frac{272{\cdot}95 \times 6}{106} = £15\ 9s.$ *Ans.*

Ex. 11.—*The true discount on a sum of money for 4 months is £10, and the commercial discount for the same time at the same rate is £10 4s. What is the sum and what is the rate per cent.?*

Banker's discount − true discount = interest on true discount.

∴ 4s. = interest on £10 for 4 months.

∴ $£\frac{1}{5} = £\frac{10 \times 4 \times \text{rate}}{12 \times 100}$. ∴ rate = 6 %.

Again, banker's discount = interest on the sum of money (or bill).

∴ $£10\frac{1}{5} = £\frac{\text{sum of money} \times 4 \times 6}{12 \times 100}$.

∴ required sum = £510. And the rate is 6 %. *Ans.*

Ex. 12.—*The banker's discount on £604 at 5 % is the same as the true discount on £611 11s. for the same time and at the same rate. Find the time.*

Interest on £604 = true discount on £611 11s.

= interest on (true P.W. of £611 11s.).

∴ £604 = true P.W. of £611 11s.

∴ £7 11s. = true discount on £611 11s.

∴ £7 11s. = interest on £604 at 5 % for a certain time.

∴ $£\frac{151}{20} = £\frac{604 \times \text{no. of years} \times 5}{100}$.

∴ required time = $\frac{1}{4}$ of a year = 3 months. *Ans.*

EXAMPLES CXXV.

(Miscellaneous Examples on Discount, &c.)

1. If £6 be the true discount on £156 for a certain time at a certain rate, find the true discount on £260 for the same time at the same rate.

2. If £12 10s. is the true discount on £137 10s. for a certain time, what is the true discount on the same sum for double the time, the rate being the same in both cases?

3. If the true discount upon a sum of money (due after a certain time) at 5 % simple interest is five-sixths of the interest for the same time at the same rate, find the time.

4. If the compound interest on a certain sum of money for 2 years at 4 % is £45 18s., what is the true discount on the same sum of money due 2 years hence at 4 % simple interest?

5. The true discount on a sum of money for 3 years and the simple interest on the same sum for $2\frac{1}{2}$ years at the same rate per cent. are each £80. What is the sum?

6. If £4 is the true discount on £34 for 6 months, for how long is £4 the true discount on £64 at the same rate per cent.?

7. Find the true discount (to the nearest penny) on £574 3s. 4d. due in 2 years 3 months at $2\frac{3}{8}$ % simple interest (not reckoning 3 days of grace).

8. Find the exact present value of £1,137 2s. $0\frac{1}{2}$d. due 2 years hence, reckoning compound interest at $3\frac{4}{7}$ % per annum.

9. Calculate (to the nearest sovereign) the present value of £100 due 100 years hence, compound interest being reckoned at $2\frac{1}{2}$ %. (It may be assumed that in 20 years at $2\frac{1}{2}$ % £100 will be worth £163 17s. 3d.)

10. The true discount on a sum of money for 3 months is £15, and the commercial discount for the same time and rate is £15 2s. 3d. Find the sum of money and the rate per cent.

11. The difference between the simple interest and true discount on a sum of money due in $2\frac{1}{2}$ years is £3. If interest be at the rate of 4 % per annum, find the sum of money.

12. The difference between banker's and true discount on a sum of money for 3 months at 5 % is 8s. 4d. Find the sum.

13. The interest on a certain sum for 10 years is greater than the true discount on the same sum due in 10 years by one-quarter of the discount. Find the rate per cent. per annum (simple interest), which is the same in both cases.

14. An annuity of £3,675 for 5 years is agreed to be discharged by a single payment at the end of the third year. Interest at 5 % per cent. per annum is to be paid on the first two instalments overdue, and discount at the same rate is to be deducted from the instalments paid in advance. The amount of the first instalment and the present value of the last are to be reckoned by compound interest. What will the whole payment amount to?

15. The true discount on a certain sum due in 3 months at 4 % is £17 5s. Find the banker's discount (exactly) and the sum.

16. The banker's discount on £450 at 4 % is the same as the true discount on £459 for the same time at the same rate. When is that time?

17. The true discount on a sum of money for a certain time at a certain rate is £6 5s., and the interest on this true discount for the same time and rate is $\frac{1}{50}$th of this amount. Find what the sum of money is.

18. A banker discounts a 4 months bill of £1,250 at such a rate of discount as produces for him interest at the rate of $5\frac{5}{59}$ % per annum on the price at which he buys the bill. What is this price?

19. Find (to the nearest penny) the present value of £324 16s. 9d. due in 4 years at $3\frac{1}{2}$ % compound interest.

20. If the simple interest on the true discount of a sum of money be $\frac{1}{100}$th part of the true discount, find the ratio between the sum of money and its simple interest, the time and rate being the same throughout.

21. Find the present value, to the nearest penny, of £11,000 due in 2 years 8 months, compound interest being at the rate of 3 % per annum.

22. If the simple interest on £13,338 6s. 8d. at 5% be the same as the true discount on £13,638 8s. 11d. for the same time and at the same rate, find when the latter sum is due.

23. The banker's discount on a certain sum is £9 0s. 3d., and the simple interest on the true present value of the same sum is £8 15s. for the same time at the same rate. What is the sum?

24. If the compound interest on £5 at 5% be equal to the true discount on £5 10s. 3d. for the same time at the same rate of compound interest, find when the latter sum is due.

25. The simple interest on £650 for 3 months is equal to the true discount on £495 12s. 6d. due in 4 months time, the rate being the same in both cases. Find the rate, and also these two equal sums of money.

26. What sum of money ought to be paid now in order to receive £365 at the end of each year for the next three years, allowing compound interest at $4\frac{1}{2}$%. (Answer to nearest penny.)

CHAPTER XXVII.

SHARES AND STOCKS.

210. When a man engages in business on his own account he employs a certain sum, called *Capital*, with which he trades. and the excess of his receipts over his expenditure is called his *Profit*. If two men form a partnership, bringing, say, £500 and £800 respectively to the business, then the profits are divided between them in proportion to the capital they each invested or supplied. It very commonly happens, however, that a business or an undertaking (*e.g.* the working of a mine or the construction of a railway) is on too large a scale for one or two men to supply all the money required. A *Company* is then formed by the combination of a number of persons, who, amongst themselves, provide or "subscribe" the necessary capital. This capital is divided into a number of equal portions, called *Shares*, and the individual members of the company are called *Shareholders*, who elect some of their number, called *Directors*, to carry on the business. The profits of the company are *divided* at intervals (generally each half-year) amongst the shareholders in proportion to the number of shares each possesses, and each person's share of the profits is called his *Dividend*.

Thus, suppose a company formed with a capital of £60,000, divided into 6,000 shares of £10 each, and that the profits allow of a dividend being declared at the rate of 6% per annum. These shares are bought or subscribed for in varying numbers by different persons, who are said to *invest* their money in the undertaking, and who then become entitled to receive their proportion of the profits—viz., $\frac{6}{100}$ of £10, or 12s. on each £10 share. Thus a person who bought 5 shares would pay £50, and receive each year a dividend of 12s. × 5, or £3.

He could not, however, recover from the company this £50, but he could sell his shares to some one else, which sale would, at the same time, surrender to the purchaser the right to receive the annual dividend. Moreover, just as the price of a ton of coal, or of iron, is continually changing, so also is the price of a share. If the company is prosperous, a man may be willing to pay for a share more than its original value (£10), so as to secure what he considers a good investment. But this sale and purchase of shares is a private transaction, and does not affect the original capital of the company.

For instance, suppose A sells his 5 shares to B for £12 each; then A (not the company) makes a profit of £2 × 5, or £10, and B becomes entitled to receive from the company the dividend £3 which was before paid to A. It must, however, be carefully noted that B has laid out or invested £60 to secure this £3 dividend, and that, therefore, the percentage on *his outlay* is $\frac{3}{60}$ of 100, or 5 %. B has spent £60; but, in the books of the company, he is credited with 5 shares, value £50, exactly as A was.

The original value of a share is called its *Nominal* Value.

The price at any particular time is known as its *Market* (or current) Value. The shares are said to be *quoted* at this market value.

According as this market value is *more* than, *equal* to, or *less* than the nominal value, so the shares are said to be at a *Premium*, at *Par*, or at a *Discount*. Thus, a £10 share when bought for £12 is at a premium of £2, or 20 % (*i.e.* $\frac{2}{10}$ of 100); similarly, when bought for £6 is at a discount of £4, or 40 % (*i.e.* $\frac{4}{10}$ of 100).

If the company does not require for immediate use the full £60,000 of its capital, but only £48,000, or four-fifths of the whole, then each shareholder contributes at first only £8 (*i.e.* $\frac{4}{5}$ of £10) on each share, but he must contribute the remaining £2 when called upon to do so. The dividend will then be declared only on that portion of the share which has been "paid up," and not on the full £10.

211. Ex. 1.—*A man lays out £500 in the purchase of £5 shares, and, after they have risen to £7 10s., sells out. What is his profit?*

The number of shares bought is £500 ÷ £5 = 100. On each share he gains £2 10s.; therefore total profit is £2 10s. × 100 = £250.

Ex. 2.—*The £5 shares of a company are quoted at £6. How many shares can a man purchase for £240?*

Each share costs £6; therefore number of shares = £240 ÷ £6 = 40.

Note that the £5 does not enter into the calculation.

Ex. 3.—*A man invests (i.e. lays out or spends) £1,024 in a company paying 5% per annum when its £10 shares can be bought for £16. Find (1) his annual dividend, (2) his percentage on his investment.*

(1) The market value of a share is £16; therefore number of shares bought = £1,024 ÷ £16 = 64; therefore the nominal value of his shares is £640. On this he receives dividend at the rate of 5%; therefore required dividend = £$\frac{640 \times 5}{100}$ = £32.

(2) To secure this £32 he has invested £1,024; therefore his percentage on his investment is $\frac{32}{1024} \times 100 = 3\frac{1}{8}$.

This latter result might be obtained thus: As he is paying a higher price (£16) for what originally cost only £10, his percentage will be less in ratio of 10 : 16; therefore required percentage = $\frac{10}{16}$ of $5 = 3\frac{1}{8}$.

Ex. 4.—A *is the original holder of 200 £5 shares (£4 only being paid up) in a company paying 10% per annum. B purchases these from A at £6 10s. for each share. What did these shares cost A and B respectively, and what percentage does B get on his outlay?*

A pays £4 for each share (he is liable for the remaining £1 at any time as long as he is a shareholder); therefore A pays £4 × 200 = £800.

B pays to A £$6\frac{1}{2}$ for each share; therefore B pays in all £$6\frac{1}{2} \times 200$ = £1,300.

B is credited in the books of the company with £800, being A's original amount (B is now liable for the remaining £1 on each share), and on this £800 he receives dividend at the rate of 10%; therefore his dividend = $\frac{10}{100}$ of £800 = £80.

To secure this he has spent £1,300; therefore his rate per cent. *on his investment* = $\frac{80}{13} = 6\frac{2}{13}$.

EXAMPLES CXXVI.

1. The £10 shares in a brewery company are selling for £10 10s. each. What is the nominal value, and what the market value, of 50 such shares?

2. £1,620 is laid out in purchasing £25 shares in a company when each can be bought for £27. How many are bought and what is their nominal value?

3. The capital of a company paying 6% per annum is £48,000, divided into shares of £30 each. What is the annual dividend of a man who owns 20 such shares? Also what is the total dividend paid by the company?

4. What profit is made by selling 20 shares of £5 each when they are quoted at £7 10s.? What is the dividend derived from these shares at the rate of 3s. per share?

5. What must be paid for 100 £1 shares in a tramway company when they are selling at £$1\frac{9}{16}$?

6. A man who holds 50 shares in a gas company sells them at £6 12s. each, thereby making a profit of £30. What price did he himself pay for each share? If the company was paying a dividend of 10s. per share, how much was he receiving?

7. Find the annual income derived from 1,000 £2 shares in a company paying a dividend of (1) 7 % per annum, (2) 2s. 6d. per share per annum.

8. Coal shares of £10 are sold at a discount of £2 10s. How much is paid for 25 of these shares?

9. A man buys 200 £1 shares in a mining company paying 5 % per annum. They are quoted at 10 % above par. What is his annual income, and what percentage does he obtain on his investment?

10. The £5 shares of a company paying a dividend of $7\frac{1}{2}$ % are selling for £8. What percentage does an investor get who buys at the latter price?

11. The holder of 100 £10 gas shares sells them when they have gone down to £8 and invests the proceeds in buying at par £5 shares in a company paying 3 % per annum. How many shares does he buy and what is his annual income from them?

12. £1,560 is invested in purchasing £50 bank shares quoted at £52. What dividend is obtained if the bank pays 18 %.

13. By investing £850 in a company paying 10 % an income of £50 is obtained. What price is paid for each £10 share?

14. By purchasing £20 water shares for £25 each, a man gets 3 % on his investment. What rate per cent. is the company paying? What is his dividend if he buys 30 shares?

15. The £10 shares of a company, of which £8 only are paid up, are selling for £14. If the company pays 8 %, what income will a man get who invests £560?

16. The holder of 60 £10 shares in a gas company paying a dividend of 7 % sells out when they have risen to £16. He invests the proceeds in the purchase of bank shares quoted at £48, when the bank is paying a dividend of 6 %. If his income is increased £6 by the transaction, what is the nominal value of a bank share?

17. A is the holder of £1,000 of the capital of a company paying a dividend of 6 % per annum. B invests £1,000 in a company paying 8 % when their £10 shares can be bought for £12 10s. By how much does B's dividend exceed A's?

18. How much must be invested in a company paying $3\frac{3}{4}$ % whose £5 shares (£4 only being paid up) are quoted at £5 10s. so as to obtain a dividend of £6? What must be paid by the owner of the shares if the remaining £1 per share is called up?

STOCKS.

212. Only an *integral* number of *shares* in a company may, as a rule, be bought or sold.† The term *Stock* is applied to the capital of a company, *any* portion of which, large or small, may be bought or sold. Such capital is not, therefore, divided into a definite number of equal portions as in the case of *shares*. The selling or buying price of the stock (called its *Market Value*) fluctuates from day to day, just as the price of any commodity, such as coal or iron (see § 210 on shares); and as a coal merchant, in quoting his prices, uses some definite quantity (say a ton) as a unit, so, in transactions with stocks, we must use some definite quantity of the stock as a unit, although we may not always express it in so many words. The unit selected is always £100 of the stock. Thus, where we have previously spoken of a £5 *share*, we now speak of £100 *stock*. Hence arises such an abbreviated expression as "3 % stock *@* 102." The meaning of this is that a company is paying a dividend of 3 % on its capital or stock, £100 of which is at the time selling for £102.

The expression is still further shortened to "the 3 per cents. @ 102."

It should be clearly understood that this £100 stock is not 100 sovereigns cash, but only a portion of the company's capital, the possession of which gives the holder the right to receive certain dividends; whereas the £102 is so much money or cash, which is used in purchasing the £100 of capital and the rights associated with it.

Compare the case of a ton of coal and the *price* of that ton, 25s., say.

Problems on stocks are of the same nature as those on shares; for we can regard the total capital of the company as divided into a certain number of equal portions or units the *nominal* value of which is £100. The sale of stock involves the surrender to the buyer by the seller of the latter's right to receive any dividends which the company may declare. According as the *market* value is greater than, equal to, or less than the *nominal* value (£100) so is the stock said to be at a

† "Fractional shares," *e.g.* $\frac{1}{3}$ of a share, are often issued as dividends; but the proprietor is generally required to make these fractions up to an integer by purchase of other fractions, and is not usually registered as the proprietor of fractions.

premium, at *par*, or at a *discount*. Thus, if £100 of stock is bought for £105, the stock is at a premium of 5%, or 5 above par; but, if bought for £96, the stock is at a discount of 4%, or 4 below par.

213. Ex. 1.—*How much is obtained from the sale of* £1,420 *of* $2\frac{1}{2}$% *stock at* 105?

This means that £100 portions of stock can be sold for £105 each.

As the £1,420 contains 14·2 such portions, therefore the whole sells for £105 × 14·2 = £1,491. *Ans.*

Notice that the $2\frac{1}{2}$% does not affect the working, this number being used only when the question of income or dividend is involved.

Ex. 2.—*How much* $3\frac{1}{2}$% *stock at* 98 *can be purchased for* £1,600 13s. 4d *?*

Here the £1,600 13s. 4d. is cash.

£98 cash will purchase one portion of stock, *i.e.* £100 stock.

And £1,600 13s. 4d. or £$1,600\frac{2}{3}$ contains £98 $1,600\frac{2}{3} \div 98$, or $16\frac{1}{3}$ times therefore required amount of stock is $16\frac{1}{3}$ portions, or

£100 × $16\frac{1}{3}$ = £1,633 6s. 8d. *Ans.*

Ex. 3.—*What is the dividend from* £1,275 *of* 4% *stock at* 106?

One portion of stock (£100) gives a dividend of £4.

∴ £1,275 ÷ £100, or 12·75 portions, give a dividend of £4 × 12·75, or £51. *Ans.*

Ex. 4.—*Find the annual dividend obtained by investing* £1,351 10s. *in* 4% *stock at* 106.

For every £106 cash invested (*i.e.* used in purchasing stock) one portion of stock (£100) is bought, and this produces a dividend of £4.

∴ each £106 cash which is used produces a dividend of £4.

∴ £1,351 10s. will produce a dividend of £4 × 1351·5 ÷ 106

= £4 × 12·75 = £51. *Ans.*

Compare this example carefully with Ex. 3.

Ex. 5.—*What percentage on his money does the investor in Ex.* 4 *obtain?*

By investing £106 cash he secures a dividend of £4.

∴ by investing £100 cash he secures a dividend of £4 × $\frac{100}{106}$ = £$3\frac{41}{53}$%.

Ex. 6.—*What is the price of the* 3 *per cents. when an income of* £117 *is obtained from investing* £4,095 *therein?*

£117 is the dividend from £4,095 invested.

∴ £3 is the dividend from £4095 × 3 ÷ 117 = £105.

i.e. £105 is the market value or price of the standard portion of stock (£100).

EXAMPLES CXXVII.

1. How much cash is obtained by selling £1,750 of the stock of a company if £100 of the stock sells for £116?

2. £1,261 is laid out in purchasing stock when £100 of the stock can be bought for £97. How much stock is bought?

3. The capital of a company paying 6% per annum is £120,000. What is the annual dividend of a man who holds $\frac{1}{64}$th part of the whole stock?

4. How much money is obtained by selling out £1,941 13s. 4d. from the 4% stock at 110? What dividend was the owner of the stock receiving?

5. What amount of $3\frac{1}{2}$% stock at 79 can be purchased for £1,975 cash, and what annual dividend will be obtained?

6. The holder of $31\frac{1}{2}$ portions of a company's stock, each portion being £100, sells out at £120. What are the proceeds of the sale?

7. What will be the price of £850 stock quoted at $7\frac{1}{2}$ above par?

8. What income is obtained from spending £1,484 in the purchase of stock, a £100 portion of which costs £106 and produces a dividend of £2 10s.?

9. A buys £6,133 6s. 8d. of 6% stock at 92. B invests £6,133 6s. 8d. in the same stock. (1) What does A pay for his purchase? (2) How much stock does B get? (3) What are their respective incomes?

10. By investing £1,597 10s. in $2\frac{1}{2}$% stock an income of £37 10s. is obtained. Find the market value of £100 of the stock.

11. Find how much must be paid for the following:—(1) £850 stock at 72; (2) £1,425 6s. 8d. gas stock at 240; (3) £948 railway stock at 160; (4) £8,000 of 8% stock at 108.

12. How much stock can be obtained by investing?—

(1) £150 in 3% stock at 80.
(2) £7,960 in railway stock at $66\frac{1}{3}$.
(3) £925 in a tramway company at 111.
(4) £189 11s. in 4 per cents. at $113\frac{1}{3}$.

13. What is the *half*-yearly dividend obtained from?—

(1) The possession of £1,807 5s. in $2\frac{1}{2}$% stock at 99.
(2) Investing £880 13s. in the $2\frac{3}{4}$ per cents. at 103.
(3) Holding £9,423 16s. 8d. of the $7\frac{1}{2}$ per cents. at 204
(4) The purchase of £4,500 of $6\frac{1}{4}$% railway stock at 150.

14. Find the market value per £100 stock—

(1) When £10,000 stock can be bought for £9,575.

(2) When £292 19s. will purchase £1,860 of 2 % stock.

(3) When £1,000 invested in the $4\frac{1}{2}$ per cents. produces an annual dividend of £30.

(4) When £27 9s. is the *half*-yearly income from investing £1,381 13s. in the $3\frac{3}{4}$ per cents.

15. How much cash must be invested to secure an annual income of ?—

(1) £135 from the 4 per cents. at 126.

(2) £22 10s. from 9 % railway stock at 200.

(3) £43 6s. 8d. from the $4\frac{1}{3}$ per cents. at 114.

(4) £1,050 from the $2\frac{5}{8}$ per cents. at par.

16. What percentage on his money is obtained by a man ?—

(1) Who buys £600 of 5 % stock for £800.

(2) Who gets an income of £30 from investing £1,000.

(3) Who buys any quantity of 6 % stock at 125.

(4) Who purchases £8,000 railway stock for £14,000 and derives from it an income of £560.

214. The stock (or the shares) of a company may consist of various kinds, which are selected by individuals according to their personal views of the security offered or the risks incurred. Generally speaking, where the security is very good, the rate of interest would be low, *i.e.* at present about 3 %.

In buying *Debentures*, a man gets the right to a *fixed* rate of interest according to agreement, and in case of the company's failure he can claim his share of the money produced by the sale of the company's property, such as warehouses, shops, machinery, &c. A holder of *Preference Stock* (*or Shares*) also has the right to interest at a *fixed* rate, payable from the profits *before* the *ordinary* stock or share holders receive anything. The remainder of the profits is divided amongst the ordinary stock or share holders, and, if the company is doing well, they frequently receive a higher rate of interest than the holders of debentures or preference stock. Thus the ordinary stock or share holders run greater risks, but hope for higher dividends.

The term *Stock* is applied also to the various amounts of money which have been borrowed by different Governments (frequently in times of war) or by the Corporations of cities and boroughs. In order to pay the interest on different sums of money borrowed from time to time, the British Government

set aside certain *funds* (raised by taxation). The terms *Funds* or *Funded Debt* is now applied to the larger part of the Public Debt of this country. Strictly speaking, however, the Government does not thus *borrow* money, for the lender has not the right to reclaim his money whenever he wishes, but is entitled to receive at stated periods interest or dividend at certain fixed rates (see Ex. cxxix., 18). The term *Consols* (an abbreviation for consolidated annuities) is also applied to this stock. The dividends on these and on other portions of the Public Debt are, however, paid *quarterly*, viz., on January 5th, April 5th, July 5th, October 5th, whilst commercial companies generally pay their dividends every six months.

Bonds.—This term is applied to some foreign Government stocks and some foreign railway stocks. The bondholder receives a paper, which is divided into smaller portions, called *coupons*, on presentation of which at the proper times he receives his dividend. At the expiration of a certain number of years, as agreed, the bond is redeemed, *i.e.* the money is paid back again. A bond is thus similar to a terminable annuity.

Deposit is the name applied to money placed with a banking or other commercial company to produce interest. It can be withdrawn by giving certain notice according to agreement.

Ex dividend (abbreviated x.d.).—As the day approaches for the payment of dividend on any stock, the market value of such stock would have a tendency to rise, and just after that day has passed the price would tend to fall. The letters x.d. are placed after the quotation of the price of stock when the stock is offered for sale near the dividend day without entitling the purchaser to receive the dividend immediately following.

215. The sale and purchase of stocks (or shares) are generally effected by means of a *Stockbroker*, who makes a certain charge for his services, called *Brokerage or Commission*, calculated as a percentage on the amount of *stock* (or *shares*) bought or sold. A *Stockjobber* makes it his business to buy stock at a low rate and sell it again at a higher rate. The stockbroker is a kind of middleman between the stockjobber and the private investor. The usual brokerage, in the case of the public funds, is 2s. 6d. (or £$\frac{1}{8}$) per £100 of *stock*. In the quotations given in the daily papers two prices are generally named—thus, 128, $\frac{1}{4}$. This means that a jobber is willing to *buy* £100 stock from a broker at the cheaper rate, £128, or to *sell* £100 stock to him at the higher rate, £128$\frac{1}{4}$. The jobber thus gains £$\frac{1}{4}$ on each £100 stock.

The broker would thus buy from the private investor, and would give him £128 less £$\frac{1}{8}$ (his brokerage), *i.e.* £$127\frac{7}{8}$, or would sell to him for £$128\frac{1}{4}$ + £$\frac{1}{8}$ (his brokerage), *i.e.* £$128\frac{3}{8}$.

Hence we have the important rule.—*In buying stock, the investor must add the brokerage to the market value of the* £100 *stock; in selling stock, he must subtract the brokerage from the market value.* See, however, note on p. 368.

In all the exercises given, so far, the brokerage has been supposed to be included in the prices named.

(Generally, in examination questions, the brokerage is supposed thus included, unless otherwise stated.)

216. Ex. 7.—*What must be paid for* £1,250 *stock at* 106, *allowing for brokerage of* $\frac{1}{8}$% *?*

£100 stock costs the buyer £106; the broker also has £$\frac{1}{8}$ for each £100 on account of his services;

∴ £100 stock costs altogether £$106\frac{1}{8}$;

∴ £1,250 stock costs altogether £$106\frac{1}{8} \times 12\frac{1}{2}$ = £1,326 11s. 3d. *Ans.*

Ex. 8.—*How much cash is obtained by the sale of* £1,250 *stock at* 106, *brokerage being* $\frac{1}{8}$% *?*

£100 stock can be sold for £106; but out of this the broker takes his charges;

∴ for each £100 stock the owner gets £$105\frac{7}{8}$;

∴ for £1,250 stock the owner gets £$105\frac{7}{8} \times 12\frac{1}{2}$ = £1,323 8s. 9d.

Notice these two examples very carefully. Had no brokerage been charged in either case, the two answers would have been exactly the same—viz., £$106 \times 12\frac{1}{2}$, or £1,325.

Ex. 9.—*How much stock at* $97\frac{7}{8}$ *is obtained by investing* £9,579 10s. (*brokerage* 2s. 6d. *per* £100 *stock*) *?*

£100 stock itself costs £$97\frac{7}{8}$; the broker requires $\frac{1}{8}$ in addition for himself;

∴ £100 stock costs altogether £$97\frac{7}{8}$ + £$\frac{1}{8}$ = £98;

∴ by investing £98 a man can obtain £100 stock;

∴ by investing £9,579 10s. a man can obtain

£9579·5 × 100 ÷ 98 = £9,775 stock.

Ex. 10.—*How much stock at* $97\frac{7}{8}$ *is sold for* £9,579 10s. (*brokerage* $\frac{1}{8}$%) ?

£$97\frac{7}{8}$ − £$\frac{1}{8}$ or £$97\frac{3}{4}$ cash is got by the sale of £100 stock;

∴ £9,579 10s. cash is got by the sale of

£9579·5 × 100 ÷ 97·75 = £9,800 stock.

Note carefully the difference between Examples 9 and 10.

Ex. 11.—*Find the annual income derived from investing* £18,200 *in* $4\frac{1}{2}$ *per cents. at* $149\frac{7}{8}$, *brokerage being* $\frac{1}{8}$.

To secure an income of £$4\frac{1}{2}$ a man must buy £100 stock, for which he must pay in all £$149\frac{7}{8}$ + £$\frac{1}{8}$, or £150;

∴ required income is £$4\frac{1}{2} \times \frac{18200}{150}$ = £546.

Ex. 12.—*Which is the better stock to invest in, the* $3\frac{1}{2}$ *per cents. at* 107 *or the* 5 *per cents. at* $154\frac{1}{2}$ *?*

In the first stock £$3\frac{1}{2}$ income is obtained by investing £107;

∴ £1 ,, ,, ,, £107 ÷ $3\frac{1}{2}$ = £$30\frac{4}{7}$.

In the second stock £5 ,, ,, ,, £$154\frac{1}{2}$;

∴ £1 ,, ,, ,, £$154\frac{1}{2}$ ÷ 5 = £$30\frac{9}{10}$.

But $30\frac{4}{7}$ is less than $30\frac{9}{10}$; ∴ the first stock is the better, for less money is laid out in it to produce £1 than is needed in the second stock.

The student may verify that the return per cent. on capital invested is 3·27... in the first, and 3·23... in the second stock, and that, if brokerage ($\frac{1}{8}$ %) is reckoned, these figures become 3·26... and 3·23... respectively.

EXAMPLES CXXVIII.

1. How much must be given for £3,500 stock in the $3\frac{1}{2}$ per cents. at $96\frac{1}{4}$? What is the annual income? (Brokerage $\frac{1}{8}$ %.)

2. A man sells out £2,000 from the 3 per cents. at $94\frac{5}{8}$. What does he obtain? (Brokerage $\frac{1}{8}$ %.)

3. How much 4 % stock at $115\frac{7}{8}$ can be bought for £11,827 13s., and what will be the half-yearly dividend? (Brokerage $\frac{1}{8}$.)

4. What quantity of the $2\frac{3}{4}$ per cents. at 99 can be purchased for £5,194 3s.? (Brokerage $\frac{1}{8}$.)

5. How much must be invested in the same stock as in Ex. 4 to secure a dividend of £144 2s.? (Brokerage $\frac{1}{8}$.)

6. (1) When brokerage is £$\frac{1}{8}$ for each £100 stock, what is the quarterly dividend on £2,000 invested in the $2\frac{1}{2}$ % Consols at $107\frac{3}{4}$? (Answer to nearest penny.) (2) How much must be paid to secure £2,000 of these Consols, with the same rate of brokerage; and (3) what would be the quarterly dividend then?

7. How much is obtained by selling £1,000 of G.W.R. debentures at 168? (Brokerage $\frac{1}{8}$.)

8. If a broker charges $\frac{1}{4}$ % on the nominal value of shares, what will it cost to purchase 600 £5 shares quoted at £5 10s.?

9. How much cash will a man receive who sells through a broker 600 £5 shares quoted at £5 10s.? (Brokerage $\frac{1}{4}$.)

10. Which will give the higher rate of interest—the 3 per cents. at 71 or the 4 per cents. at 101? (No brokerage.)

11. Reckoning brokerage $\frac{1}{8}$ %, which is the better stock—4 per cents. at 157 or 6 per cents. at 236 ?

12. Which of the following stocks produces the highest rate of interest, which the next, and so on ?—(1) $2\frac{3}{4}$ % Consols at 109, (2) $3\frac{1}{2}$ per cents. at 131, (3) 7 per cents. at 250, (4) $1\frac{1}{4}$ per cents. at 30 % discount, (5) 3 per cents. at par. (No brokerage.)

217. Ex. 13.—£11,000 *of* $2\frac{3}{4}$ % *Consols are sold at* $108\frac{5}{8}$, *and the proceeds are invested in the* 3 *per cents. at* $96\frac{3}{4}$. *Find* (1) *how much of the second stock is bought;* (2) *the change in income.* (*Brokerage* $\frac{1}{8}$ *in each case.*)

(1) £100 of first stock is sold for £$108\frac{5}{8}$ − £$\frac{1}{8}$ = £108·5 cash ;

∴ £11,000 of first stock is sold for £108·5 × 110 = £1,085 × 11 cash.

£$96\frac{7}{8}$ cash will purchase £100 of the second stock.

∴ £1,085 × 11 cash will purchase

£100 × 1085 × 11 ÷ $96\frac{7}{8}$ = £12,320 stock. ***Ans.***

(2) Income from first stock = £$2\frac{3}{4}$ × 110 = £302 10s.

Income from second stock = £3 × 123·2 = £369 12s.

∴ There is an increase of £67 2s. *Ans.*

Ex. 14.—*How much must I invest in the* $2\frac{1}{2}$ % *Consols at* 108 (*brokerage included*) *to obtain a net income of* £195 *after paying income tax at the rate of* 6d. *in the* £ *?*

By investing £108 I secure a dividend or income of £2 10s. The tax on this is 6 × $2\frac{1}{2}$d. = 1s. 3d. ; therefore net income = £2 8s. 9d. = £$2\frac{7}{16}$.

∴ £$2\frac{7}{16}$ is the net income from investing £108.

∴ £195 is the net income from investing £108 × $\frac{195}{2\frac{7}{16}}$ = £8,640. *Ans.*

Ex. 15.—*If I invest in the* 3 % *stock and, after paying* 8d. *in the* £ *income-tax, obtain* $2\frac{1}{2}$ % *on my investment, at what price did I buy the stock ?*

Suppose I buy £100 stock; the net dividend from this will be £3 less 2s., or £2·9. Then $2\frac{1}{2}$ % on buying price = £2·9 ; therefore buying price = £116. *Ans.*

Ex. 16.—*What will be the present value of a bond for* £100 *paying* 6 % *per annum redeemable at par at the end of* 11 *years, reckoning interest at* 4 % *per annum ?* (*Simple interest in each calculation.*)

Each year there is a dividend of £6 on the bond, and at the end of 11 years the full value £100 will be paid to the holder. We must therefore find present values of £6 due in one year, £6 due in 2 years, £6 due in 3 years, etc., ..., and £106 due in 11 years, interest at 4 % (simple).

Ex. 17.—*A man invested* £5,173 11s. 3d., *partly in Consols at* $101\frac{1}{8}$ *and the remainder in railway stock at* $146\frac{1}{2}$. *When the Consols were at* $102\frac{1}{4}$ *and the railway stock at* $143\frac{1}{8}$, *he sold out, and received altogether the sum he had originally invested. How much money did he invest in Consols?*

By the transactions he loses £$3\frac{3}{8}$ on every £100 railway stock and gains £$1\frac{1}{8}$ on every £100 Consols. If he neither loses nor gains on the whole, he must have 3 times as much Consols as railway stock, since $3\frac{3}{8} = 3 \times 1\frac{1}{8}$. But to purchase £300 of Consols and £100 of railway stock he must invest £$303\frac{3}{8}$ in Consols, £$146\frac{1}{2}$ in railway stock, *i.e.* altogether £$449\frac{7}{8}$. Hence if he invests £$449\frac{7}{8}$, he puts £$303\frac{3}{8}$ in Consols; therefore, since he invests £$5,173\frac{9}{16}$, he puts in Consols £$5,173\frac{9}{16} \times 303\frac{3}{8} \div 449\frac{7}{8}$ = £3,488 16s. 3d. *Ans.*

EXAMPLES CXXIX.

(Unless otherwise stated, brokerage is supposed to be included in the prices of stocks.)

1. A man invests £1,890 in the 4 per cents. at 126. What is his dividend?

2. How much 3 % stock sold at $92\frac{1}{2}$ will produce £3,700? If this sum of £3,700 be then invested in railway stock at $115\frac{5}{8}$, what must be the dividend from £100 of this stock in order that the same income may be received as from the original investment?

3. A man holding £625 stock in the 4 per cents. sells out at $81\frac{3}{8}$, and lends the proceeds at 5% per annum. Find the alteration in his annual income?

4. If a person invests £1,496 5s. in 3 % stock at $99\frac{3}{4}$, what income will he receive? And if he afterwards sells the stock at $100\frac{1}{8}$, what will he get for it?

5. Find the income arising from an investment of £250 in a 3 % stock at $66\frac{2}{3}$.

6. What is the price of $2\frac{3}{4}$ % stock when an investment of £2,109 7s. 6d. produces an income of £61 17s. 6d. per annum?

7. Find the difference in incomes obtained by investing £7,700 in $2\frac{3}{4}$ % Consols at $96\frac{1}{4}$, and an equal sum in New South Wales 4 % Government stock at 110.

8. Determine the increase in income obtained by transferring £4,000 of stock from the $3\frac{1}{4}$ per cents. at 87 to the $6\frac{1}{4}$ per cents. at 145.

9. A man has £8,750 New Zealand $3\frac{1}{2}$ % stock. He sells out at 109. £2,500 of the proceeds he lends on mortgage at 7 %, and the remainder he invests in railway shares yielding 4 % at 136. Find the change in his income. (Answer to nearest penny.)

10. What sum must be invested in the $3\frac{1}{2}$ per cents. at 104 to obtain an income of £329?

11. How much money must be invested in the 3 per cents. at 87 in order to produce a net income of £295 after deducting income tax at 4d. in the £?

12. How much $2\frac{3}{4}$ % stock will produce an income of £638 per annum?

13. A man invested money in 3 % stock at 105; after paying income-tax at 6d. in the £ his income therefrom was £146 5s. What sum was invested?

14. I invest £4,278 in the 5 per cents. at $114\frac{7}{8}$, but afterwards sell at $135\frac{1}{8}$ and invest the proceeds in the 4 per cents. at $92\frac{7}{8}$. What is the change in my income? (Brokerage $\frac{1}{8}$ on each transaction.)

15. A man invests £1,668 15s. in 3 per cents. at $89\frac{7}{8}$. His first year's income he invests in the same stock but at a different price. His next year's income is £57 10s. At what price did he invest in the second case? (Brokerage $\frac{1}{8}$ in both cases.)

16. A person invests £2,852 in railway shares at 115, the nominal value of a share being £100, and the annual dividend on it being £5. What is his yearly income?

17. Find the buying price of 3 % stock when an income of £28 10s. is obtained by investing £869 5s.

18. The Chancellor of the Exchequer proposed (1888 A.D.) to give £102 stock paying interest at $2\frac{3}{4}$ %, or £108 stock paying interest at $2\frac{1}{2}$ % in exchange for £100 of the current 3 % stock. Find the loss of income on each £100 of stock held which would be incurred by accepting these offers respectively, and show that the losses are in the ratio of 13 to 20.

19. A man has £2,000 of a 3 % stock, which he sells out at 90. He then invests £1,000 of the proceeds in the 5 per cents. at 125. What rate of interest must he get for the remainder in order that his income may be unaltered?

20. A person bought railway stock at 88, and, after receiving the *half*-year's dividend at $4\frac{1}{2}$ % per annum, sold out at $92\frac{3}{8}$, making a total profit of £198 15s. How much stock did he buy?

21. A man invests £1,059 2s. in the 3 per cents. at 89. What is his annual dividend?

22. A person invests £2,362 10s. in 3 % Consols, which he sells when they have risen $\frac{5}{8}$ %, thereby gaining 15 guineas. At what price did he buy them?

23. £21,000 invested in 4 % stock produces annually £216 13s. 4d. more than if it had been invested in $2\frac{3}{4}$ % stock at 99. Find the price of the 4 % stock.

24. £9,000 invested in a certain 5 % stock produces approximately £139 0s. 4d. annually more than if it had been invested in $2\frac{3}{4}$ % Consols at 112. Find the price of the 5 % stock.

25. A man has £3,430 stock in the $3\frac{1}{2}$ per cents. When they are at $85\frac{1}{2}$ he sells out and invests his capital in the 4 per cents. at 98. Find the alteration in his income.

26. Find the income obtained from investing £1,118 15s. in the 3 per cents. at $89\frac{1}{2}$.

27. A man invests a sum of money sufficient to pay for the stock and brokerage in the 3 per cents. at 110, and, after receiving the dividend for half a year, sells out at $110\frac{1}{8}$, thus gaining £110 on the whole transaction. What was the sum invested? (Brokerage $\frac{1}{8}$ in each case.)

28. A man invests £1,700 in the 3 % stock at 85. What amount of stock does he obtain? After receiving a half-year's dividend he sells out when the stock is at 86. How much better off is he than if he had lent the original £1,700 for half a year at 3 % per annum?

29. A man invests £1,404 16s. in $2\frac{3}{4}$ per cents. at 99. How much must he invest in $3\frac{1}{2}$ per cents. at 105 that the rate of interest on his whole investment may be 3 %?

30. I invest equal sums in a 4 % stock and in a 3 % stock and get 5 % for my money; the 4 per cents. are at 90. What is the price of the 3 per cents.?

31. What sum of money must be invested in the $2\frac{3}{4}$ % stock at 105 to produce an income of £638 per annum after an income tax of 8d. in the £ has been deducted?

32. A person invests half his capital in the 3 per cents. at 99, and half in the 4 per cents. at 120. Which investment produces the larger income?

33. A man sold £1,000 of the 4 % stock at 120, and invested the money realized in the 6 % stock at 150. What was the change in his income?

34. What money (including broker's commission) must be invested in the 4 per cents. at 85 to bring in a net income of £529 per annum, when the income tax is 10d. in the £, brokerage being $\frac{1}{8}$?

35. A man invests £5,000 in the $3\frac{1}{2}$ per cents. when they are at $73\frac{1}{2}$. What is his income? What percentage on his investment does he obtain? (Answers to the nearest penny).

36. A man invests £1,980 in the $3\frac{1}{2}$ per cents. at 99 and £3,220 in the $4\frac{1}{2}$ per cents. at 105. Find the average rate of interest on his whole investment.

37. A person buys £500 stock at 66 and afterwards £500 of the same stock at 69; he sells out the whole when the price has risen to 89. Find the increase in his capital.

38. A man sells out $2\frac{3}{4}$ % Consols at $96\frac{1}{4}$, and by investing the proceeds in shares which pay an annual dividend of £4 per share raises his income 5 %. What was the price of each share?

39. A person bought Consols at 94 and sold the same at $95\frac{5}{8}$, thereby gaining £55. What cash did he receive for the stock, reckoning $\frac{1}{8}$ % brokerage on each transaction?

40. A person holding £8,400 of 4 % stock sold out at $110\frac{1}{2}$ and invested the proceeds partly in $2\frac{1}{2}$ % stock at 85 and partly in $5\frac{1}{4}$ % stock at $136\frac{1}{2}$; his income was unaltered. How much of each stock did he buy?

41. How much stock does a man hold if, by selling out at 97 and investing the proceeds in $4\frac{1}{2}$ % stock at 114, he obtains an income of £14 11s.?

42. A person has a certain amount of 4 % stock. He sells it at $117\frac{1}{3}$ and invests half the proceeds in 2 % stock at 96, and the rest in 3 % stock at 99. He then finds that his annual income is reduced by £20. What amount of the original stock had he?

43. Find the quarterly income (to the nearest penny) derived from £3,500 in the $2\frac{3}{4}$ % Consols, after a deduction of income tax at the rate of 4d. in the £ per annum. [N.B. The whole annual tax is to be paid.]

44. If I invest £8,064 7s. in the 3 % Consols at $102\frac{3}{8}$, what income shall I get after deducting an income tax of 6d. in the £? (Answer to nearest penny.)

45. What income would be obtained from the investment of £10,500 half in 4 % stock at 105, and half in 5 % stock at 125?

46. What amount of $2\frac{3}{4}$ % Consols will yield a net income of £152 17s. 1d. when the income tax is 8d. in the £?

47. A man possesses £22,400 of $2\frac{3}{4}$ % Consols. He sells them at $99\frac{3}{8}$. He then invests one-half the proceeds in 4 % railway debenture stock at $132\frac{1}{2}$, lends £7,266 13s. 4d. on mortgage at $4\frac{1}{2}$ % and loses the rest. What alteration is produced in his annual income?

48. A man invests the following sums of money, viz., £2,500 in 3 % stock at $95\frac{1}{2}$, £8,000 in 4 % stock at par, and £6,000 in 5 % stock at $92\frac{1}{2}$. What rate of interest does he receive on his whole investment? (Answer to nearest penny.)

49. By investing a certain sum of money in the 4 % stock at 89 I get the same income as if I had invested £168 less in the 5 per cents. at 85. What is the income?

50. A person sells out £1,000 of 3 % stock at $83\frac{1}{3}$, and invests the proceeds in the purchase of £100 railway shares which are quoted at £110, thus increasing his half-yearly income by £4 3s. 4d. What is the annual dividend on each share? (Answer to nearest penny.)

51. A man has £3,100 in the 3 per cents. at 92, but sells out, and, with the proceeds, purchases 5 % railway stock at 115. Find the change in his income.

52. Find the income produced by investing £253 15s. in $3\frac{1}{2}$ % stock at $101\frac{1}{2}$ and the amount realized by subsequently selling out at 99.

53. By selling 4 % stock at 91, and investing the proceeds in railway shares when £100 share is worth £146 5s., I increase my income in the ratio of 7 to 5. What dividend per cent. per annum did the railway company pay?

54. I purchase £600 worth of Indian 3 % stock at 120. How much Canadian 5 % stock at 150 must I purchase to get an average of 3 % on the two investments?

55. I possess £12,000 of 3 % Consols, bought when they were at $99\frac{1}{4}$. They are now at $100\frac{1}{8}$. The Chancellor of the Exchequer offers me £108 of $2\frac{1}{2}$ per cents. for each £100 of the 3 per cents., and I agree so to exchange $\frac{5}{8}$ of my stock. The remainder I sell out and invest in $3\frac{1}{2}$ % debentures. What must be their price in order that my income may be unaltered? And to what price must the $2\frac{1}{2}$ per cents. attain in order that, if I presently sell them and the debentures, when the latter have fallen to $94\frac{1}{8}$, I may exactly recover my original capital? ($\frac{1}{8}$ % brokerage is to be charged for all sales and and purchases, but not for the exchange.)

56. A person holding £4,205 of $2\frac{1}{2}$ % stock calculates that, by selling at 84 and investing the proceeds in 5 % debentures at the current price, he will improve his income by £63 1s. 6d. What is the price of the debentures?

57. A man has an income of £415 derived from capital invested in 4 % stock. He sells out this stock at 119, and reinvests the proceeds in a 5 % stock. What price must he pay for this latter stock if his new income is £425?

58. I pay income tax at the rate of 6d. in the £ on the income arising from 4 % stock. When the income tax is raised to 8d. I sell out at $146\frac{3}{8}$ and buy $3\frac{1}{2}$ % stock. At what price must I buy this stock if my net income remains the same as before? (Brokerage $\frac{1}{8}$ in both selling and buying.)

59. If money invested in the 3 % Consols gives exactly 3 % after paying 5d. in the £ income tax, what is the price of the Consols?

60. A person derived an income of £462 from $2\frac{3}{4}$ % Consols. He sold out at $101\frac{1}{4}$, and invested the proceeds in 4 % railway stock, thereby increasing his income by £69 11s. 3d. What was the price of the railway stock?

61. A man invested £25,200 in 4 % railway stock. He sold out when the stock was at 110, and invested the money realized in 3 % stock at 105. After allowing 6d. in the £ for income tax in each case, he found that his income was reduced £175 10s. by the transfer. What price was paid for the railway stock?

62. A person invested money in 3 % stock at $95\frac{7}{8}$, and derived a net income, after paying income tax at 5d. in the £, of £146 17s. 6d. He afterwards sold the stock at $102\frac{1}{8}$ and invested the proceeds in $2\frac{1}{2}$ % stock, and, the income tax having meanwhile been raised to 6d. in the £, he found that his net income was 12s. 6d. less than before. How much money did he originally invest and at what price did he buy the $2\frac{1}{2}$ % stock? ($\frac{1}{8}$ % brokerage in each sale and purchase.)

63. I invest in $2\frac{3}{4}$ % stock at 110. After 2 years more I sell out at 105. Show that, if brokerage of $\frac{1}{8}$ % be paid on investing and selling out, my capital, with accumulated simple interest, has increased only by about 4s. $6\frac{1}{2}$d. per £100.

64. The 3 per cents. are at $96\frac{7}{8}$ and $3\frac{1}{4}$ per cents. at 105. What amount invested in the former would produce an annual income of £1 more than if invested in the latter, $\frac{1}{8}$ % brokerage being charged on each investment?

65. What sum must be invested in 4 % Spanish stock at $63\frac{1}{2}$ to give an income of £340 4s. 8d.?

66. A man invests a sum of money in 3 % stock at $87\frac{1}{4}$, and, at the end of 3 months, sells out at 90, having in the interval received a half-year's dividend. He then spends 20 guineas in taking a holiday, and finds that the sum remaining for reinvestment (including the dividend) is to the sum originally invested in the ratio of 25 to 24. If the brokerage was 2s. 6d. per £100 on both buying and selling the stock, and if the income tax deducted from the dividend was at the rate of 5d. in the £, how much was originally invested?

67. A person, having bought a certain amount of $2\frac{3}{4}$ % stock at 95, afterwards sold it, and, with the proceeds, bought $3\frac{1}{2}$ % stock. He obtained £900 less stock than before, but his income was unchanged. How much money did he originally invest?

68. A man invested a sum of money partly in 3 % stock at 99 and partly in 4 % stock at 101. When the 4 per cents. had risen to 114 he sold out of this stock, and invested the money realized in $2\frac{1}{2}$ % stock at 76. He thus diminished his income by £12 10s., but increased his amount of stock by 25 %. What was the sum first invested?

69. The difference between the incomes derived from investing a certain sum in 5 % stock at 127 and $5\frac{1}{2}$ % stock at 135 is £4 14s. Find the amount invested and the incomes resulting from each investment.

70. On a certain day $2\frac{1}{2}$ % Consols were at $96\frac{1}{4}$ and Egyptian 3 per cents. at $102\frac{1}{2}$. A man who had a certain sum to invest calculated that, if he invested it in one of these stocks, he would have a larger income by £5 4s. than if he invested it in the other. What sum had he to invest?

71. Investing a certain sum in a railway paying £6 per £100 share at a time when each share cost £125, I find my income £10 14s. 6d. better than when I invested the same amount in 3 % Consols at 95. Determine the amount.

72. A person finds he can obtain £5 more per annum by investing in the $3\frac{1}{2}$ per cents. at 96 than in the 3 per cents. at 88. How much has he to invest?

73. A man has £3,000, which he invests partly in a 3 % stock at 96 and partly in a 6 % stock at 108. He makes $3\frac{1}{2}$ % on the whole investment. What sums does he invest in each stock? (Answer to nearest penny.)

74. A man invests a certain sum in the ordinary stock of a railway at 90, which pays a dividend of $3\frac{1}{2}$ %. At the end of a year he sells out at 86, and invests the proceeds in a 7 % stock at 98, and thus, in the second year, receives £647 10s. more for interest than in the first year. How much did he originally invest, assuming that the prices of the stocks given above include brokerage?

75. Find what amount must be invested in the $2\frac{1}{2}$ per cents. at $95\frac{3}{4}$ to produce a clear income of £100 a year, brokerage being $\frac{1}{8}$ % and income tax being 6d. in the £1.

76. A person sells out of the 3 per cents. and realizes £18,000. Would it be more to his advantage to reinvest this money in the $3\frac{1}{2}$ per cents. at $101\frac{7}{8}$ or in a mining company paying a dividend of $1\frac{3}{4}$ % per annum, of which the £10 shares were quoted at $5\frac{3}{8}$?

77. I wish to transfer £50,000 in the 4 % preference stock at 105 to 5 % debentures at 131. It is calculated that the result of each sale of £5,000 will be to lower the price of the stock, which I am selling by $\frac{1}{16}$ % (*i.e.* the first £5,000 will be sold at 105, the next at $104\frac{15}{16}$, &c.), and that each purchase of £5,000 will raise the price of that which I am buying by $\frac{1}{8}$ %. Find my change of income to the nearest penny, assuming that the charges for brokerage amount altogether to £62.

78. Which is the better stock to invest in, the 5 per cents. at $142\frac{1}{2}$ or the 4 per cents at $113\frac{3}{4}$? What sum of money must a man have so that by investing one-third of it in the former and two-thirds of it in the latter stock, he may obtain an income of £455 13s. 4d.?

79. A man invests £16,213 partly in shares at 105, bearing a dividend (free from income tax) of 8%, and partly in a mortgage at 5 % interest, on which he pays income tax at 8d. in the £. His net income from each investment is the same. What is his whole income?

80. A person holding £2,600 in the 4 per cents. sells out when his stock is at 90 and buys into $3\frac{1}{2}$ per cents. at 80. How is his income altered, and how much $3\frac{1}{2}$ % stock does he hold?

81. If I buy 3 per cents. at $78\frac{3}{8}$ and $3\frac{1}{2}$ per cents. at $95\frac{3}{16}$, which is the better investment? If I invest £6,000 in each, and the former rise and the latter fall $\frac{1}{16}$, how much should I lose or gain if I sell out? (Answer to the nearest shilling.)

82. G.W. $4\frac{1}{2}$ % debentures being at 146 and L. & N.W. 4 % debentures at 133, a man who already held £26,600 of the former stock invests £3,059 in the latter. What is his income from these two sources and what difference would be made in it, if he were to sell his G.W. stock and invest the proceeds in L. & N.W.?

83. A person invests £7,878 in $2\frac{3}{4}$ % Consols at 101, and pays income tax at 8d. in the £. On the stock rising to 103 he sells out, and invests the proceeds in railway stock at 156, yielding 5 %, free of income tax. Find the increase in his net income.

84. I invest £2,820 in Italian 5 per cents. at 94, sell the stock again at 95, and reinvest the proceeds in Spanish 4 per cents. at 57. If I receive a quarter's dividend on the Spanish stock, and afterwards sell it at 58, what sum shall I realize by the dividend and sale together?

85. A man holding a quantity of 4 % stock sells out at 91, and, after deducting £350 from the proceeds of the sale, invests the remainder in $4\frac{1}{2}$ % stock at $87\frac{1}{2}$. If his annual income be thereby increased by 2 % of its former amount, what quantity of 4 % stock did he originally hold?

86. By selling out railway $3\frac{1}{2}$ % stock at 96, and buying tramway $4\frac{1}{4}$ % stock at 108, a man increased his annual income by £25. What amount of stock did he sell?

87. I invest a certain sum, half in a 4% stock at par and half in a 5 % stock at 120. If I had invested one-quarter of the sum in the first stock, and the remainder in the second stock, I should have had £5 more annual income than in the former case. What sum had I to invest?

88. A man invests £10,000 in $2\frac{3}{4}$ % Consols at $111\frac{3}{4}$, inclusive of commission. Find within a penny (1) the amount of Consols he will obtain; (2) the sum he will receive each quarter for interest.

89. A man invests £5,830 in a certain stock at $99\frac{1}{4}$, and an equal sum when the price has risen to 103. He subsequently sells out the whole of this stock, and obtains only £4 more than the whole sum which he invested. Find the price at which he sold out, reckoning brokerage $\frac{1}{8}$ % on each transaction.

90. A man invests in $4\frac{1}{2}$ % stock at $123\frac{1}{4}$. What percentage is he actually receiving upon his outlay after a deduction of 8d. in the £ has been made for income tax? (Answer to nearest penny.)

91. A man sold £45,000 of 3 % debenture stock, and invested the money realized in $2\frac{1}{2}$ % Consols at 103, brokerage of $\frac{1}{8}$ being charged on each transaction. After allowing for the deduction of income tax at 6d. in the £ on the first income and at 8d. in the £ on the second, he found that his net income was reduced by £83 15s. At what price were the debentures sold?

92. A man lays out £12,000 in three investments—viz., $\frac{4}{15}$ths in a business which pays $7\frac{1}{2}$ % per annum, $\frac{5}{8}$ths in a speculation by which he gets $1\frac{1}{5}$ %, and the remainder at such a rate as gives him 3 % on the whole of his money. Find that rate.

93. An investment in 3 % Consols standing at 102 is partially converted into $2\frac{1}{2}$ % Consols at 94. The part which is converted is half as large again as the part which remains in the 3 per cents., and the loss in net income, after the deduction in each case of an income tax at 6d. in the £, is £155. Write down an arithmetical expression for the amount of the original investment, but do not evaluate it.

94. A person possessed of £2,500 invested £348 13s. 9d. in 3 % stock at $99\frac{1}{2}$, £1,651 17s. 6d. in foreign 6 % securities at 110, and the remainder on loan at 5 %. At the end of a year he sold the stock at 97, securities at $102\frac{1}{4}$, and recovered only 75 % of the loan. He paid $\frac{1}{8}$ % brokerage on both purchase and sale of stock and securities, and received a year's dividends on them, as well as interest on the loan. How much was he possessed of at the end of the year?

95. Having invested a sum of money in 3 % Consols at $100\frac{3}{8}$, I held the stock until I had received a half-year's dividend. I then sold it at $99\frac{3}{8}$, and invested the proceeds of the sale, together with the dividend, so as to produce $3\frac{3}{4}$ %, and my income was thus improved by £9 19s. 6d. per annum. If brokerage amounting to $\frac{1}{8}$ % be charged upon sales and purchases, what sum did I originally invest?

96. By selling out of the 3 per cents. when they are $91\frac{1}{8}$, and then buying $3\frac{1}{4}$ % stock, I find that I obtain £1 0s. $1\frac{3}{7}$d. of income for every £1 which I had previously received. Find the price of the $3\frac{1}{4}$ % stock, $\frac{1}{8}$ % brokerage being charged in each transaction.

97. A holder of railway stock receives for one year a 4 % dividend, and spends $\frac{7}{8}$ths of his net income, after paying an income tax of 6d. in the £. The next year, the dividend being $4\frac{3}{8}$ % and the income tax being raised to 8d., he spends only $\frac{4}{5}$ths of his net income, and finds that he has a surplus income of £22 7s. 11d. greater than in the previous year. How much stock did he possess?

98. A man calls in a sum of £10,000 lent on mortgage at 4 %, and invests the money in £3,000 L. & N.W. Railway 4 % debenture stock at 133 (including brokerage) and £3,000 G.W. Railway 5 % stock at 167 (including brokerage). The balance of the £10,000 is placed on deposit at $2\frac{1}{2}$ %. What is the change in the income derived from the £10,000?

99. A and B have each £7,200 to invest. A buys Costa Rica stock, which pays no dividend, at 15; B buys Portuguese 3 per cents. at 48. At the end of a year A sells at $15\frac{3}{4}$; B, having received the year's dividend, sells at $47\frac{1}{2}$. Which had made the better investment?

100. A person intending to purchase enough $2\frac{1}{2}$ % stock to yield an income of £579 15s. per annum, when the stock is at 95, delays till the price has risen to $95\frac{3}{4}$. How much more will he have to pay?

101. A invests in the 3 % stock at par, which he can sell at par any time, and B invests in the 4 per cents. at 110, redeemable at par in 15 years. Which is the better investment? (Reckon simple interest.)

102. How much must a man invest in the $3\frac{1}{2}$ per cents. at 98 to found a scholarship of the annual value of £75?

103. A bill of £10,000, due in two months time, is discounted by a banker at 3 %. The proceeds are invested in the purchase of 3 % stock at a premium of 2 %. Find to the nearest penny the net income after the payment of income tax of 6d. on each complete pound of the gross income. (Brokerage $\frac{1}{8}$.)

104. Interest being at $4\frac{1}{2}$ % per annum, what should be given for a 7% bond redeemable at par at the end of 3 years? (Answer to nearest penny. Use simple interest in each calculation.)

105. The capital of a company consists of £50,000 debentures paying 4%, £40,000 preference shares paying 5%, and £60,000 ordinary shares. The annual profits are £10,100. If the directors decide to put aside £1,600 to the reserve fund, at what rate per cent. can they declare a dividend on the ordinary shares?

106. A man has £7,000 in the $2\frac{3}{4}$ per cents. He sells out when they are quoted at $97\frac{1}{8}$, $\frac{3}{8}$, and with the proceeds purchases 5% stock quoted at 115, $\frac{1}{4}$. If, in the former case, he pays an income tax of 6d., and in the latter 7d., on each complete pound of gross income, find, to the nearest penny, the alteration in his net income. (Brokerage $\frac{1}{8}$).

107. The rent of a house is £60 a year. The owner sells it at 12 years' purchase on the rent, and invests the money he thus obtains in the purchase of £10 shares in a company paying 8% per annum. If the shares are bought at a premium of 10%, what is the change in his income? (Answer to nearest penny.)

Note on Brokerage.

It is customary in Arithmetic text-books and in examination papers to reckon the brokerage at $\frac{1}{8}$ per cent., that is, as 2s. 6d. per £100 of *Stock*. The actual rules of the Stock Exchange are, however, much more complicated, and the $\frac{1}{8}$ per cent. rule applies only to British Government securities, Indian Government stocks, Foreign Government bonds, and the ordinary stocks of railways whose market value is £50 or less. On other stocks, the brokerage is either $\frac{1}{4}$ per cent. or $\frac{1}{2}$ per cent. On Bank of England and Bank of Ireland stocks, it is $\frac{1}{4}$ per cent. on the *price*, and on registered stocks, other than railway ordinary stocks, $\frac{1}{2}$ per cent. on the *price*. On shares, the brokerage may be roughly stated as averaging 1d. for each 10s. in the value of the share. The official scale for country brokers is roughly one-half the London scale.

CHAPTER XXVIII.

EXCHANGES.

218. Suppose a Frenchman living in England has brought with him some of the gold coins of his own country, and starts a banking account in London by paying these coins over the counter: what ought the banker to allow him for them? This depends wholly on the amount of pure gold they contain. If the banker can do nothing else with them, he can, at all events, take them to the Mint and have them recoined into English money. The calculation of what they are worth, regarded merely as precious metal—as "*bullion*"—is an easy one.

Ex.—40 *lbs. troy of British standard gold,* $\frac{11}{12}$ *fine, are coined into* 1869 *sovereigns.* 1 *Kg. of French standard gold,* $\frac{9}{10}$ *fine, is coined into* 155 *napoleons. If* 1 *napoleon* = 20 *francs and* 1 *oz. troy* = 31·103 *g., find in francs and centimes the intrinsic value of* £1, *and in shillings and pence the value of* 1 *napoleon.*

1869 sovereigns contain 480 oz. troy standard gold $= \frac{11}{12} \times 480$ oz. troy fine gold $= 440 \times 31{\cdot}103$ g. fine gold; therefore 1 sovereign contains $\frac{440}{1869} \times 31{\cdot}103$ g. fine gold. (1)

155 napoleons contain 1000 g. French standard gold $= \frac{9}{10} \times 1000$ g. fine gold $= 900$ g. fine gold; therefore 1 napoleon contains $\frac{180}{31}$ g. fine gold. (2)

From (1) and (2) 1 sovereign $= \frac{440}{1869} \times 31{\cdot}103 \times \frac{31}{180}$ napoleons $= 25{\cdot}221$ francs (nearly).

Similarly 1 napoleon = 15s. $10\frac{1}{4}$d. (nearly).

By treating the coinage of the various countries in this way we reach what is called the *Mint Par of Exchange, i.e.* the sum which would be given for the gold coin of one country at the Mint of another country. The following table gives this

result for the most important countries of Europe. It must be clearly understood, however, that it refers *only to the gold coins* of the countries named. For further details reference should be made to *Whitaker's Almanack.*

Country.	Coins.	Approx. Value.	£1 =
Latin Union.			
France	1 *Franc* = 100 *Centimes*	$9\frac{1}{2}$d.	25·22
Belgium			
Switzerland			
Italy	1 *Lira* = 100 *Centesimi*		
Greece	1 *Drachma* = 100 *Lepta*		
Scandinavian Union.			
Denmark	1 *Krone*† = 100 *Öre*	1s. $1\frac{1}{4}$d.	18·2
Norway & Sweden			
United States	1 *Dollar* = 100 *Cents*	4s. $1\frac{1}{4}$d.	4·87
Germany	1 *Mark* = 100 *Pfennige*	$11\frac{3}{4}$d.	20·43
Austria-Hungary	1 *Krone*† = 100 *Heller*	10d.	24·0
Holland	1 *Florin* = 100 *Cents*	1s. 8d.	12·0
Russia	1 *Rouble* = 100 *Kopecks*	3s. 2d.	6·30
India	1 *Rupee* = 16 *Annas*	1s. 4d.	15·00

EXAMPLES CXXX.

Using the table just given (the column headed "Approx. Value" for A and the last column for B) determine—

A. The value in English money of—

1. 4746 francs 25 centimes. **2.** 10900 roubles. **3.** 1856 dollars. **4.** 2146 mark 50 pfennig. **5.** 1000 rupees.

B. The value in foreign money of—

1. £857 13s. 4d. in francs and centimes. **2.** £2450 18s. 6d. in marks, &c. **3.** £1000 in dollars. **4.** £1500 in roubles. **5.** £17 6s. 8d. in kroner and öre.

219. Suppose, now, that an English wine merchant has ordered in a stock of Bordeaux wines and of champagne from

† Note that this word is *krone* in Austria, with plural *kronen*; *krone* in Norway and Denmark, with plural *kroner*; *krona* in Sweden, with plural *kronor*. But in foreign languages the singular is often used where we should expect the plural, just as we say "two *foot* six" for "two *feet* six," or "a ten-*shilling* piece." The student may use an English form for the plurals of these words if he wishes, and say krones, marks, roubles, &c.

a French wine dealer, and that the French dealer's charge is 125000 francs, how is the English merchant to pay him? He could, of course, calculate, as we have just been doing, how much English gold is required to make up this total in French gold. He could then pack up this amount of gold and send it to France. But there are many objections to this. In the first place, the gold required would weigh about 90 lbs. avoirdupois, and the cost of carriage and of insurance of this would be a serious addition to his bill. Then the French merchant would have the trouble of sending this gold to the French Mint, and of waiting until it was coined into French money. To avoid this recourse is had to *Bills of Exchange*. The English merchant will go to one of the foreign bankers or bill-brokers in London and purchase a "bill on Paris" for the requisite amount. This bill will be an order from the bill-broker to his correspondent in Paris to pay to the French merchant, who will present it, the required sum in French money, either "*at sight*," *i.e.* "on demand," or "*at three months*" or "six months' notice." Bills of the first kind belong to what is called "*the Short Exchange*," of other kinds "*the Long Exchange*." The English merchant will, of course, have to pay the bill-broker a little for his trouble in undertaking to transmit the money, and, thus, £1 in London will not purchase a bill for 25 francs 22 centimes on Paris, but rather less, say about 25 francs 15 centimes. But will not the bill-broker have to send the gold to France after all? Is it simply a process of transferring the trouble from one man to another? By no means; for, while the wine merchant is ordering wine from France, the French tradesmen are ordering goods – such as coal, iron, cloth, cottons, &c.—from England. Payment is made by them in bills on London, and the bill-broker and his French correspondent can balance these against the other. If France purchased from England just as much as England purchased from France, the bills would exactly balance at the end of a year, and there would be no gold to be sent from one country to another. In any case, only the balance need be sent. If there is a balance to be sent from France to England, the *balance of trade* is said to be "against France" and "in favour of England."

Let us now look at the question of the charge to be made by the broker upon the English merchant. What determines

this? Regard the broker and his French correspondent as being one firm; then, if this firm has issued more bills on Paris than it has issued on London, it is clear that it is in some danger of having to send gold from London to Paris after all. The cost of transmitting 1 sovereign to Paris averages about 1d. or 10 centimes. £1 gold in London would, therefore, fetch in Paris 25·22 francs, less the cost of carriage, *i.e.* about 25·12 francs. Hence, if the bill-broker thinks it certain that he will have to send gold to Paris to equalize his accounts, he must give for £1 in London a bill for 25·12 only on Paris. He *cannot usually give less* than this; for, if he did, the English merchant would find it cheaper to send the gold direct to the French merchant. This sum is called the *Specie Point.* But there is another specie point; for, if he were to offer more than 25·32, French merchants would begin to send gold across from France to England. Between these two specie points, 25·12 and 25·32, the "Exchange on Paris" may vary from day to day, and even from hour to hour, according as the broker receives more or fewer bills on one city or the other. There are many other circumstances which help to fix the *Rate of Exchange*, for which special books should be consulted. The Rates of Exchange are published in the chief newspapers. Thus, on Tuesday, August 14, 1900, the exchange in London on Paris was 25·17½, on Vienna 24·57, while in Paris the exchange on London was 25·18, and in Vienna 24·27.

It must be carefully noted that these figures refer to the coinage of the country in question. Thus, the figure for Vienna is in *kronen*, not in *francs*.

Since the rate of exchange varies from day to day and from city to city (*e.g.* on August 14, 1900, the rate in London on Paris was 25·17½, but on Marseilles 25·37½), it will often happen that a merchant who has a payment to make to Paris will find it cheaper, instead of buying a bill on Paris, to buy a bill on Berlin, say, send this to his German agent, and direct him to remit the money to Paris by purchasing at Berlin bills on Paris. He thus makes an *Indirect Exchange* between London and Paris, and the rate of exchange between London and Paris calculated *via* Berlin, or other cities, is called the *Arbitrated Rate of Exchange.*

Ex.—*On August* 16, 1900, *the rate in London on Berlin* (*sight*) *was* $20{\cdot}49\frac{1}{2}$, *on Paris was* $25{\cdot}17\frac{1}{2}$; *in Berlin for bills on Paris* 81·35. *I have to pay* 25000 *francs in Paris. Ought I to buy bills on Berlin or on Paris direct? What difference, in English money, is there between the two courses? What is the arbitrated rate between London and Paris?*

Via Berlin—£1 = 20·495 marks, 81·35 marks = 100 francs; therefore 1 mark = $\frac{100}{81{\cdot}35}$ francs; therefore £1 = $\frac{20{\cdot}495 \times 100}{81{\cdot}35}$ francs; therefore 1 franc costs £$\frac{81{\cdot}35}{2049{\cdot}5}$; therefore 25000 francs cost £$\frac{81{\cdot}35 \times 25000}{2049{\cdot}5}$ = £992·3152.

Direct—£1 = 25·175 francs; therefore 1 franc costs £$\frac{1}{25{\cdot}175}$; therefore 25000 francs cost £$\frac{25000}{25{\cdot}175}$ = £993·0487.

Hence it is cheaper to send the money *via* Berlin. The difference is £·7335 = 14s. 8d. The arbitrated rate of exchange is the value of £1 sent *via* Berlin, *i.e.* is $\frac{2049{\cdot}5}{81{\cdot}35}$ francs = 25·19

EXAMPLES CXXXI.

1. Find the number of francs and centimes in £696 16s. 2d. at 24.57 francs for £1.

2. Express one lac (= 100,000) of rupees in English money at 1s. $2\frac{1}{4}$d. per rupee.

3. Change £583 14s. 2d. into Dutch money at 12·08 florins for £1.

4. How many United States dollars at 4s. 2d. per dollar are equivalent to £130 8s. 4d.?

5. If 20·4 marks = £1, find the equivalent in marks and pfennige of £625 13s. 4d.

6. Find the Russian equivalent for £1500 at the rate of 3s. $1\frac{1}{2}$d. for 1 rouble.

7. If 1 mark = $11\frac{3}{4}$d., express 250 mark 80 pfennig in £ s. d.

8. What is the equivalent in £ s. d. of 850 roubles if 40 roubles = 19 ducats, 3 ducats = 31 francs, 25 francs = £1?

9. When £1 = 25·2 francs, 123·5 francs = 100 marks, 4·2 marks = 1 dollar, express 1000 dollars in £ s. d.

10. £6 = 35 rix-dollars, 1 moidore = 27s. How many rix-dollars are there in 400 moidores?

11. What is the arbitrated rate of exchange between London and Vienna through Berlin when £1 = 20·5 marks and 170·3 marks = 100 florins? (Answer, correct to two decimals, in florins per £.)

12. If £1 = 20·4 marks, 81·6 marks = 100 francs, 10 francs = 11 paper lire, how many paper lire are equivalent to £1000?

13. If 19 United States dollars = 80 marks, 16·1 marks = 1 napoleon (= 20 francs), how many francs are equivalent to 3059 United States dollars?

14. A merchant in New York owes 500 dollars to a merchant in London. He draws on Berlin at 1 dollar for 4 marks 27 pfennige, then on Paris at 4 marks for 5 francs, and then on London at 9·6d. per franc. What does the English merchant receive?

15. On leaving England a man changes £100 into French money at the rate of 25·25 francs for £1. He spends 1200 francs in France, and changes the rest into marks at 5 francs for 4 marks. He spends 620 marks in Germany, and, on landing in England, has £21 10s. 10d. left. At what rate did he change his remaining marks into English money? (*Ans.* pence per mark.)

16. Exchange 15000 kroner into £ s. d. at 18·19 kroner per £1, and £576 18s. 3d. into florins at 12 florins $7\frac{1}{2}$ cents per £.

17. A person in Paris owes another in Vienna 6670 Austrian florins. Instead of paying direct he pays through a broker in London, and thereby gains, in English money at current rates, £1 5s. $9\frac{3}{25}$d. The exchange between London and Paris being 25 francs for £1, and between London and Vienna 11·60 florins for £1, find the rate of exchange between Paris and Vienna. (*Ans.* francs per 100 florins.)

18. How many English sovereigns must I have in order that, if I take them to France and change them into francs at an exchange of 25 francs 45 centimes for each sovereign, I can spend as much of them as is equal to £16 while there, and by changing the rest into sovereigns on leaving at an exchange of 25 francs 25 centimes have as many sovereigns as I started with?

19. If a merchant has to remit upon Paris at 25·63 and to draw upon Lisbon at 56, at what rate may he draw upon Lisbon if the Paris rate falls to 25·52?

20. What is the par of exchange between England and Germany when 40 lbs. troy of 22-carat gold are coined into 1869 sovereigns, and 279 marks can be coined from 100 grammes of pure gold, given that 1 oz. troy = 31·1035 grammes? (Answer to two decimals in marks per £.)

21. The United States gold coin of 10 dollars, $\frac{9}{10}$ fine, weighs 258 grs. and 1 oz. of English standard gold, $\frac{11}{12}$ fine, is worth £3 17s. $10\frac{1}{2}$d. How many dollars (to three decimal places) contain the same amount of fine gold as £1?

22. One dollar weighs 412·5 grs. and is $\frac{9}{10}$ fine, and 1 lb. troy of British standard silver, $\frac{37}{40}$ fine, is coined into 66 shillings. Find the number of pence in 1 dollar when British standard silver has a marketable value of 5s. 1d. per oz.

23. If the cheque exchange in London on Paris is 25·30, and the rate of discount for three months' bills in London is at 5 % per annum, what debt in Paris can be discharged by a person in London who holds a three months' bill on London for £1000?

Note on "Chain Rule."

220. It is customary in connexion with this subject to explain an application of the Unitary Method, which is called the Chain Rule. Its use will save a little, but very little, time in some of the questions just given. We will illustrate it by an example.

Ex.—*If* 4 *chickens are worth* 3 *ducks*, 7 *ducks worth* 2 *geese, and* 9 *geese worth* 7 *turkeys, what is the price of a chicken when a turkey costs* 18s. *?*

We may arrange our data thus:

4 chickens = 3 ducks,
7 ducks = 2 geese,
9 geese = 7 turkeys,
1 turkey = 18s.,

and deduce at once 1 chicken $= \dfrac{18 \times 7 \times 2 \times 3}{1 \times 9 \times 7 \times 4}$ shillings = 3s.

This is really only an abbreviated form of "1 chicken is worth $\frac{3}{4}$ of a duck, but 1 duck = $\frac{2}{7}$ goose; therefore 1 chicken = $\frac{2}{7} \times \frac{3}{4}$ goose, &c., &c." But the notation "4 chickens = 3 ducks" for "are worth 3 ducks" is not to be recommended.

EXAMPLES CXXXII.

1. If £1 is equivalent to 25 francs, 36 ducats to 375 francs, and 19 ducats to 40 roubles, find how many roubles are equivalent to £95.

2. If 8 lbs. of coffee cost as much as 5 lbs. of tea, and 7 lbs. of coffee as much as 40 lbs. of sugar, what is the price of each when 1 lb. each of tea, of coffee, and of sugar together cost 4s. $7\frac{1}{2}$d. ?

3. When 52 lbs. of coffee are worth as much as 12 lbs. of tea, 22 lbs. of tea are worth as much as 572 lbs. of sugar, a cask of sugar costs 2 guineas, and 1 cwt. of coffee costs 8 guineas, what is the weight of a cask of sugar ?

4. If £1 be equal to 25·5 francs, and 1 Kg. to 2·2046 lbs. avoirdupois, find, to the nearest penny, the price per hundredweight of a substance 2 Kgs. of which cost 9·5 francs.

5. If £1 = 25·25 francs and £1 = 11·75 gulden, find the value of 354 gulden in francs and centimes, to the nearest centime, 1 franc being 100 centimes.

6. If 6 tons of coal cost £8 11s. and 15 cwts. of coke cost 9s., how many tons of coke are equivalent in price to 16 tons of coal ?

7. If 35 lbs. of tea can be exchanged for 600 lbs. of sugar, 1 cwt. of sugar for 14 lbs. of coffee, find the price of 14 lbs. of the same tea when 10 lbs. of the same coffee cost 11s. 8d.

8. A plot of land is valued at £1936 per acre. What is its worth in francs per square metre? (£1 = $25\frac{1}{7}$ francs, 1 metre = $39\frac{3}{8}$ ins.)

9. When silk is sold at 19·25 francs the metre, find the price per yard in shillings and pence. (£1 = 25·20 francs, 1 metre = 39·371 ins.)

CHAPTER XXIX.

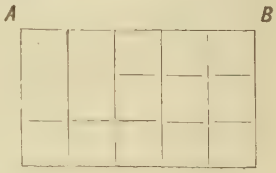
RECTANGULAR AREAS.

221. A **square** is a surface bounded by four equal sides, with its four angles all right angles; *e.g.* the different spaces on a chess-board.

A *square unit* (*i.e.* square inch, square foot, &c.) is a square each of whose sides is one unit (*i.e.* inch, foot, &c.) in length.

The *area* of a surface (*i.e.* the amount of surface included within its bounding lines) is measured by the number of square units which it contains.

A **rectangle** is a four-sided figure with its opposite sides equal and its four angles all right angles; *e.g.* the floor and walls of most rooms, a sheet of note-paper. A square is thus a particular form of a rectangle. The *dimensions* of a rectangle are its length and breadth. The *perimeter* of a rectangle is the sum of its four sides, *i.e.* the distance you would walk if you started from one corner and walked along its four sides till you returned to your starting point.



Let *ABCD* represent a rectangle whose length is 5 units and whose breadth is 3 units. Divide *AB* into 5 equal parts and *AD* into 3 equal parts; then each of these parts will represent the unit of length (inch, foot, &c.). Through the various points of division draw lines parallel to the sides, as in the figure. We thus obtain 15 smaller rectangles, each of whose sides represents one unit of length. The area of the whole is, therefore, 15 square units (square inches, feet, &c.).

If the length and breadth do not contain some chosen unit (say an inch) an exact number of times, *e.g.* length $\frac{5}{6}$ in. breadth $\frac{1}{2}$ in., the above illustration still applies; for we may take as our unit of length $\frac{1}{6}$ in.; then the length and breadth contain 5 and 3 of such units respectively, and the whole would contain 15 square units.

Hence the area of a rectangle is found by multiplying the *number* of units of length in its length by the *number* of units of breadth in its breadth. The product is so many corresponding square units. This may be briefly expressed thus :—

Area of a rectangle = *length* × *breadth*, these two being both expressed in terms of any (the same) linear unit, and the area in terms of the corresponding square unit. This is equivalent to either of the following :—

Length of a rectangle = *area* ÷ *breadth.*

Breadth of a rectangle = *area* ÷ *length.*

Just as a number referring to linear inches, multiplied by a number referring to linear inches produces a number referring to square inches, so a number referring to square inches divided by a number referring to linear inches produces a number referring to linear inches.

A rectangle whose length and breadth are 4 ft. 6 ins. and 3 ft. 4 ins. respectively is sometimes spoken of as a rectangle 4 ft. 6 ins. by 3 ft. 4 ins., or $4' 6'' \times 3' 4''$.

222. Ex. 1.—*What is the area of a rectangle whose length and breadth are* $3\frac{1}{2}$ *yds. and* 2 *ft.* 6 *ins. respectively ?*

We must first express the two dimensions in the same unit, viz. feet.

Length = $3\frac{1}{2}$ yds. = $10\frac{1}{2}$ ft. Breadth = 2 ft. 6 ins. = $2\frac{1}{2}$ ft.

∴ Area = $10\frac{1}{2} \times 2\frac{1}{2}$ sq.ft. = $\frac{21}{2} \times \frac{5}{2} = 26\frac{1}{4}$ sq.ft. = 26 sq.ft. 36 sq.ins.

Ex. 2.—*What is the length of a rectangular enclosure of* $3\frac{1}{2}$ *acs. whose width is* 80 *yds.* 2 *ft.?*

Reduce the acres to square yards and the width to linear yards. The answer is linear yards.

$3\frac{1}{2}$ acs. = $3\frac{1}{2} \times 4840$ sq.yds. Width = $80\frac{2}{3}$ yds.

∴ Required length = $3\frac{1}{2} \times 4840 \div 80\frac{2}{3} = \frac{7}{2} \times \frac{4840}{1} \times \frac{3}{242}$ = 210 yds.

N.B.—Be careful to distinguish between a 10-foot square and 10 square feet.

A 10-*foot square* means a square each of whose sides is 10 ft. long. Its area is therefore 100 sq.ft.

EXAMPLES CXXXIII.

Find the areas of the following rectangles, whose lengths and breadths are respectively :—

1. 7 ft., 5ft. **2.** 17 ft., 3ft.

3. 16 yds., 4 yds. (Answer in square yards.)

4. 3 furls., 25 yds. (Answer in square yards.)

5. 250 yds., 18 ft. (Answer in square yards.)

6. 14 furls., 8 furls. (Answer in acres.)

Express the areas of the following rectangles in square yards, &c. :—

7. 10 ft. 6 in. by 5 ft. **8.** 3 yds. 2 ft. by 2 yds.

9. 4 ft. 11 ins. by 3 ft. **10.** 17 yds. by 4 ft. 6 in.

11. 5 yds. 2 ft. 6 in. by 3 yds. 1 ft. **12.** $7\frac{1}{2}$ yds. by $3\frac{3}{4}$ ft.

Find the acreage of the following rectangular fields, whose lengths and breadths are respectively (answer in acres, roods, square poles, square yards) :—

13. 500 yds., 250 yds. **14.** 70 chains 25 links, 40 chains 45 links.

15. 374 yds., 132 yds. **16.** 55 chains 25 links, 30 chains.

17. Find the breadth of a rectangle whose area is 11 sq.yds. 3 sq.ft. 30 sq.ins. and whose length is 12 yds. 1 ft. 2 ins.

18. A building stands on a square base, whose side is 7 yds. 2 ft. 4 ins. What is the area of the base?

19. What is the length of a rectangle which has a breadth of 5 yds. 11 ins. and an area of 42 sq.yds. 1 sq.ft. 50 sq.ins.?

20. The floor of a room 15 ft. square is covered with carpet at 4s. 6d. per square yard. Find the cost of the carpet.

21. A rectangular playground covers $2\frac{1}{2}$ acres. If the length is 363 ft., what is the number of yards in the width?

22. What will be the cost of paving a court which measures 83 ft. by 20 ft. 3 ins. at 3s. 6d. a square yard?

23. A plot of ground is laid with turfs measuring 18 ins. by 10 ins. The total cost at 7s. per 100 is £2 16s. If the length of the plot is 50 ft., what is its width?

24. How many pieces of matting, each 4 ft. long and 1 ft. 8 ins. broad, will cover the floor of a room 13 ft. 4 ins. long and 12 ft. broad?

25. A strip of building land on the side of a straight road has a depth of 180 ft. It is sold at the rate of £5 5s. per foot of road frontage. What price is this per acre?

26. A building estate is to be let in plots so as to return a ground rent of £65 per acre. What will be the ground rent of a rectangular plot 49 ft. 6 ins. wide and 286 ft. deep?

27. A parish contains 6 mls. of roads of an average breadth of 44 ft. Find the extent of these roads in acres.

28. How many acres are there in a field 160 yds. long and 121 yds. broad?

29. How many plots, each containing 2 rs. 16 sq.pls. 8 sq.yds., are there in 10 acres 36 sq.pls. 15 sq.yds.?

30. How many trees, each standing in $2\frac{1}{4}$ sq.yds. of ground. can be planted out in a space containing 13 acs. 2 rs. 9 sq.pls.?

31. A field is 121 yds. long and 86 yds. broad. What will be its value at £80 an acre?

32. How many square yards in a field 300 ft. square?

33. A plot of ground 22 ft. 6 ins. by 20 ft. has a flower-bed cut out in the centre 9 ft. by 7 ft. 6 ins. What fraction of the whole is occupied by the flower-bed?

34. A carpet costing 3s. 6d. a square yard covers the whole of the floor of a room whose length is 14 ft. If the whole cost of the carpet is £2 9s., what is the width of the room?

35. A paper having a daily circulation of 121000 is published six times a week. If it contains 4 sheets, each measuring 2ft. 6 ins. by 2 ft., how many square miles would 10 weeks' issue cover?

36. How much was paid for painting a wall $16\frac{1}{2}$ ft. long and $12\frac{1}{4}$ ft. broad at 3s. per square yard?

37. A parish map is drawn on the scale of a foot to a mile. Express as the decimal of a square inch the size on this map of a field which contains an acre.

38. A room is 69 ft. 4 ins. long, and, if it were 10 ft. wider than it is, its area would be 364 sq.yds. What is its present width?

39. A certain area contains 33376 sq.ft. $130\frac{3}{8}$ sq.ins.; its length is 217 ft. $3\frac{1}{2}$ ins. What is its breadth?

40. The area of a rectangular enclosure being 33 sq.pls. 1 sq.yd. 6 sq ft. 108 sq. ins., and its length 9 pls. 1 ft. 6 ins., what is its width?

41. Find the area of a closed cubical box whose edge is 8 ins.

42. A rectangular enclosure contains 285 sq.yds. What possible lengths and breadths may it have when each is an integral number of yards?

Carpeting Rooms, &c.

223. Take a rectangular sheet of paper; fold it double along its length, and again fold double the same way. Cut it along the creases into four strips, which will be all equal. Suppose these strips placed with their narrow edges just fastened together so as to form a continuous length of paper. Now the sheet of paper has not been altered in area, but only in shape. Conversely, if a continuous length be cut up into a number of equal lengths, which are then placed side by side, the area remains the same.

Now carpet is usually made in one continuous piece, from which lengths

are cut suited to the size of the floor. These are placed side by side and sewn together. In the process, however, owing to the fitting of the pattern, &c., some is wasted. The carpet is usually sold at so much per linear yard, depending on the quality and the width.

224. Ex. 1.—*How many yards of carpet* 27 *in. wide will cover a room the width of which is* 16 *ft. and the length* $27\frac{1}{2}$ *ft.? Also what will be the cost at* 4s. 6d. *a yard?*

Area of room $= 27\frac{1}{2} \times 16$ sq.ft. $= 27\frac{1}{2} \times 16 \div 9$ sq.yds.

Area of carpet = (Required length in yards $\times \frac{3}{4}$) sq.yds.

$\therefore$ Required length in yards $\times \frac{3}{4} = 27\frac{1}{2} \times 16 \times \frac{1}{9}$.

$\therefore$ Required length in yards $= 27\frac{1}{2} \times 16 \times \frac{1}{9} \div \frac{3}{4}$

$$= \frac{55}{2} \times \frac{16}{1} \times \frac{1}{9} \times \frac{4}{3} = \frac{1760}{27} = 65\frac{5}{27} \text{ yds.}$$

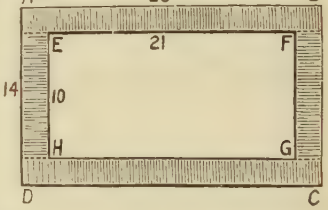
Again, the cost $= 54\text{d.} \times 65\frac{5}{27} = \frac{54}{1} \times \frac{1760}{27} = 3520\text{d.} = £14\ 13\text{s.}\ 4\text{d.}$

We thus see that the required length of carpet = area of room ÷ width of carpet, care being taken to have all measurements in corresponding units, *i.e.* feet and square feet, yards and square yards, &c.

If the cost only is required, it will not always be necessary to work out in full the length, as frequently certain common factors in the numerator and denominator can be removed.

225. The following example and figure should be carefully studied, as a large number of examples of a similar type occur. A figure should be drawn to help in the solution.

Ex. 2.—*Find the expense of paving a pathway* 6 *ft. wide round and immediately outside a court* 21 *yds. long and* 10 *yds. broad at* $9\frac{1}{4}$d. *per square yard.*



The inner portion represents the court; the shaded portion the pathway. The width of the pathway is 2 yds. The length of the line *AB* is found by adding *double* the width of the pathway to 21, *i.e.* 25 yds. So the length of the line *AD* is 10 yds. + *double* the width of the pathway = 14 yds.

Hence the area of the figure *ABCD* is 25×14 sq.yds.

But the area of the inner figure *EFGH* is 21×10 sq.yds.

$\therefore$ The area of the pathway is $25 \times 14 - 21 \times 10 = 140$ sq.yds. The cost at $9\frac{1}{4}$d. per sq.yd. is $9\frac{1}{4}\text{d.} \times 140 = 1295\text{d.} = £5\ 7\text{s.}\ 11\text{d.}$

Similarly, if we had been given the dimensions of the outer figure *ABCD*, we should have obtained those of the inner figure *EFGH* by subtracting from each *double* the width of the shaded part.

EXAMPLES CXXXIV.

Find the cost of carpeting the following rooms:—

1. Length 18 ft. 9 ins., breadth 16 ft., with carpet 30 ins. wide at 4s. a yard.

2. Length 17 ft. 6 ins., breadth 15 ft., with carpet 28 ins. wide at 6s. 3d. a yard.

3. Length $26{\cdot}\dot{6}$ ft., breadth $17{\cdot}1\dot{6}$ ft., with carpet $\frac{3}{4}$ yd. wide at 4s. 6d. a yard. (Answer to nearest penny.)

4. Length and breadth 15 ft. each, with carpet 30 ins. wide at 5s. 6d. a yard.

5. How many yards of carpet 2 ft. 3 ins. wide will be required to cover a floor 22 ft. by 18 ft.?

6. Brussels carpet 21 ins. wide costs 6s. a yard, and Kidderminster 36 ins. wide costs 4s. 6d. a yard. What is the difference in expense in carpeting a room 21 ft. by 14 ft. 8 ins. with the two kinds?

7. What is the cost of carpeting a room 18 ft. 9 ins. long and 17 ft. 6 in. broad with carpet 2 ft. wide at 4s. 9d. a yard? (Answer to nearest penny.)

8. The floor of a room 22 ft. by 15 ft. 9 ins. is covered with carpet 27 ins. wide. How many yards are required? Find the cost at 3s. 6d. a yard, reckoning the odd fraction of a yard as a whole yard.

9. A room measures 25 ft. by 20 ft. The central part is covered with carpet 19 ft. by 14 ft. with a border to it all round 1 ft. wide, the carpet (including border) costing 5s. 6d. a square yard. The rest of the floor is stained at $4\frac{1}{2}$d. a square yard. Find the total cost of carpet and stain.

10. A room is 24 ft. 6 ins. long and 17 ft. broad. Find the cost of covering the floor with carpet $\frac{7}{8}$ yd. wide at 3s. 3d. a yard.

11. A room $15\frac{3}{4}$ ft. long and 11 ft. broad was covered with new carpet 27 ins. wide, costing 4s. 6d. a yard. If the old carpet was sold for 22s. and the cost of laying the new carpet was 3d. a square yard, find the net cost.

12. A floor is $27{\cdot}\dot{3}$ ft. long and $20{\cdot}1\dot{6}$ ft. broad. What will be the cost of covering it with matting $2{\cdot}\dot{4}$ ft. broad at 2s. 6d. a yard?

13. What will the tiling of a rectangular floor cost at 20 guineas per 100 sq.ft., the length of the floor being $2\frac{1}{2}$ pls. and its width 1 pl.?

14. A carpet $19\frac{1}{2} \times 15\frac{3}{4}$ ft., and costing 8s. per square yard, is laid down in a room measuring 23×17 ft., and the rest of the floor is covered by floor-cloth at $5\frac{1}{4}$d. per square foot. Find the total cost correct to the nearest penny.

15. The cost of carpeting a room 18 ft. 6 ins. by 15 ft. 3 ins. with carpet at 4s. 6d. a yard is £9 8s. 1d. What is the width of the carpet?

16. When 54 yds. 2 ft. 6 ins. of carpet exactly cover a floor which is 23 ft. 6 ins. long and 15 ft. 9 ins. broad, what must be the width of the carpet?

17. The carpet in the centre of a room consists of six lengths, each 30 ins. wide, sewn together. The cost of the carpet at 5s. a yard amounts to £9. The rest of the floor is covered by a border of linoleum 18 ins. wide all the way round. Find the cost of the linoleum at 2s. 6d. a square yard.

18. A rectangular court is 120 ft. long and 90 ft. broad; a path of the uniform width of 10 ft. runs round it on the inside. Find the cost of covering the path with flag-stones at 4s. 6d. per square yard; also the cost of covering the remainder of the court with turf at 6s. 6d. per 100 sq.ft.

19. A rectangular plot of grass 60 ft. by 40 ft. has a gravel walk 3 ft. wide all round it on the outside. Find the size of the walk in square feet.

20. A rectangular plot of ground, whose dimensions are 60 ft. by 40 ft., has a grass plot in the centre, surrounded by a gravel path 3 ft. wide. What is the size of the gravel path in square feet?

21. A square recreation ground is bordered by a gravel path 9 ft. wide on the inside, and this path covers 3 acs. Find the extent of the recreation ground in acres and square yards.

22. A rectangular courtyard is 100 ft. long and 60 ft. broad. Two paths cross it at right angles—one from end to end and the other from side to side. Each of these is 5 ft. wide. Find the cost of laying down the remaining area with turf at 6d. per square yard.

23. How many flag-stones, each 4 ft. long and 3 ft. wide, are needed to pave a cloister 12 ft. wide surrounding an inner rectangular court which is 45 yds. long and 42 yds. wide?

24. A rectangular grass plot 45 ft. long and 15 ft. broad is surrounded by a path 5 ft. wide. What is the area of the path?

25. A path 3 ft. 6 ins. wide surrounds a rectangular lawn. The lawn is 270 ft. long and 80 ft. wide. Find the area of the path in square feet.

26. A path $4\frac{1}{2}$ ft. wide is made round a lawn 12 yds. long and 9 yds. wide. How many square yards are there in the path?

Papering Walls.

226. Take a rectangular strip of paper and bend it into the form of a rectangular box without bottom or lid. In this form it may be taken to represent the four walls of a room, the length of the strip representing the perimeter of the room, and the width of the strip representing the height of the room.

Thus, the *area of the four walls* = *perimeter* × *height of room*

= 2(*length* + *breadth*) × *height*.

Wall-paper is usually sold in pieces, each containing 12 yds., and generally 21 ins. in width. Hence problems dealing with the papering of walls may be treated in the same way as those

dealing with the carpeting of floors. Allowance must be made in practice for doors, windows, &c., by subtracting their total area from that of the walls. Further allowance is needed in practice for waste in matching the pattern.

227. Ex.—*What length of paper 21 ins. wide will be required for the walls of a room 18 ft. long, 13 ft. 6 ins. broad, and 11 ft. 9 ins. high, allowing for a fireplace 5 ft. high and 5 ft. broad, a door and a window each 7 ft. by 4 ft.?*

Perimeter of room	$= 2$ (length + breadth) $= 63$ ft.
∴ Area of walls	$= 63 \times 11\frac{3}{4}$ sq.ft. $= 740\frac{1}{4}$ sq.ft.
Area of fireplace	$= 25$ sq.ft.
Area of door and window	$= 7 \times 4 \times 2$ sq.ft. $= 56$ sq ft.
∴ Area to be papered	$= 740\frac{1}{4} - 81 = 659\frac{1}{4}$ sq.ft.
∴ Required length of paper	$= (659\frac{1}{4} \div 1\frac{3}{4})$ ft. $= 376\frac{5}{7}$ ft.
	$= 125\frac{4}{7}$ yds.

This contains between 10 and 11 pieces of 12 yds each. The purchaser would pay for the full 11 pieces.

EXAMPLES CXXXV.

1. A room is 41 ft. long, 32 ft. 4 ins. broad, and 20 ft. high. What will be the difference in expense between papering and painting the walls, the paper being 7d. a square yard and painting 10d. a square yard?

2. Find the cost of painting the walls of a room 10 yds. long, 7 yds. 1 ft. broad, and 15 ft. high at a cost of 1s. 3d. a square yard.

3. A room is 24 ft. long, $16\frac{1}{2}$ ft. broad, and 18 ft. high. Find how many yards of paper 4 ft. wide would be required for the walls, and the cost at 6d. a yard.

4. A room is 20 yds. 1 ft. 6 ins. long, 15 yds. 1 ft. 6 ins. wide, and 17 ft. 6 ins. high. How much will it cost to paper it with paper $\frac{3}{4}$ yd. wide at $2\frac{1}{4}$d. per yard?

5. How many pieces of paper 12 yds. long and 21 ins. wide will be required for the walls of a room 18 ft. long, 16 ft. broad, and 10 ft. 6 ins. high? (Reckon the fractional piece in your answer as a whole.) What will be the cost at 2s. a piece?

6. Find the cost at 3d per square foot of painting the walls of a room 18 ft. long, 9 ft. high, and 15 ft. wide, allowing for a window 5 ft. by 4 ft., two doors each 7 ft. by 3 ft., and a fireplace 4 ft. by 5 ft.

7. A room is 19 ft. 6 ins. long, 16 ft. 8 ins. broad, and 12 ft. high. What is the total area in square feet of the four walls?

8. Find the cost of papering a room 37 ft. 3 ins. long, 18 ft. 4 ins. wide, and 14 ft. high with paper 2 ft. 4 ins. wide at 6d. a yard.

9. A room is 18 ft. 4 ins. long, 13 ft. $6\frac{1}{2}$ ins. broad, and $10\frac{1}{2}$ ft. high. Find the cost of papering it with paper $\frac{3}{4}$ yd. wide at 3d. a yard.

10. What will be the cost of painting and decorating the walls of a room 22 ft. 6 ins. long, 17 ft. 3 ins. wide, and 15 ft. high at 3s. 6d. per square yard?

11. What will be the expense of covering the walls of a room 17 ft. long, 13 ft. wide, and 14 ft. high with paper 2 ft. 6 ins. wide, costing $4\frac{1}{2}$d. a yard?

12. A room 22 ft. 6 in. long and 15 ft. 9 ins. wide is to be covered with floor-cloth $1\frac{1}{2}$ yds. wide. How many yards are required, and what will be the cost at 3s. 6d. per square yard?

13. A room is 14 ft. long, 13 ft. 6 ins. wide, and 7 ft. 8 in. high. What would be the cost of covering the walls with paper $\frac{1}{2}$ yd. wide at 13s. 6d. per dozen yards?

14. Find the cost of varnishing the four walls of a room whose length is 24 ft. $8\frac{1}{2}$ ins., breadth 16 ft. $3\frac{1}{2}$ ins., and height 12 ft. 6ins. at 6d. the square yard.

15. Find the cost of papering the walls of a room 24 ft. 10 ins. long, 13 ft. 5 ins. wide, and 11 ft. 4 ins. high with paper 1 ft. 11 ins. broad, costing $7\frac{3}{4}$d. per yard, an area of 108 sq.ft. being occupied by doors and windows.

16. A room is 20 ft. long, 13 ft. 6 in. broad, and 12 ft. 6 ins. high. In carpeting it a margin of 9 ins. is left all the way round the room, and the paper reaches to 6 in. from the floor. Allowing for a fireplace 5 ft. high by 4 ft. broad, two windows (2 ft. above the floor), each 7 ft. 6 ins. by 3 ft. 6 ins., and a door the same size as a window, find the total cost of the carpet at 3s. 6d. per square yard and the paper at $2\frac{1}{4}$d. per square yard.

17. Find the cost of painting a rectangular room 21 ft. long, 15 ft. wide, and 11 ft. 4 ins. high, the walls and doors being painted at 1s. 3d. and the ceiling at 9d. per square yard, allowance being made for two windows 7 ft. high by 3 ft. wide and a fireplace 6 ft. wide by $4\frac{1}{2}$ ft. high.

18. A room whose length, breadth, and height are 24 ft. 3 ins., 18 ft. 9 ins., and 15 ft. 3 ins. respectively contains a fireplace $6\frac{1}{4}$ ft. by 5 ft. and a door and window each $7\frac{1}{2}$ ft. by $3\frac{3}{4}$ ft. Find the cost of papering the room with paper $\frac{3}{4}$ yd. wide at $4\frac{1}{2}$d. per yard.

19. How many yards of paper 20 ins. wide will be required for the walls of a room 24 ft. long, 18 ft. wide, and $12\frac{1}{2}$ ft. high, the doors and windows occupying 120 sq.ft.?

20. A room 25 ft. 7 ins. long and 16 ft. 9 ins. broad has three windows, each 5 ft. by 3 ft. 6 in., a door 7 ft. by 4 ft., and a fireplace 5 ft. by 6 ft. The cost of papering the walls with paper $\frac{5}{8}$ yd. wide at $4\frac{1}{2}$d. per yard is £3 8s. 10d. Find the height of the room.

21. The cost of painting the four walls of a room 22 ft. long by 17 ft. wide at 4d. per square foot is £16 6s. If 36 sq.ft. be deducted for windows, find the height of the room.

22. Find the cost of painting the four sides and the bottom of a tank $2\frac{1}{2}$ yds. long, 4 ft. wide, and $4\frac{1}{2}$ft. deep at 4d. per square foot.

23. What will it cost to cover with zinc the sides and bottom of a cistern whose length, breadth, and depth are respectively 7 ft. 10 ins., 5 ft. 4 ins., and 1 ft. 9 ins. at 6s. 9d. per square yard?

24. What will it cost to cover with lead an open cistern whose depth, length, and breadth are 3 ft. 6 ins., 7 ft. 10 ins., and 5 ft. 4 ins. respectively at 10s. $1\frac{1}{2}$d. per square yard?

25. How many square feet, &c., of zinc would be required to line a closed box whose internal dimensions are length 3 ft. 6 in., breadth 2 ft., and depth 3 ft. 3 ins?

Miscellaneous Problems on Areas.

228. Ex.—*If a foot-rule by which the sides of a rectangle are measured is too long by ·06 of an inch, find by what decimal of the calculated area the true area will exceed the calculated.*

Suppose the length and breadth of the rectangle contained the *faulty* foot-rule any convenient number of times—say 3 and 2; then the *calculated* area would be 6 sq.ft. or 6×12^2 sq.ins.

The true length of the rule is, however, 12·06 ins.

∴ The true length and true breadth are respectively $12{\cdot}06 \times 3$ ins. and $12{\cdot}06 \times 2$ ins.

∴ The true area is $(12{\cdot}06)^2 \times 6$ sq. ins.

∴ The excess of the true over the calculated area is

$$\{(12{\cdot}06)^2 - 12^2\} \times 6 \text{ sq.ins.}$$

∴ The required ratio is

$$\frac{(12{\cdot}06)^2 - 12^2}{12^2} = \frac{(12{\cdot}06 + 12)(12{\cdot}06 - 12)}{12 \times 12} = \frac{24{\cdot}06 \times {\cdot}06}{12 \times 12} = \frac{4{\cdot}01 \times {\cdot}01}{2 \times 2}$$

$$= {\cdot}010025. \quad \textit{Ans.}$$

EXAMPLES CXXXVI.

1. Find the area of the total surface of a block of wood 2 ft. long, 9 ins. wide, and 1 ft. 1 in. in height.

2. If it cost £49 14s. 6d. to decorate a wall space measuring 69 ft. 4 ins. by 6 ft. 9 ins., what will it cost for one measuring $22\frac{1}{5}$ yds. by $3\frac{1}{3}$ yds., the style of decoration used in the second case being half as expensive again as in the first case?

3. A boy measures the length and breadth of a rectangular room with what he supposes is a yard measure, and from his measurements he calculates the area of the room to be 180 sq.yds. But his "yard measure" is found to be $\frac{1}{2}$ in. short of a true yard. How many square inches is he wrong in his estimate of the area of the room?

4. A district which is in shape a square of 10 mls. each side has been partly mapped to the scale of 6 ins. to the mile and partly to a scale of 4 ins. to the mile. The area of these two portions together is 2500 sq.ins. What area has been mapped to each scale?

5. In a courtyard 67 ft. 6 ins. long by 42 ft. 9 ins. wide there is a footway 5 ft. 6 ins. wide on one longer side of the yard. What is the cost of paving the whole, the price per square yard for the footway being 3s. 6d. and for the remainder 3s.?

6. A 7 ft. strip along two adjacent sides of a rectangular garden forms a border for flowers. If the dimensions of the garden be 42 ft. by 34 ft., what is the exact proportion of the strip to the whole area?

7. Within a rectangular court of length 200 ft. a gravelled path 10 ft. wide is laid down along the four sides, the cost of the gravel being 3d. per square foot. If the path had been twice as wide, the gravel would have cost £81 more. Find the breadth of the court.

8. The map of a district is drawn to a scale of 10 mls. to an inch. What area in the map will represent a lake 200 acres in extent?

9. Find in acres, roods, and poles the area of an L-shaped enclosure the outer edges of which are 330 yds. and 110 yds. respectively and the inner edges 297 yds. and 77 yds.

10. A room is 21 ft. 4 ins. long, 15 ft. 9 ins. broad; the door and windows together occupy 65 sq.ft. The cost of papering the remaining part of the surface of the walls with paper 25 ins. wide at 3s. 9d. per piece of 12 yards is £2 8s. 8d. Find the height of the room.

11. If land is worth £16 5s. 6d. an acre, determine the value of a rectangular field the sides of which are respectively 27 pls. 1 yd. 1 ft. 6 ins. and 14 pls. 3 yds. 2 ft. in length.

12. The interior surface of a room 24 ft. long, 17 ft. broad, and 15 ft. high is subjected to a pressure from the atmosphere of 15 lbs. per square inch. What is the total amount of the pressure?

13. A rectangular room 20 ft. 6 ins. long and 12 ft. 6 ins. wide is carpeted with a certain material. If the room had been 2 ft. longer and 1 ft. 6 in. wider, the additional cost would have been £2 7s. How much did it cost to carpet the room?

14. A room 30 ft. long and 24 ft. wide has the lower part of its walls papered to a height of 15 ft., the upper part being covered by a painted frieze. The paper is 27 ins. wide and costs 6d. a yard, and the painting costs 4d. per square foot. The total cost of papering and painting is £9 4s. $4\frac{1}{2}$d. Assuming that the doors and windows occupy one-sixteenth of the frieze and one-sixth of the lower part of the walls, find the height of the room.

15. How many sods of turf will be required to cover a plot of ground 40 yds. long and 100 ft. wide if each sod is 1 yd. long and 1 ft. broad?

16. A rectangular lawn 51 ft. long is surrounded by a path which is 4 ft. 6 ins. wide. If the path is 96 sq.yds. in area, find the breadth of the lawn.

17. If one side of a rectangle be measured with a foot-rule ·06 in. too long, find by what vulgar fraction of an inch another foot-rule must be too short if, the other side being measured with it, the calculated area is correct.

18. The perimeter of a rectangle is 240 yds; the length is to the breadth as 5 : 3. Find its area in square yards.

The following examples involve the finding of a root (Chap. xxxiii.) :—

19. A square field of $2\frac{1}{2}$ acs. is to be enclosed with iron railings. What length of railings will be required?

20. The sides of two squares contain 77 yds. 1 ft. 9 ins. and 7 yds. 2 ft. 4 ins. Find the side of a square whose area is equal to the sum of the areas of the two squares.

21. Find the length of the side of a square field which is larger by 3 acs. than another square field, the length of a side of which is 91 yds.

22. A railing encloses a rectangular field of 15 acs. The length of the field is to its breadth as 3 : 2. What is the whole length of the railing?

23. A rectangular field which is twice as long as it is wide costs 1·0138s. per square yard to turf. If the whole cost is £191 17s. $0\frac{3}{4}$d., find the lengths of the sides of the field.

24. How far will a man have to walk while ploughing 10 acs. if the distance from the middle of one furrow to the middle of the next be 15 ins.? Do not make any allowance for turning.

25. A square field contains 13 acs. 19 sq.pls. $9\frac{1}{4}$ sq.yds. Find the length of each side in yards.

26. How long will it take to walk round a square field containing 52 acs. 324 sq.yds. at the rate of $3\frac{1}{2}$ mls. an hour? (Answer in minutes and vulgar fraction.)

27. In the centre of a room 21 ft. square there is a square of Turkey carpet, and the rest of the floor is covered with oilcloth. The carpet and oilcloth cost respectively 16s. 6d. and 8s. 6d. per square yard, and the whole cost of the carpet and oilcloth is £35 4s. 6d. Find the width of the oilcloth border.

28. The area of a square field is 10 acs. How long will it take a man to walk round the edge of it at the rate of 3 mls. an hour?

CHAPTER XXX.

VOLUMES OF RECTANGULAR SOLIDS.

229. A **cube** is a body bounded by six equal square faces. Any one of the sides bounding any face is called an *edge* of the cube.

A *cubic unit* (*i.e.* cubic inch, cubic foot, &c.) is a cube each of whose edges is one unit (*i.e.* inch, foot, &c.) of length.

The *volume* of a body (*i.e.* the amount of space included within its bounding faces) is measured by the number of cubic units which it contains. It is sometimes called its *capacity* or *cubical content.*

A **rectangular solid** is a body bounded by six rectangular faces, *e.g.* a box with plane lid. Opposite pairs of faces are equal. A cube is a particular form of a rectangular solid. The *dimensions* of a rectangular solid are its length, breadth, and depth (height or thickness).

230. Suppose we take a rectangular solid whose dimensions are 5, 4, 3 ins. We may divide it up by planes parallel to its faces by a method similar to that in § 53, and show that its volume would be $5 \times 4 \times 3$, or 60 c.ins.

Or we may build up the solid by taking a number of inch cubes of wooden building bricks. Make a layer containing 5 in its length and 4 in its breadth. The whole layer will contain 5×4 or 20 bricks, and the area of its bottom surface will be 20 sq.ins.

On this place another and then another exactly similar layer. We have now that which represents a rectangular solid containing $5 \times 4 \times 3$ or 20×3 or 60 bricks, and its volume is 60 c.ins.

Hence the volume of a rectangular solid is found by multiplying together the three numbers expressing (by the same unit of length) its three dimensions. This may be briefly expressed thus :—

Volume = *length* × *breadth* × *height*, where the last three are all expressed in the same linear unit (inches, feet, &c.) and the volumes in the corresponding cubic unit (cubic inches, cubic feet. &c.).

Since length × breadth = area of one face (see § 221), the above may be written as

Volume = area of one face (or base) × corresponding height.

The following are also equivalent forms:—

$$\textit{Any one dimension} = \frac{\textit{Volume}}{\textit{Product of the other two dimensions}}.$$

$$\textit{Area of one face} = \frac{\textit{Volume}}{\textit{Corresponding height}}.$$

The volume of a cube is found by cubing the number of units of length in any one of its edges.

A rectangular solid whose length, breadth, and depth are 2 ft. 6 ins., 1 ft. 6 ins., and 9 ins. is sometimes spoken of as a solid 2 ft. 6 ins. by 1 ft. 6 ins. by 9 ins. or $2' \ 6'' \times 1' \ 6'' \times 9''$.

231. Ex. 1.—*Find the volume of a block of stone measuring* 10 *ft. by* 3 *ft. by* 1 *ft.* 6 *ins.*

Expressing the dimensions in the same unit, we have required volume in cubic feet $= 10 \times 3 \times 1\frac{1}{2} = 45$ c.ft.

Ex. 2.—*A rectangular room contains* 3240 *c.ft. of air. If the dimensions of the floor are* 18 *ft. and* 15 *ft., what is the height of the room?*

(Required height × 18 × 15) c.ft. = 3240 c.ft.

∴ Required height in feet = 3240 ÷ (18 × 15) = 12 ft.

Ex. 3.—*A gallon of water weighs* 10 *lbs., and* 1000 *oz. measure* 1 *c.ft. How many cubic feet of water will be required to fill a tank which holds* 375 *galls.?*

1 gall. weighs 10 lbs.; therefore 375 galls. weigh 3750 lbs.

∴ *Number* of cubic feet in tank = 3750 lbs. ÷ 1000 oz.

$= 3750 \times 16 \div 1000 = 60$ c.ft.

Ex. 4.—*Determine the number of cubic inches of wood in a box, with a lid, made of wood* $\frac{3}{4}$ *in. thick and measuring externally* 3 *ft. by* 2 *ft. by* 1 *ft.* $5\frac{1}{2}$ *ins.*

By a method similar to that in § 225, we have

Internal length = external length − *twice* the thickness of wood

$= 36 - \frac{3}{4} \times 2 = 34\frac{1}{2}$ ins.

So internal breadth $= 24 - \frac{3}{4} \times 2 = 22\frac{1}{2}$ ins.

And internal depth $= 17\frac{1}{2} - \frac{3}{4} \times 2 = 16$ ins.

But volume of box (by its external measurements) $= 36 \times 24 \times 17\frac{1}{2}$ c.ins.

$= 15120$ c.ins.

And internal volume $= 34\frac{1}{2} \times 22\frac{1}{2} \times 16$ c. ins. $= 12420$ c.ins.

∴ By subtraction we get the required volume of the wood to be 2700 c.ins.

EXAMPLES CXXXVII.

1. A paving stone measures 6 ft. long, 3 ft. 6 ins. wide, and 4 ins. thick. How many cubic feet does it contain?

2. If a cubic foot of stone weighs 249 lbs., what is the weight in pounds of a block of stone 3 ft. 6 ins. long, 2 ft. 6 ins. wide, and 2 ft. thick?

3. Find the value of a stack of wood 18 yds. long, $6\frac{1}{2}$ yds. wide, and 12 yds. high at 8d. the cubic foot.

4. Find the volume of a slab of marble 8 ft. long, 7 ft. 6 ins. wide, and 6 ins. thick.

5. How many cubic feet of air does a room contain whose length is 25 ft. 4 ins., breadth 19 ft. 6 ins., and height 12 ft.?

6. What is the height of a wall which is 50 yds. long, 1 ft. $1\frac{1}{2}$ ins. thick, and contains 15300 bricks, each 9 ins. long, $4\frac{1}{2}$ ins. wide, and 3 ins. thick?

7. A gallon of water weighs 10 lbs., and 1000 oz. of water occupy 1 c.ft. How many gallons of water are there in a tank 12 ft. by 10 ft. by 2 ft. 6 in.?

8. Calculate in tons the weight of water in a rainfall of ·29 in. over an acre of ground, given that 1 c.ft. of water weighs $62\frac{1}{2}$ lbs.

9. Find the weight of water (1000 oz. to a cubic foot) in a half-empty rectangular cistern, whose internal dimensions are 10 ft. 6 ins., 4 ft., and 3 ft.

10. How many gallons of water would cover half a square mile of ground to the depth of 2 ins. if 1 gall. contain $277\frac{1}{4}$ c.ins.? (Answer to nearest integer.)

11. A rectangular tank is 6 ft. long and 4 ft. broad (inside measure), and the depth of water in it is 2 ft. 4 ins. If a cubic foot of water weigh 1000 oz. avoirdupois, find the weight of water in the tank in tons, hundredweights, quarters.

12. If a gallon contains 277·274 c.ins., how many cubic yards are there in 100 bushs.?

13. If a school-room is 25 ft. long and 20 ft. wide, how many children will it accommodate, allowing for each of them 8 superficial feet at the least? And if the room is 10 ft. 4 ins. high, what cubical space is there for each?

14. A tank is 30 ft. 9 ins. long, 16 ft. 7 ins. wide, 6 ft. 4 ins. deep. Find how much water it will hold in cubic feet and cubic inches.

15. Find the total surface of a cube whose edge is 15 ins.

16. If a cubic inch of rain-water weigh $252\frac{1}{2}$ grs., what weight of water falls upon a statute acre for every inch of rainfall? (Answer in tons and pounds.)

17. In a coalfield 500 sq.mls. in area the coal has an average thickness of 60 ft. For how many years would this last, if the annual consumption be 75,000,000 tons and a cubic yard of coal weigh 1 ton?

18. A quadrangle 120 ft. by 100 ft. has a grass plot 60 ft. by 50 ft. in the centre. Find the cost of gravelling the rest of it to a depth of 6 ins. at 6s. a cubic yard.

19. Find the surface in square feet of a rectangular solid $13\frac{1}{3} \times 7\frac{1}{2} \times 5\frac{1}{4}$ ft.

20. A rectangular plot of garden is 100 ft. long and 60 ft. broad. The outer edge of it is covered to a width of 5 ft. with gravel 3 ins. deep, costing 5s. 3d. per cubic yard. Find the total cost of the gravel?

21. A safe 2 ft. 6 ins. long, 2 ft. wide, and 2 ft. 9 ins. high, measured externally, is made of iron 2 ins. thick and weighs $23\frac{1}{4}$ cwts. Determine the height of a safe 2 ft. 10 ins. long, 2 ft. 1 in. wide, made of iron $2\frac{1}{2}$ ins. thick and weighing $35\frac{1}{4}$ cwts.

22. The external dimensions of a box (with a lid) are as follows:—Length 2 ft., breadth $1\frac{1}{2}$ ft., height 1 ft. How many cubic inches of air are there in it if the walls of the box are $1\frac{1}{2}$ ins. thick?

23. Find the cost of excavating a ditch 1800 ft. long, 7 ft. wide, and 5 ft. deep at the rate of 1s. 6d. for every 9 c.yds.

24. How many tons of water are there in a reservoir 30 ft. 6 ins. long, 16 ft. 6 ins. wide, and 20 ft. deep? (A cubic foot of water weighs 1000 oz.)

25. Find in tons what weight of water falls upon $85\frac{1}{3}$ acs. of ground during a rainfall of $1\frac{3}{4}$ ins., a cubic foot of water being taken to weigh 1000 oz.

26. Find in tons, hundredweights, pounds, the weight of water contained in a tank 6 yds. long, 2 yds. high, and 8 ft. broad, the tank being full. (1 c.ft. of water = 1000 oz.)

27. The external length, breadth, and thickness of a closed box are 4 ft., 2 ft., and 16 ins. respectively, and the wood of which it is made is 1 in. thick. Find (1) the number of cubic inches of wood in the box, and (2) the cost of lining it with metal at $4\frac{1}{2}$d. per square foot.

28. What must be the depth of a cistern 8 ft. long and 4 ft. wide which shall contain 400 galls. if a gallon of water be supposed to weigh 10 lbs. and a cubic foot of water 1000 oz.?

29. The water in a cistern 10 ft. square is estimated to weigh 38·95 tons on the supposition that a cubic foot of water weighs 62·32 lbs. What is the depth of the water?

30. A rectangular tank with sides and a bottom 1 in. thick is 21 ft. long, 7 ft. 8 ins. broad, and 11 ft. 1 in. deep when measured externally. How many ounces of water will it contain? (1 c.ft. = 1000 ozs.)

31. If $6\frac{1}{4}$ galls. of water equal 1 c.ft., and water while freezing increases by expansion ·089 of its bulk, how many gallons of water are required to cover a rink 110 yds. long and 96 yds. wide with ice $5\frac{1}{2}$ ins. thick?

32. If 2 c.ins. of mercury weigh 1 lb. avoirdupois, and if 100 c.ins. of air weigh 31 grs., find how many miles in height a column of air will be which has the same weight as, and stands on a base of the same area as, a column of mercury 29·388 ins. high.

33. A rectangular cistern $10\frac{1}{8}$ ft. in length, $6\frac{1}{4}$ ft. in breadth, and $3\frac{1}{2}$ ft. in depth contains $140\frac{5}{8}$ c.ft. of water. What is the least number of bricks each 9 ins. long by $4\frac{1}{2}$ ins. wide by 3 ins. thick that must be thrown into the cistern to make the water rise to the top, a brick being found to absorb water to the extent of one-fifth of its volume?

34. Find the cost of making a road $\frac{1}{2}$ ml. long, 36 ft. wide, the soil being first excavated to a depth of 1 ft. at a cost of 1s. per cubic yard, rubble being then laid in 9 ins. deep at a cost of 1s. 6d. per cubic yard, and 3 ins. of gravel at 3s. 3d. per cubic yard being laid on the top, and the whole consolidated by a steam-roller at a cost of 2d. per square yard.

35. A room is 14 ft. 8 ins. by 10 ft. 6 ins. by 12 ft. If each cubic inch of air in it has a mass of ·310117 gr., calculate correctly to two places of decimals the number of pounds (Av.) of air in the room.

The following examples involve the extraction of a root (Chap. xxxiii.):—

36. The surface of a cube is 86·64 sq.ft. What is the length of an edge?

37. A reservoir contains 3217428 c.ft. of water; its depth is one-third of its length, and its breadth is half the difference between the length and one-third of the depth. Find the dimensions.

38. Determine in feet the depth of a tank capable of holding 10977·12 galls., the length and breadth of the tank being each seven times the depth. (A gallon contains 277·2 c.ins.)

39. An ordinary wooden match-box contains 70 matches. The section of each match is a square; 5 matches are placed side by side, forming a rectangle $\frac{1}{2}$ in. wide and 2 ins. long. Find the internal cubical contents of the box, supposing one-eighth of the whole space inside is wasted.

CHAPTER XXXI.

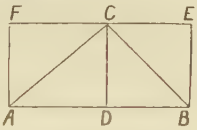
AREAS AND VOLUMES OF SIMPLE FIGURES.

232. We give here formulæ for the areas and volumes of some other simple figures. For further details reference must be made to Briggs' and Edmondson's "Mensuration."

Triangle.—By the *altitude* or *height* of a triangle is meant any one of the three lines which can be drawn from the three vertices perpendicular to the opposite sides. The side to which the altitude is drawn may be called the *base.*

Area of the $\triangle ABC$ is half the rectangle on the same base AB and of the same height.

$= \frac{1}{2}$ **base** $AB \times$ **height** CD.



Ex. 1.—Thus, if AB = 6 ft. and CD 4 ft., then area = 12 sq.ft.

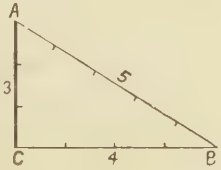
A **right-angled triangle** is one which contains a right angle.

The side opposite the right angle is called the **hypotenuse.** In Euclid, Book I., Prop. 47, it is proved that the square on the hypotenuse AB is equal to the sum of the squares on the two sides AC and BC.

Ex. 2.—Thus, if AC be 3 ft. and BC be 4 ft., the square on AC would contain 9 sq.ft., and the square on BC would contain 16 sq.ft.

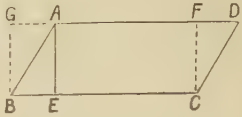
$\therefore$ The square on AB would contain 25 sq.ft.

$\therefore$ The length of AB would be 5 linear feet.



A **parallelogram** is a plane four-sided figure with its opposite sides parallel. It can be proved that the opposite sides are also equal. When the angles of the parallelogram are right angles, the figure becomes a rectangle.

The *area of a parallelogram* $ABCD$ = area of a rectangle $GBCF$ on the same base BC = **side** $BC \times$ **height** BG *or* AE.



A **rhombus** is a parallelogram with all four sides equal

Ex. 3.—If two opposite sides of a parallelogram be each 7 ft. and their perpendicular distance from one another be 5 ft., the area of the parallelogram will be 35 sq.ft.

233. **A Circle.**—The ratio of the circumference of any circle to its diameter is a constant ratio, and is denoted by the Greek letter π. Its exact value cannot be determined; but $\frac{22}{7}$, $\frac{355}{113}$, and 3·1416 are three close approximations, the second being the closest.

Denoting the diameter by d and the radius by r, we have circumference $\div d = \pi$ or *circumference* $= \boldsymbol{\pi \times d}$ *or* $\mathbf{2\pi \times r}$.

Ex. 4.—The diameter of a halfpenny is 1 in.; therefore its circumference contains very nearly $\frac{22}{7}$ or $3\frac{1}{7}$ ins.

The area of a circle $= \boldsymbol{\pi . r^2}$. The following are equivalent forms:—

$$\text{Area of a circle} = \pi \times \frac{d^2}{4} = \frac{3{\cdot}1416}{4} \times d^2 = \text{square of diameter} \times {\cdot}7854.$$

$$\text{Area of a circle} = \frac{2\pi r \times r}{2} = \tfrac{1}{2}\ \text{circumference} \times \text{radius}.$$

Ex. 5.—*Find the radius of a circle whose circumference is* 33 *ins.*

By the above, circumference $= 2\pi . r$; therefore $2\pi . r = 33$; therefore

$$r = 33 \div 2\pi = \tfrac{33}{1} \times \tfrac{7}{44} = \frac{3 \times 7}{4} = 5\tfrac{1}{4}\ \textit{ins.}$$

234. **Cylinder.**—Take a rectangular sheet of paper and roll it round till two opposite sides meet edge to edge. The other two will become the circumferences of two equal circles. The figure thus formed is a thin hollow cylinder open at both ends. We may conceive, therefore, of the curved surface of a cylinder as capable of being unrolled so as to assume the shape of a rectangle, the circumferences of its two ends becoming straight lines.

Therefore *area of the curved surface of a cylinder* = area of the rectangle formed by unrolling it = *circumference of base* × *height* = $\mathbf{2\pi.r.h}$, where r is the radius of the circular ends and h is the height of the cylinder.

The *whole* area of a cylinder (including its two circular ends) is $2\pi rh+2\pi r^2$.

The *volume of a cylinder* = *area of base* × *height* = $\boldsymbol{\pi.r^2.h}$.

Ex. 6.—*A hollow cylindrical tube open at both ends is made of iron* $\frac{1}{2}$ *in. thick. If the internal diameter be* 5 *ins. and the length of the tube be* 8 *ft., find the number of cubic inches of iron in it.*

Internal diameter = 5 ins. ; therefore internal radius = $2\frac{1}{2}$ ins.

∴ External radius = 3 ins.

∴ Required volume of iron = difference in volume between two cylinders of same height (or length, 96 ins.) having radii 3 and $2\frac{1}{2}$ ins. respectively $= \pi.96.(3)^2-\pi.96.(2\frac{1}{2})^2$

$$= \pi.96\left\{(3)^2-(2\tfrac{1}{2})^2\right\} = \tfrac{22}{7}\times\tfrac{96}{1}\times\tfrac{11}{4} = \tfrac{5808}{7} = 829\tfrac{5}{7} \text{ c.ins.}$$

Sphere.—The *area of the surface of a sphere* = $\mathbf{4\pi.r^2}$, where r is the radius.

The *volume of a sphere* = $\frac{4}{3}\pi.r^3$, or $\frac{4}{3}.\pi.\left(\frac{d}{2}\right)^3 = \frac{1}{6}.\pi d^3$,

where d is the diameter.

Cone.—The *area of the curved surface of a cone* = $\boldsymbol{\pi.r.l}$, where r is the radius of its base and l is the slant height.

The *area of the plane circular base is* $\boldsymbol{\pi.r^2}$.

The *volume of a cone* = $\frac{1}{3}$ × *area of base* × *height*

= $\frac{1}{3}$ *of cylinder of same base and height*

$= \frac{1}{3}\pi.r^2.h$.

235. *The areas of similar figures* (such as squares) *are in the same ratio as the squares on their corresponding sides.*

Ex. 7.—*A certain district has been mapped on the scale of 6 ins. to a mile. What is the actual area in acres of a field which is represented on the map by $\frac{1}{2}$ sq.in.?*

$$\frac{\text{Area of field}}{\text{Area on map}} = \left(\frac{1 \text{ ml.}}{6 \text{ ins.}}\right)^2 = (5280 \times 2)^2.$$

$\therefore$ Area of field $= \frac{1}{2} \times (5280 \times 2)^2$ sq.ins. $= 8\frac{8}{9}$ acs.

236. So *the volumes of similar solids* (such as cubes and spheres) *are in the same ratio as the cubes on their corresponding sides (radii, &c.).*

Ex. 8.—*In a cubical box whose edge is 1 ft. how many smaller cubes can be neatly placed, the edge of each smaller cube being 2 ins.?*

$$\frac{\text{Volume of large cube}}{\text{Volume of small cube}} = \left(\frac{\text{Edge of large cube}}{\text{Edge of small cube}}\right)^3.$$

$\therefore$ Required number $= \left(\frac{1 \text{ ft.}}{2 \text{ ins.}}\right)^3 = (6)^3 = 216.$

EXAMPLES CXXXVIII.

Throughout these examples π is taken as $\frac{22}{7}$, except in Ex. 27. Examples marked with an asterisk involve the extraction of a root. (See Chap. xxxiii.)

1. Find the area of a triangular field one of whose sides is 150 yds. and the corresponding altitude is 80 yds.

2.* A rectangle whose sides are respectively 30 ins. and 40 ins. is divided by its diagonal into two equal parts. Find the area of each part and the length of the diagonal.

3.* The two sides containing the right angle in a right-angled triangle are 24 ft. and 7 ft. Find the area of the triangle, and the length of the hypotenuse.

4.* The area of a rectangular field is 8 acs. 2 rs. 16 pls. 30 yds. 108 ins., and the length of each of the longer sides is 1 furl. 2 pls. 4 yds. 2 ft. Find the length of (1) its shorter sides, (2) its diagonal.

5. In measuring a four-sided field it is found that the distance between two opposite corners is 7 chains 50 links, and the perpendicular distances of the two other corners from this line are 3 chains 43 links and 5 chains 32 links respectively. Find the area of the field in acres, roods, square poles.

6. The two longer sides of a field in the form of a parallelogram are each 140 yds. 2 ft., and the perpendicular distance between them is 81 yds. What is the area of the field in acres and square yards?

7. The perimeter of a rhombus is 440 yds., and a pair of opposite sides are 88 yds. apart. How many acres does the rhombus enclose?

8. A hollow cylinder of thin material when unrolled assumes the form of a rectangle 60 ft. by 44 ft. Find the number of square feet in the curved surface of the cylinder. If the height of the cylinder is 60 ft., what is the radius of its base?

9. What is the area in square yards of a circle whose diameter is 63 ft.?

10. Taking the circumference of the earth (supposed a perfect sphere) as 25000 mls., what is its diameter in miles?

11. A wheel 30 ins. high is found to have made during a journey 67802 revolutions. Find the length of the journey in miles and vulgar fraction of a mile.

12. If the fore wheel of a carriage is 7 ft. and the hind wheel is 10 ft. in circumference, how far will the carriage have gone when the fore wheel has made 1584 revolutions more than the hind wheel?

13. A road when measured by a chain 100 ft. long is found to contain exactly 68 lengths less than when measured by a chain 66 ft. long. Find the length of the road in miles.

14. The driving wheels of a locomotive are 7 ft. 8 ins. in diameter, the smaller wheels 3 ft. 8 ins. in diameter, and one spoke in each wheel is painted red. At the start all the painted spokes are upright and above the axle. How many more times in the first 5 mls. will they all be upright and above the axle at the same instant?

15. Find the curved surface of a cylinder whose height is 20 ft. and whose circumference is $7\frac{3}{5}$ ft.

16. The water from a roof 20 ft. by 11 ft. flows into a cylindrical tub of sectional area $7\frac{1}{14}$ sq.ft. What increase in depth (in inches) of the water in the tub will be due to this source with a rainfall of ·12 in.

17. How many cubic feet of earth must be dug out to sink a well 21 ft. deep and 4 ft. in diameter?

18. Taking the earth as a complete sphere of 4000 miles radius, find the number of cubic miles it contains and also the number of square miles in its surface.

19.* A spherical ball of lead 3 ins. in diameter is melted and recast into three spherical balls. The diameters of two of these are $1\frac{1}{2}$ ins. and 2 ins. respectively. What is the diameter of the other?

20.* Find the diagonal of a rectangular solid whose dimensions are 6 ins., 8 ins., 10 ins.

21. A cylindrical vessel whose base is 6 ins. in diameter holds 1 gall. of water. Taking a gallon of water to occupy 277·2 c.ins., what is the height of the vessel in inches?

22.* A cylindrical iron rod 8 ft. long weighs 15 lbs. 6 oz. What is its thickness, reckoning a cubic inch of iron to weigh 5·8 oz.?

23. A cylindrical tank 20 ft. in diameter contains 23495 galls. of water. What is the depth if 11 galls. of water occupy 3050 c.ins.? (Answer to nearest inch.)

24. Find (correct to the tenth of an ounce) the weight of a hollow ball of copper 3·9 ft. in external diameter and $1\frac{4}{5}$ in. thick, assuming that a ton of copper contains 3 c.ft. 1693·$\dot{7}8\dot{2}\dot{1}$ c.ins.

25.* The volumes of spheres are proportional to the cubes of their radii. Two spheres of the same material weigh 8 lbs. and 6 lbs., and the radius of the smaller is 2 ins. If the two were melted down and formed into a single sphere, what would be its radius in inches correct to two decimal places?

26.* The volume of a sphere being ·5236 of the cube of its diameter, find the diameter of a sphere of which the volume is 8·18125 c.ft.

27. A piece of lead in the shape of a peg-top (without the peg) consists of a hemisphere and a cone, the greatest length (15·25 cms.) being twice the greatest breadth. Calculate its weight to the nearest gramme, given that (1) the lead is 11·19 times as heavy as an equal volume of water, (2) the volume of a sphere is $\frac{4\pi}{3}$ × cube of radius, (3) the volume of a cone is $\frac{\pi}{3}$ × product of height and square of radius of base, (4) $\pi = 3{\cdot}1416$ approximately, (5) 1 cubic centimetre of water weighs 1 g.

28. The volume of timber in the trunk of a tree is roughly estimated by multiplying the length by the square of the mean quarter-girth. The trunk of a certain tree, if cut so as to be 27 ft. in length, would have a mean girth of 10 ft. 6 ins., and, if it were cut 10 ft. longer, the mean girth would be 9 ft. 9 ins. In the former case the value would be estimated at £5 10s. 3d. What would it be in the latter?

29. Find the number of cubic inches of metal used in making a cylindrical pipe, having an external diameter of 1 ft. 10 ins., with metal 1 in. thick, if the length of the pipe is 4 ft. 2 ins.

30. Compare the volume of the Moon with that of the Earth if the diameter of the former be to the diameter of the latter as 27 to 100.

31. The plan of a field is in the form of a triangle whose base is 4 ins. and altitude 2 ins. Find the acreage of the field if the scale of the plan is 2 ins. to 1 furl.

32.* Find the diameter of a sphere whose volume is equal to the sum of the volumes of three spheres having diameters 3 ins., 4 ins., 5 ins. respectively.

SECTION IX

PROBLEMS.

237. We proceed to discuss various types of arithmetical problems.

Problems on Work.

Ex. 1.—*A man can dig a piece of land in 2 days, a boy in $3\frac{1}{2}$ days. How long will they take if they work together, each at the same rate as before?*

The man can do whole work in 2 days; ∴ can do half of it in 1 day.

The boy can do whole work in $3\frac{1}{2}$ days; ∴ can do two-sevenths in 1 day.

Hence, by addition, the man and boy together can do

$\frac{1}{2}+\frac{2}{7}$ of it in 1 day, *i.e.* $\frac{11}{14}$ of it in 1 day, *i.e.* 11 times the whole in 14 days, *i.e.* the whole in $\frac{14}{11}$ or $1\frac{3}{11}$ days.

∴ req. time $= 1\frac{3}{11}$ days. *Ans.*

In such examples, where we are dealing with the work done by different agents, we must first reduce to one common standard of time (one day, hour, &c.) before attempting to add or subtract.

Ex. 2.—*A and B together can do a piece of work in 6 days, B and C together in 10 days, C and A together in $7\frac{1}{2}$ days. How long will each require separately to do the same work?*

We have A and B do $\frac{1}{6}$ of the work in 1 day;
B and C do $\frac{1}{10}$ of the work in 1 day;
C and A do $\frac{2}{15}$ of the work in 1 day.

Therefore, by addition, twice the work done by A, B, C together in 1 day is $\frac{1}{6}+\frac{1}{10}+\frac{2}{15}$ or $\frac{2}{5}$ of the whole.

Therefore A, B, C do $\frac{1}{5}$ of the work in 1 day.

But B and C do $\frac{1}{10}$ of the work in 1 day.

Therefore, by subtraction, A does $\frac{1}{5}-\frac{1}{10}$ or $\frac{1}{10}$ of the work in 1 day;
so B does $\frac{1}{5}-\frac{2}{15}$ or $\frac{1}{15}$ of the work in 1 day,
and C does $\frac{1}{5}-\frac{1}{6}$ or $\frac{1}{30}$ of the work in 1 day.

Therefore A, B, C do whole work in 10, 15, 30 days respectively.

Ex. 3.—*If* 25 *men with* 10 *boys can do in* 6 *days as much as* 21 *men with* 30 *boys can do in* 5 *days, compare the rates of working of a man and a boy; also, how many boys must help* 40 *men to do the same work in* 4 *days?*

Reducing to the standard of 1 day, we get

25 men with 10 boys can do in 6 days the same as 6 times (25 men with 10 boys) can do in 1 day.

So 21 men with 30 boys can do in 5 days the same as 5 times (21 men with 30 boys) can do in 1 day.

Therefore the work of 150 men + work of 60 boys = the work of 105 men + work of 150 boys.

Therefore the work of $(150-105)$, *i.e.* 45 men = the work of $(150-60)$, *i.e.* 90 boys.

Or, the work of 1 man = the work of 2 boys.

Again expressing the work of men and of boys in terms of boys' work, we have 25 men with 10 boys do as much as $25 \times 2 + 10$ or 60 boys.

The question now tells us that

in 6 days the work could be done by 60 boys;
therefore in 1 day the work could be done by 360 boys;
therefore in 4 days the work could be done by 90 boys.

But 40 men (equivalent to 80 boys) are already engaged; therefore 10 boys more are required.

EXAMPLES CXXXIX.

1. A can do a piece of work in 3 days, and B in 5 days. How long will they take to do the same when working together?

2. A and B together can mow a field in 10 hours, but by himself A requires 15 hours. How long would B take to mow the same field?

3. A and B together can perform a certain piece of work in 24 hours, A and C together in 30 hours, B and C together in 40 hours. In what time will each be able to perform it when working separately?

4. A can do a piece of work in $10\frac{2}{3}$ days, B in $9\frac{3}{5}$ days, C in $5\frac{9}{11}$ days. In how many days can A, B, and C, working together, complete the work?

5. A, B, and C can build a certain wall in 16, 20, and 24 days respectively. In how long a time would A and B together build a similar wall of double the length if C helped them for one-half of each day?

6. A and B working together can do a piece of work in $7\frac{1}{2}$ days, B and C working together can do it in 12 days, and C alone can do it in 30 days. How long will it take A, B, and C if they all work at it together? How long would A require working alone?

7. Three men together fill a pit in $1\frac{1}{2}$ hours, throwing in respectively 8, 10, and 12 equal shovelfuls of earth per minute. In what time could each man have filled up the pit by himself?

8. A's rate of working is to B's as 4 to 3, and B's is to C's as 2 to 1. How long will it take C to do what A would do in 6 days?

9. A, B, and C together can do a piece of work in 3 days; A could do it by himself in 5 days, or B could do it by himself in 12 days. In how many days would C do it by himself?

10. If a man, a woman, and a boy can do a piece of work in 3, 4, and 6 days respectively, what fraction of the work would remain unfinished after one man, one woman, and two boys had been working at it for a day?

11. A can do a piece of work in 2 days, which B can do in $3\frac{1}{2}$ days. If A's wages are £1 18s. 6d. a week, and B's £1 7s. 6d. a week, what would A receive for doing a piece of work which B would do for £5 10s.?

12. If 5 men complete all but 156 yards of an embankment, and 7 men complete all but 50 yards in the same time, find the length of the embankment.

13. The wages of one man are 3s. 9d. a day and of another 2s. 3d.; a piece of work costs £3 15s. if the first man does it, and £3 7s. 6d. if the other does it; what would it cost if they did it together?

14. If 3 men and 5 boys can do as much in 17 days as 5 men and 3 boys can do in 15 days, compare the rates of working of a man and a boy.

15. A and B together could do a piece of work in 40 days; after working for 10 days they are assisted by C, and the work is then finished in 20 days more; C does as much in 2 days as B does in 3 days. In how many days could each of them do the work alone?

16. A and B together can do a piece of work in $13\frac{7}{11}$ hours, B and C together in $11\frac{1}{9}$ hours, C and A together in 12 hours. In how many hours could they separately do the same work?

17. Eight men with 2 boys can do a piece of work in 7 days, which 25 men with 13 boys can do in 2 days. What fraction of the whole can be done in 1 day by 11 men with 20 boys?

18. A and B could separately mow a field in 6 and 5 days respectively. If they work alternate days at it, A beginning, which of them will finish the work, and what fraction of a day's work will he do on this last day?

19. A can do a piece of work in 6 days, B in 8 days, and C in 12 days. B and C work together for 2 days, and then C is replaced by A. Find when the work will be finished.

20. A alone can reap a certain field in 15 days and B in 12 days. If A begins alone, and after a certain interval B joins him, the field is reaped in $7\frac{1}{2}$ days. How long were A and B working together?

21. Four men and 10 boys together do one half of a piece of work in 6 days; two more men and two more boys having joined them, another third of the work is done in the next three days. How many more boys must be put on, if the rest of the work is to be finished in one day more?

22. Three men are employed to dig an acre of land. A can dig 38 perches in 6 days, B can dig 60 perches in 8 days; A, B, and C together can dig the whole in 7 days. What are the lowest three integral numbers expressing their relative working powers, and in what time would A and C do five-sixths of the work?

23. A piece of work is done by three men A, B, C in 5 days in the following manner:—A works the whole time, B only on the first and second days, C only on the third, fourth, and fifth days. The work might also have been done by B and C working together for 6 days without the assistance of A. If B and C working together for two days can do as much work as A can do alone in three days, find how long it would take A, B, and C each to do the work separately.

Problems on Supply Pipes.

238. Ex. 1.—*Two supply pipes can fill a cistern in* 10 *and* 12 *minutes respectively, and a waste pipe at the bottom can empty* 5 *gallons a minute. If it is empty at first, and all three pipes are opened, it fills in* $7\frac{1}{2}$ *minutes. How many gallons does it hold?*

The two supply pipes together can in one minute fill $\frac{1}{10}+\frac{1}{12}$ or $\frac{11}{60}$ of the cistern; but when all three pipes are in action $\frac{1}{7\frac{1}{2}}$ or $\frac{2}{15}$ of the cistern only is filled in one minute; therefore the waste pipe can empty $\frac{11}{60}-\frac{2}{15}$ or $\frac{1}{20}$ of the cistern in one minute. But the waste pipe can empty 5 galls. in one minute.

Therefore $\frac{1}{20}$ of the cistern contains 5 galls.

Therefore the whole cistern contains 100 galls.

Ex. 2.—*A rectangular tank* 9 *ft. by* 5 *ft. by* 2 *ft. has a supply pipe pouring in* 576 *c.ins. of water in a minute, and another pipe which can empty the whole tank in* 3 *hours. If the tank is full, and both pipes are open, how many hours will pass before the tank is empty?*

The cubical contents of the tank are $9\times5\times2\times1728$ c.ins.

Therefore the number of cubic inches of water which can pass through the waste pipe in one minute is

$$9\times5\times2\times1728\div(3\times60)=864.$$

Thus the waste pipe empties 864 c.ins. of water in one minute, and the supply pipe pours in 576 c.ins. of water in one minute.

Therefore the combined effect of the two is that 288 c.ins. are emptied in one minute.

Therefore number of minutes for emptying the whole is

$$9\times5\times2\times1728\div288=9\times5\times2\times6.$$

Therefore required number of hours is

$$9\times5\times2\times6\div60=9 \text{ hours.}$$

EXAMPLES CXL.

1. How long will two pipes together take to fill a cistern which they can separately fill in 20 and 25 minutes?

2. A vessel can be filled by one pipe in 12 minutes and emptied by another in 8 minutes. If the vessel be full, how long before it is empty, when both pipes are open?

3. Pipes A and B can fill a cistern in 3 minutes and 5 minutes respectively, and C can empty it in $7\frac{1}{2}$ minutes. In what time will the cistern be filled, when A, B, C are all open?

4. Two taps can separately fill a cistern, when the waste-pipe is closed, in 10 and 12 minutes respectively, and when the waste-pipe is open they can together fill it in 15 minutes. How long does it take the waste-pipe to empty the cistern, when the taps are not running?

5. A cistern can be emptied by three pipes in 3 hours; one pipe alone would empty it in 6 hours, another in 9 hours. How long would the third pipe take?

6. Two pipes, A and B, can fill a cistern separately in 12 and 20 minutes respectively, and a waste pipe C can carry off 15 gallons per minute; if all the pipes are opened, the cistern fills in 2 hours. How many gallons does it hold?

7. A cistern can be filled by a pipe A in 12 minutes; but, if an emptying tap B be open all the time, it takes 18 minutes to fill the cistern. If B be closed at first, but opened after A has run for 10 minutes, what is the whole time in which the cistern will be filled?

8. Of three pipes A, B, and C, A fills a cubic inch in a second, B a cubic foot in a minute, C a cubic yard in an hour. If all were running together, in what time would they fill 1069 c.ins.?

9. A cistern half full of water is being emptied by two pipes, which separately could empty the whole in 36 and 40 minutes respectively. What must be the filling capacity of a third pipe so as to fill the cistern in 20 minutes, if it be opened after the first two have been in simultaneous operation for 5 minutes, all three then remaining open?

10. A cistern can be filled by one of two pipes in 30 minutes, and by the other in 36 minutes; they are both opened together for a certain time, but, being partially clogged, only $\frac{5}{6}$ of the full quantity of water flows through the former and only $\frac{9}{10}$ through the latter. The obstructions, however, being suddenly removed, the cistern is filled in $15\frac{1}{2}$ minutes from that moment. How long was it before the full flow of water began?

11. The diameter of the second of three pipes is to that of the first in the ratio of 4 to 3. Find the ratio of the diameter of the third to that of the second, if the water which will flow through all three in 9 minutes is 50 times as much as will flow through the first alone in 1 minute, it being assumed that the quantity of water flowing through any pipe varies as the square of its diameter. (See Ex. 3, § 244.)

12. A tank 9 yds. long, $3{\cdot}0\dot{3}$ yds. wide, and $2{\cdot}05\dot{3}$ ft. deep, holds water enough for 27 men for 308 days. Find the depth of a tank $6{\cdot}2$ yds. long and $2{\cdot}0\dot{2}$ yds. wide, which will hold enough for 158 men, 48 women, and 81 children for 45 days, if a woman uses $\frac{3}{4}$, and a child $\frac{2}{3}$ as much as a man.

13. A rectangular tank 9 ft. $7\frac{1}{2}$ ins. long, 2 ft. 8 ins. deep, and 4 ft. $1\frac{1}{11}$ ins. broad, has two pipes connected with it, one of which can supply 1080 c.ins. of water per minute, while the other can discharge the whole contents of the tank in $3\frac{1}{2}$ hours. If the tank be empty, and both pipes be simultaneously opened, how long will it take to fill?

14. A bath can be filled by the cold-water pipe in 9 minutes and by the hot-water pipe in $11\frac{1}{4}$ minutes. A person leaves the bath-room after turning on both pipes simultaneously, and returns at the moment when the bath should be full. Finding, however, that the waste pipe has been open, he now closes it. In $3\frac{3}{4}$ minutes more the bath is full. In what time would the waste pipe empty it?

15. A cistern partly full, into which a steady stream of water is flowing, has a number of equal holes in the bottom, which can be opened or closed at will. If 10 were opened, the cistern would be emptied in 20 minutes; if 8 were opened, it would be emptied in 35 minutes. Twelve are opened; how soon will it be emptied?

Problems on Trains.

239. In solving problems on trains you will find a simple diagram, such as can be made by straight lines only, of great use. Fix your attention on *one point of the train*, say the front of the engine, and find how far it has to move. Reduce all the data to a single denomination, as far as possible. It will usually be best to write lengths in feet.

To reduce velocities from "miles per hour" (m.p.h.) to "feet per second" (f.p.s.):

$$\textbf{60 m.p.h.} = 60 \times 1760 \times 3 \text{ f.p.h.} = \frac{60 \times 1760 \times 3}{60 \times 60} \text{ f.p.s.} = \textbf{88 f.p.s.}$$

Learn this and reduce all others mentally by means of it, *e.g.*,

$$45 \text{ m.p.h.} = \tfrac{3}{4} \text{ of } 60 \text{ m.p.h.} = \tfrac{3}{4} \text{ of } 88 \text{ f.p.s.} = 66 \text{ f.p.s.}$$

So again $\quad 40 \text{ m.p.h.} = \frac{2}{3} \text{ of } 88 \text{ f.p.s.} = \frac{176}{3} \text{ f.p.s.}$

Leave it in this form. Do not write it $58\frac{2}{3}$ f.p.s.

Ex. 1.—*How long does a train* 88 *yds. long, moving at* 45 *m.p.h., take to pass a certain telegraph post?*

A figure will make it clear that while the train is passing the post a point E on the engine has moved through a distance equal to the whole length of the train; therefore E has moved 88 yds., *i.e.* 264 ft. But E moves 45 m.p.h., *i.e.* 66 f.p.s.; therefore it will move 264 ft. in $\frac{264}{66} = 4$ seconds. *Ans.*

Ex. 2.—*How long does the same train take to pass a station* 220 *yds. long?*

E has had to go from one end of the platform to a distance beyond the other end equal to the length of the train, *i.e.* 220 yds. + 88 yds. *i.e.* 308 yds., *i.e.* 924 ft. But the train travels 66 f.p.s.; therefore it travels 924 ft. in $\frac{924}{66} = 14$ seconds. *Ans.*

Ex. 3.—*A train was noticed to pass a station* 220 *yds. long in* 10 *seconds and another station* 330 *yds. long in* 14 *seconds. How long was the train and how fast was it moving?*

We see, as in Ex. 2, that *E* travels:—

First Station. Length of station (660 ft.) + length of train in 10 seconds.
Second Station. ,, ,, (990 ft.) + ,, ,, 14 ,,

Therefore it travels 330 ft. in 4 seconds,

$$i.e.\ \frac{330}{4} \text{ f.p.s., } i.e.\ \frac{330}{4 \times 88} \text{ of 88 f.p.s., } i.e.\ \frac{15}{16} \text{ of 60 m.p.h.,}$$

$$i.e.\ 56\tfrac{1}{4} \text{ m.p.h.}$$

Again, if the train travels $\frac{330}{4}$ f.p.s., then in 10 seconds it runs a distance of 825 ft. But this is equal to combined length of first station (660 ft.) and train; therefore the train = 825 − 660 ft. = 165 ft.

If along a level road two bicyclists start, one riding at 9 m.p.h., the other at 8 m.p.h., in the same direction, then at the end of an hour the one will have gone 9 mls., the other 8 mls.; hence the first will be gaining on the second 1 m.p.h. If at the end of the hour the first turn back and ride to meet the second, they will approach one another at 9 + 8, *i.e.* 17 m.p.h., and, as they are only 1 ml. apart, they will meet in $\frac{1}{17}$ of an hour, *i.e.* about $3\frac{1}{2}$ minutes. Similar principles apply to two trains meeting one another.

Ex. 4.—*Two trains,* A *and* B, *of lengths* 88 *yds. and* 132 *yds. respectively, are approaching one another in opposite directions at the rates of* 60 *and* 45 *m.p.h. respectively. How long do they take to pass one another?*

A's rate is 60 m.p.h., *i.e.*, 88 f.p.s.
B's ,, 45 m.p.h., *i.e.*, 66 f.p.s.

A and B are approaching one another at 154 f.p.s.
and, after they meet, leave one another at this rate.

To a passenger in A matters are the same as if he were moving at 154 f.p.s., and B were at rest. For the whole train to clear B he has to move 88 yds. + 132 yards, *i.e.*, 220 yds., *i.e.*, 660 ft. *with regard to* B, *i.e.* he has to increase his distance from B by this amount, but he is leaving B at 154 f.p.s.; therefore he gains 660 ft. in $\frac{660}{154} = 4\frac{2}{7}$ seconds. *Ans.*

Ex. 5.—*In the last Example how long does* B *take to pass a passenger in* A *?*

B is approaching A and moving past a passenger in A at 154 f.p.s. But for B to pass the passenger in A its whole length must pass him; therefore B has to move its whole length of 396 ft. at 154 f.p.s.; therefore it takes $\frac{396}{154} = \frac{18}{7} = 2\frac{4}{7}$ seconds. *Ans.* Notice carefully the difference between this question and the last.

Ex. 6.—*How long does* A *take to pass a passenger in* B *?*

The answer, similarly, is $1\frac{5}{7}$ seconds. Explain why the answers to Ex. 5 and Ex. 6, when added together, must produce that of Ex. 4.

Ex. 7.—*A man standing on a platform notes that a train going in one direction takes* 3 *seconds to pass him, a train of the same length in the other direction* 4 *seconds. How long did they take to pass one another ?*

Call the trains A and B. In 3 seconds A runs its own length; therefore in 1 second A runs $\frac{1}{3}$ of its length. So B runs $\frac{1}{4}$ of its length in 1 sec. Their lengths are the same; therefore in 1 second the trains separate $(\frac{1}{3} + \frac{1}{4})$ of the length of either, *i.e.* $\frac{7}{12}$ of the length of either. At the moment when their engines are level with one another they have to separate by *twice* the length of either in order to get clear of one another. But they separate $\frac{7}{12}$ of that length in 1 second; therefore they separate their own length in $\frac{12}{7}$ seconds; therefore twice their length in $\frac{24}{7}$ seconds, *i.e.* $3\frac{3}{7}$ seconds. *Ans.*

Ex. 8.—*A train is observed to pass a pointsman in* 3 *seconds. Later it passes a man walking at* 4 *m.p.h. in the same direction in* $3\frac{1}{3}$ *seconds. Then it passes a train half as long again moving in the opposite direction in* 4 *seconds. How long does this second train take to pass completely through a station* 440 *yds. long ?*

The man walks 4 m.p.h., *i.e.* $\frac{88}{15}$ f.p.s.; therefore in $3\frac{1}{3}$ seconds he has moved $\frac{10}{3} \times \frac{88}{15} = \frac{176}{9}$ ft. In order to pass him the train must have moved through its own length *plus* this distance. But it passes him in $3\frac{1}{3}$ seconds, and it moves through its own length (when it passes the pointsman) in 3 seconds; therefore it moves $\frac{176}{9}$ ft. in $\frac{1}{3}$ second, *i.e.* 176 ft. in 3 seconds; therefore it is 176 ft. long. The second train must be $\frac{3}{2} \times 176 = 264$ ft. long. Their combined length is $264 + 176 = 440$ ft. They take 4 seconds for passing one another. Hence they are approaching (or passing) one another at a rate of 440 ft. in 4 seconds, *i.e.* 110 f.p.s. But this rate, as we see from Ex. 4, is the sum of the separate rates of the trains. The first train moves at $\frac{176}{3}$ f.p.s. Their combined rate is 110; therefore the second train is moving at $110 - \frac{176}{3}$ f.p.s., *i.e.* $\frac{154}{3}$ f.p.s. The combined length of station and second train is 440 yds. + 264 ft. = 1584 ft.; therefore the time the second train takes to pass the station is $1584 \div \frac{154}{3} = 30\frac{6}{7}$ seconds. *Ans.*

EXAMPLES CXLI.

1. A train leaves London for Edinburgh, a distance of 405 mls., at 9.30 a.m., travelling at the rate of 45 m.p.h. Allowing 40 minutes for stops, &c., at what time will Edinburgh be reached?

2. A train from Paddington to Bath covers the distance (108 mls.) in 2 hours 15 minutes. What is the average speed (1) in m.p.h., (2) in f.p.s.?

3. If the interval between two telegraph posts be 66 yds., and a passenger in a train notes that it takes $3\frac{3}{8}$ seconds to cover this interval, at what rate is the train moving in m.p.h.?

4. The rates of a coach and a train are as 3 : 16. The train takes $1\frac{1}{2}$ hours to perform a certain journey, including 15 minutes for stops, &c. The coach stops three times for 10 minutes each. How long does the coach take to run the journey?

5. The driver of a train, 88 yds. long, travelling at 40 m.p.h., blows his whistle 3 seconds before his engine reaches a tunnel. The last carriage emerges from the tunnel 36 seconds later. Find the length of the tunnel in feet.

6. A train, 160 ft. long, is travelling at the rate of 25 m.p.h. Find how long it will be in passing completely over a bridge 240 yds. long.

7. A train passed through a station with a velocity which continued uniform until it passed another station 40 mls. away, and at the latter station was 2 minutes late. Had the velocity been 50 m.p.h., it would have been 10 minutes late. At what rate did the train travel?

8. A train travels from a station A to station D, stopping on its way at stations B and C. The distance AB, BC, CD, are 45, 50, and 35 mls. respectively, and the corresponding average rates at which the train travels between these stations are 40, 48, and 42 m.p.h. The train stops 5 minutes at B, and 10 minutes at C. Find the time taken over the journey, and the average rate of the train, (1) excluding the time of stopping at the stations, (2) including it.

9. A train takes 20 minutes longer to do a journey when it is running 27 m.p.h. than when it is running 30 m.p.h. Determine the length of the journey.

10. A train whose length is 176 ft. is moving at the rate of 45 m.p.h. and overtakes a second train going at the rate of 30 m.p.h. How long does the first train take to pass a passenger in the second train? And, if the first train takes $\frac{1}{4}$ minute to pass the second train, find the length of the second train.

11. A train going 30 m.p.h. passes a man walking in the same direction at the rate of 3 m.p.h. and goes by him in 10 seconds. What is the length of the train?

12. A train 88 yds. long meets a man going in the opposite direction at the rate of 3 m.p.h., and passes him in 8 seconds. At what rate is the train going?

13. A passenger train moving at the rate of 45 m.p.h., overtakes a mineral train $1\frac{1}{2}$ times as long moving at the rate of 27 m.p.h., and passes it completely in 25 seconds. How long would the passenger train take to pass completely through a station 165 yds. in length?

14. Two trains, A and B, travelling at the rate of 60 m.p.h. approach each other on the same line of rails. A (*i.e.* the last carriage) leaves a certain point at noon, and 2 minutes later B (*i.e.* the engine) reaches a point exactly 100 miles distant. The collision takes place at 12 hours 50 minutes 58 seconds. If both trains (*i.e.* their last carriages) had left the points at noon, the collision would have occurred at 12 hours 49 minutes 57 seconds. Find the length of each train.

15. A goods train 120 yds. long starts from a station at 2 o'clock and travels at the rate of 20 m.p.h. A fast train, 40 yds. long, starts 3 hours afterwards from the same platform, and on the same line of rails, travelling at the rate of 45 m.p.h. Find how many miles each engine will have travelled when the fast train reaches the goods train. (The last carriage in each train is supposed to start from the same spot.)

16. The numbers expressing in yards the lengths of a fast and slow train are three times the numbers expressing in m.p.h. the speeds of the slow and fast trains respectively, these last numbers consisting of the same two digits, but in the reverse order. Find (1) the number of seconds they would take in completely passing one another when journeying in opposite directions on parallel lines, (2) the difference in the times taken by two persons, one in each train, in passing the other train, journeying in the same direction.

17. A train, travelling at the rate of 40 m.p.h., whilst inside a tunnel meets another train of half its length travelling at 60 m.p.h., and passes it completely in $4\frac{1}{2}$ seconds. Find the length of the tunnel if the first train passes completely through in 4 minutes $37\frac{1}{2}$ seconds.

18. The distance from A to B is 170 mls. A train leaves A at 9 a.m., travelling at the rate of 40 m.p.h., at the same time another train leaves B travelling at the rate of 50 m.p.h. If the latter train is stopped from 10 to 10.30 a.m. and then taken on again at the same rate as before, when and where will they meet?

19. Two trains start simultaneously from A and B and travel toward each other at the rates of 45 and $27\frac{1}{2}$ m.p.h. respectively. When they meet one has gone 28 mls. further than the other. Find the distance from A to B.

20. A train is travelling at the rate of 7 furls. in 2 minutes. By how many feet per second must its speed be increased, if it is to gain 330 yds. in 1 minute on another train travelling in the same direction at the rate of 45 m.p.h.?

21. Two trains, A and B, are travelling in opposite directions and each whistles as it enters a tunnel 70 yds. less than 2 mls. in length. A, on entering, hears B's whistle. B, $17\frac{1}{4}$ seconds after entering, hears A's whistle. The velocity of sound being 1150 f.p.s., find the number of m.p.h. B is travelling.

22. Two trains leaving London at 9.45 and 10 a.m. reach Grantham, 105 mls. distant, at 11.47 and 12.2 respectively, each running uniformly but losing 1 minute in starting and 1 minute in stopping. An up train meets the first express at 10 hours 42 minutes $31\frac{1}{2}$ seconds, and the second at 10 hours 50 minutes 24 seconds. If it proceeds uniformly, when will it reach London?

23. My house is half-way between two towns, A and B (which are 10 mls. apart), and is situated beside the tram-line which connects them. The trams start at 8 a.m., and leave A and B every 5 minutes, completing the journey from A to B in 30 minutes. I leave my house at 8.17 a.m. and walk at the rate of 5 mls. an hour towards B, proposing to travel by that tram which overtakes me nearest to the mid-point between my house and B. Which of the trams that overtake me should I travel by, and what length of time should I have to spend in it?

24. Two men walk with uniform speed along a railway in the same direction. A train travelling uniformly overtakes one of them, who walks at 4 m.p.h., and the whole train, which is 132 yds. long, passes him in $7\frac{1}{2}$ seconds; $6\frac{3}{4}$ seconds after it began to pass the above named man, the train begins to pass the other and occupies $7\frac{1}{8}$ seconds in passing him. When will one man overtake the other? (Reckon your answer from the time the train begins to pass the first man.)

25. On AB, a section of railway $\frac{1}{4}$ ml. long, there are 4 lines of rails. Trains 88 yds. long, travelling at the rate of 15 mls. an hour, cross the section from A to B, entering it at 3 minutes past each hour and thereafter at intervals of 10 minutes. Others, 110 yds. long, travelling at the rate of 45 m.p.h. in the same direction, enter the section at 10 minutes past each hour, and at intervals of 15 minutes thereafter. A precisely similar succession of trains crosses the section in the opposite direction, each of them entering the section at B 1 minute later than the corresponding train enters at A. Show that *some* portion of the section will be occupied for 16 minutes 8 seconds in each hour.

26.* A train leaves A for B every morning at 9 and takes $7\frac{1}{3}$ days for the journey; a train leaves B for A every day at noon and takes $7\frac{2}{3}$ days for the journey. The speed of these trains on the level is the same, but immediately after leaving B there is a steady incline, the effect of which is that a train *from* B can travel uniformly while on the incline at seven-eighths only of its rate on the level, while the train *to* B runs uniformly down the incline one-eighth faster than it did on the level. If I leave A for B on Monday morning, at what times do I meet trains from B?

Races and Games of Skill.

240. Ex. 1.—*In* 1000 *yds. handicap* B *has* 6 *yds. start and* C 15 *yds. When* A *reaches the winning-post* B *is* 4 *yds. and* C *is* 5 *yds. behind. How many yards start can* B *give* C *in* 1000 *yds. so as to make a dead heat?*

B has 6 yds. start, and is 4 yds. from the end at the close of the race; he has therefore run only $1000-10$, or 990 yds. Similarly at the close of the race, C has run only $1000-20$, or 980 yds.;

$\therefore$ B runs 990 yds. while C runs 980 yds.;

$\therefore$ B runs 1000 yds. while C runs $\frac{980}{990} \times 1000$, or $989\frac{89}{99}$ yds.

Thus, for a dead heat, C must have $1000-989\frac{89}{99} = 10\frac{10}{99}$ yds.; or we might have argued thus:

In a race of 990 yds. B can give C 10 yds. start;

$\therefore$ in a race of 1000 yds. B can give C $\frac{10}{990} \times 1000 = 10\frac{10}{99}$ yds.

Ex. 2.—A *can beat* B *by* 20 *yds. in a race of* 200 *yds., and in a race of* 250 *yds.* B *can beat* C *by* 10 *yds. By how many yards can* A *beat* C *in* 100 *yds., assuming that their respective rates are the same in the various races?*

A runs 200 yds. while B runs 180 yds.

B runs 250 yds. while C runs 240 yds.

B thus forms the connecting link between A and C. Find the L.C.M. of the two distances run by B, viz., 180 and 250. This L.C.M. is $18 \times 25 \times 10$.

Then
A runs 200×25 yds. (= 5000 yds.) while B runs $18 \times 25 \times 10$ yds., and B runs $18 \times 25 \times 10$ yds. while C runs 240×18 yds. (= 4320 yds.).

i.e. A runs 5000 while C runs 4320;

$\therefore$ A runs 100 while C runs $86\frac{2}{5}$;

$\therefore$ A can give C $100 - 86\frac{2}{5}$, or $13\frac{3}{5}$ yds. start.

A similar method can be adopted in the solution of problems on games of skill.

Ex. 3.—*A boat's crew can row a* 4*-mile course in* 20 *minutes in still water, and in* 16 *minutes with the tide. How long would it take to row the course against the tide?*

In still water the crew can row 4 mls. in 20 minutes;

$\therefore$ in 16 minutes they can row $4 \times 16 \div 20$, or $3\frac{1}{5}$ mls.

But they go altogether 4 mls. in these 16 minutes;

$\therefore$ the tide must flow at the rate of $4 - 3\frac{1}{5}$, or $\frac{4}{5}$ ml. in 16 minutes, *i.e.* 1 ml. in 20 minutes;

$\therefore$ the crew, when rowing *against* the tide, go $4 - 1$, or 3 mls. in 20 minutes;

$\therefore$ they row the 4 mls. *against* the tide in $\frac{20 \times 4}{3}$, or $26\frac{2}{3}$ minutes.

Ex. 4.—*A cyclist, travelling at the rate of* 12 *m.p.h., overtakes a traveller* A, *walking at the rate of* 4 *m.p.h.; 9 minutes later he passes another man* B, *travelling at the rate* $3\frac{1}{2}$ *m.p.h. If the rates of travelling are uniform, find at what interval after* B *has been passed by the cyclist he will be overtaken by* A.

Let A be the position of the first man when overtaken by the cyclist, and B the position of the second man at the same moment.

Now the cyclist *gains* the distance AB in 9 minutes

But the cyclist *gains* on B $12 - 3\frac{1}{2}$, or $8\frac{1}{2}$ m.p.h.

$$\therefore\ AB = 8\tfrac{1}{2} \text{ mls.} \times \tfrac{9}{60} = 8\tfrac{1}{2} \times \tfrac{3}{20} = \tfrac{51}{40} \text{ mls.}$$

Again, the first man *gains* on the second man $4 - 3\frac{1}{2}$, or $\frac{1}{2}$ ml. an hour;

$\therefore$ first man *gains* the distance AB in $(\frac{51}{40} \div \frac{1}{2})$ hours = 2 hours 33 minutes;

$\therefore$ the first man overtakes the second man in 2 hours 33 minutes $-$ 9 minutes, or 2 hours 24 minutes from the moment when the cyclist overtook B.

EXAMPLES CXLII.

1. A can beat B by 4 yds., and B can beat C by 2 yds. in a 100 yds. race. By how many yards can A beat C in the same race?

2. In a game of billiards A scores 70 to C's 100. B can score 70 while A scores 60. How many points can C give B in a game of 600?

3. A works 10 arithmetical examples while B works 7 of the same kind; A could work 15 while C could work 11. Which is the quicker worker, B or C?

4. In a game of skill A scores 70 while B scores 60, B scores 10 to each 11 of C while D scores 23 to every 22 of C. By how many can A beat D in a game of 420?

5. At a game of rackets the skill of three players A, B, C is represented by the numbers 5, 4, 2, respectively. How many points can A give B, and A give C, in a game of 15, so as to make an even match?

6. A and D play against B and C, the latter two winning by 20 in a game of 300. If the powers of A, B, C be represented by the numbers 10, 9, 6, respectively, what number represents D's skill?

7. In a 100 yds. race A can beat B by 1 yard, and B can beat C by 1 yard. By how much can A beat C in the same race?

8. A, B, C, run a $\frac{1}{4}$-mile race, A being "scratch," B having 10 yds. start, and C 15. B wins by 5 yds. against both A and C How many yards start could A give C and yet beat him by 5 yds.?

9. Three men run a mile race; the first beats the last by $76\frac{12}{23}$ yds., the second beats the last by 11 seconds, and the pace of the first is to that of the second as 45 : 44. In what time does each run the mile?

10. A and B ride a race of 31 mls. on bicycles. The driving-wheel of A's machine makes 3410 revolutions per hour, and has a circumference of 168 ins.; that of B makes 3520 revolutions per hour, and has a circumference of 162 ins. Who will win, and by how much?

11. In a race of 1 mile A beats B by 32 yds., and C by 48 yds. By how much can B beat C in a race of 1080 yds.?

12. A can beat B by 1 foot if he gives him 3 yds. start in a race of 100 yds.; if B gives C a start of 8 yds. in a race of 120 yds. C wins by 28 ins. Who will win, and by how much, if A gives C a start of 10 yds. in a race of 100 yds.?

13. A and B can run at the rate of $12\frac{1}{2}$ and $12\frac{7}{16}$ mls. an hour respectively. If A give B 10 yds. start, in what time will he overtake him? When A has run a mile will he be in front of or behind B?

14. A can run a mile in 4 minutes 33 seconds, and B in 4 minutes 40 seconds. How many yds. start can A give B in a mile race to make a dead heat?

15. Three men enter for a mile race and start even. When the first man reaches $\frac{1}{2}$ ml. he is 20 yds. ahead of the second man, and when the second man reaches $\frac{1}{2}$ ml. he is 22 yds. ahead of the third man. What start could the first have given the third?

16. A and B run a mile race round a course of 8 laps to a mile, A being "scratch," and B having 30 yds. start. At the end of the third lap they are both level. They then increase their rates in the ratio of 66 to 67 and 21 to 22 respectively. By how many yards and by how many seconds does A win, given that A runs the whole mile in 4 minutes $25\frac{1}{2}$ seconds?

17. Two boats row a race over a straight course, 1 ml. 995 yds. long, their rates being 12 mls. and $11\frac{23}{25}$ mls. per hour respectively. Assuming that sound travels at the rate of 1140 ft. in a second, find how much the faster boat will be ahead of the other when the sound of the gun fired at starting is heard at the winning-post.

18. A can give B 50 yds. start in a mile and beat him by 5 yds. in $5\frac{1}{2}$ minutes. At the same rate A can give C 10 seconds start and make a dead heat of the race. How many yards start must be given to B that he may win the mile by 11 yds. against C?

19. In 1000 yds. handicap A, with 68 yds. start, won by 5 yds. from B, with 75 yds. start, in 2 minutes 16 seconds. If A had been at "scratch," what start could he have given B so as still to have won by 5 yds.? What would have been A's time?

20. Two boats start in a race, the one rowing 36 strokes per minute and advancing 33 ft. in each stroke, while the other rows 39 strokes per minute, advancing 27 ft. in each. When the first has rowed one half of the course, both crews suddenly reduce their strokes to 33 per minute. If the first boat thenceforth clears 39 ft. per stroke, what must the second clear in order that the race may be a dead heat?

21. During the first part of a boat-race over a course of $1\frac{1}{2}$ mls., the two crews are each rowing 34 strokes a minute, and when the leading crew has just completed half the distance the other crew is 40 ft. behind. The former now quickens its stroke to 35 a minute, and the latter takes a slower stroke of 33 a minute; but at each stroke the former boat travels a shorter distance by 4°/₀, and the latter boat a longer distance by 4°/₀. Which boat wins, and by how much (to the nearest inch)?

22. In 100 yds. race A can beat B by 4 yds.; in $\frac{1}{4}$ ml. race C can beat A by 11 yds. By how much can C beat B in a mile race, supposing that the average speeds of each man when running 100 yds., $\frac{1}{4}$ ml., and 1 ml. are proportional to 9, 8, 7?

23. A, B, C run a mile race; A is at scratch, B has 85 yds. start, and C has 150 yds.; $3\frac{1}{4}$ minutes after the start B passes C, and $\frac{1}{2}$ minute later A passes C; 200 yds. beyond the spot where A passes C he would have passed B, but when 100 yds. from the winning-post he increases his pace by 6 ft. a second. Who will win, what time will he take, and what will be the position of the other two when the winner passes the winning-post?

24. A cruiser pursuing a privateer, both moving in a straight line at their highest speed, hears the report from one of the privateer's guns 15 seconds after seeing the flash. If sound travels at the rate of 1100 f.p.s., and if a knot be taken as 2000 yds., how many knots an hour does the cruiser steam so as to overtake the privateer $5\frac{1}{2}$ hours after hearing the gun, the privateer's speed being $15\frac{1}{2}$ knots an hour?

25. A and B run a 5 ml. race over a course $\frac{1}{5}$ ml. round. B is allowed a start of $\frac{1}{5}$ of a round, and is overtaken when he passes for the third time through the point from which he started. A wins by 1 minute 49 seconds. Find the uniform rate at which each man runs.

26. Three bicyclists, A, B, C, ride at the rate of 639, 615, 603 yds. per minute respectively. If they start together in the same direction round a circular track $\frac{1}{4}$ ml. in circumference, find how long after the start, at what part of the track, they will next be all together, and how often A will have passed C in the meantime.

27. A and B run a mile race ending in a dead heat. At first A runs 20 % faster than B. B then quickens his pace and for the remainder of the race is running 20 % faster than A. How many yards had A run before B quickened his pace?

28. A starts at 2 o'clock and walks twice round a circular path of 1 mile with uniform speed. B starts from the same point as A at $6\frac{1}{2}$ minutes past 2 and walks twice round the path with uniform speed different from that of A and in the opposite direction. They pass each other when B has completed 500 yds. of his first mile, and again when A has completed 500 yds. of his second mile. Find when they will again pass, and when each will finish his walk.

Problems on Clocks.

241. [In this Section M.H. will be used for "minute hand," H.H. for "hour hand," and often M.S. for "minute-space," *i.e.* the distance on the clock dial between two minute-dots, a distance corresponding to 6° of angular measurement.]

Ex. 1.—*A clock which loses 5 seconds an hour is set right on Sunday at noon.* (*a*) *What time will it indicate on Monday at noon?* (*b*) *What will be the correct time when it indicates noon on Monday?* (*c*) *When will the clock next indicate correct time?*

(*a*) 24 hours have elapsed; therefore the clock has lost 5 × 24 seconds, *i.e.* 2 minutes; therefore it indicates 11.58 a.m.

(*b*) There are $60 \times 24 = 1440$ minutes in the day; the clock has lost 2 minutes; therefore in 1440 minutes its M.H. has moved through $1440 - 2 = 1438$ M.S. But the clock will indicate noon after its M.H. has moved through 1440 M.S. This will be after $\frac{1440}{1438} \times 1440$ true minutes $= (1\frac{1}{719}) \times 1440$ true minutes $= 1$ day $+ \frac{1440}{719}$ true minutes. Hence the true time will be $\frac{1440}{719}$ true minutes after noon, *i.e.* $2\frac{2}{719}$ minutes after noon.

(*c*) The clock can only indicate correct time after it has lost 12 hours. It loses 5 seconds in 1 hour; therefore 1 minute in 12 hours; therefore 12 hours in $12 \times 60 \times 12$ hours $= 360$ days. *Ans.*

Ex. 2.—*Two clocks are set right simultaneously at* 12 *noon, one of which loses* 6 *seconds per hour, and the other gains* 3 *seconds in* 50 *minutes.* (*a*) *How long will it be before the minute hands are again in the same direction?* (*b*) *What will then be the time a.m. or p.m. by each clock?* (*c*) *When will both clocks simultaneously indicate correct time?* (*d*) *When will both clocks first indicate the same time?*

(*a*) Call the clocks A and B. Since there are 3600 seconds in an hour, the M.H. of A moves $3600 - 6$, *i.e.* 3594 second-spaces in 60 minutes, therefore $\frac{3594}{60}$ second-spaces in 1 minute.

The M.H. of B moves $3000 + 3$, *i.e.* 3003 second-spaces in 50 minutes; therefore $\frac{3003}{50}$ second-spaces in 1 minute;
therefore in one minute the M.H. of B gains $\frac{3003}{50} - \frac{3594}{60}$ second-spaces on the M.H. of A, *i.e.* $\frac{4}{25}$ second-space; therefore the M.H. of B gains 1 second-space on M.H. of A in $\frac{25}{4}$ minutes; therefore 1 minute-space in $(60 \times 25)/4$ minutes. The M.H.'s are in "the same direction" when that of B has gained 30 M.S., *i.e.* half the dial, on that of A. This will happen in $30 \times 60 \times 25 \div 4$ minutes $= 187\frac{1}{2}$ hours $= 7$ days $19\frac{1}{2}$ hours. This is the answer to (*a*). A true clock would indicate 7.30 a.m.

(*b*) A loses 6 seconds per hour; therefore in $187\frac{1}{2}$ hours it loses $6 \times 187\frac{1}{2} = 1125$ seconds, *i.e.* 18 minutes 45 seconds. But, since a true clock indicates 7.30 a.m., A indicates 7 hours 30 minutes $-$ 18 minutes 45 seconds, *i.e.* 7 hours 11 minutes 15 seconds a.m. Similarly B indicates 7 hours 41 minutes 15 seconds a.m.

Note.—If "in the same direction" is strictly interpreted, we must suppose B's M.H. has gained 60, not 30, M.S. on A's. It is easy to prove that this happens after 15 days 15 hours; that the true time is 3 a.m.; A's time is 2 hours 22 minutes 30 seconds a.m., B's 3 hours 22 minutes 30 seconds a.m.

(*c*) If A is to indicate *correct time*, it must have lost 12 hour-spaces. But this will happen in 7200 hours. B will gain 12 hour-spaces in 12,000 hours. The L.C.M. of 7200 and 12,000 is 36,000; therefore in 36,000 hours, *i.e.* 1500 *days*, both will be again correct at noon. *Ans.*

(*d*) We have shown in (*a*) that B gains 1 M.S. on A in $(60 \times 25)/4$ minutes; therefore it will gain 1 hour on A in 375 hours; therefore it will gain 12 hours on A, *i.e. will indicate the same time as* A, in 12×375 hours $= \frac{375}{2}$ days $= 187\frac{1}{2}$ days. Verify for yourself that the time indicated by both clocks is 4.30.

In problems referring to the times at which the hands occupy stated positions with regard to one another, as, for instance, when they are coincident, are at right angles, &c., it must be remembered that in every hour (*a*) the hands cross once, (*b*) they are *twice* at right angles, or at 60° or any other angle with one another, (*c*) they are *twice* in the same line [including (*a*)]. The key to all these questions is to regard them as race-questions in which the M.H. is racing after, catching up, and passing the H.H.

Now the M.H. moves 60 m.s. in 1 hour, the H.H. only 5; therefore the M.H. gains $60-5$, *i.e.* 55 m.s. in every hour; therefore it gains 1 m.s. in $\frac{1}{55}$ hour; therefore it gains 60 m.s. in $\frac{60}{55}=\frac{12}{11}$ hours. But when it has gained 60 m.s. it is in *precisely the same position* relative to the H.H. as it was before. Hence, if the M.H. and H.H. coincide at any moment, they will coincide again $\frac{12}{11}$ hours later. Similarly with all other positions. They coincide at 12.0.; hence we can write down at once *all* their coincidences.

	h. m.	h. m.	h. m.	
Coincidences:	12.0;	$1.5\frac{5}{11}$;	$2.10\frac{10}{11}$;	&c.

They are at right angles at $\begin{cases} 3.0 \text{ when the M.H. is behind the H.H.} \\ 9.0 \text{ when the M.H. is in front of the H.H.} \end{cases}$

	h.m.	h.m.	h.m.	
Hence the right-angle positions are	3.0;	$4.5\frac{5}{11}$;	$5.10\frac{10}{11}$;	&c.
	9.0;	$10.5\frac{5}{11}$;	$11.10\frac{10}{11}$;	&c.

Be careful not to take 6.15 or 12.15 as right-angle positions.

Ex. 3.—*At what time between* 7 *and* 8 *do the hands cross?*

At 7 o'clock the M.H. is 35 m.s. behind the H.H. It gains 55 m.s. in the hour; therefore it will gain 35 m.s. in $\frac{35}{55}$ of an hour $=\frac{7}{11}$ hour $=38\frac{2}{11}$ minutes. Hence the M.H. has caught up and will cross the H.H. at 7 hours $38\frac{2}{11}$ minutes.

Ex. 4.—*At what time between* 5 *and* 6 *is the* M.H. *just as far behind the* H.H. *as* 10 *minutes later it is in front of it?*

In 10 minutes the M.H. moves 10 m.s., but it moves 12 times as fast as the H.H.; therefore in 10 minutes the H.H. has moved $\frac{10}{12}$ m.s. $=\frac{5}{6}$ m.s. It is now as far behind the M.H. as it was in front; therefore originally it was $\frac{10-\frac{5}{6}}{2}$ m.s. in front of the M.H. $=\frac{55}{12}$ m.s. But at 5 o'clock the M.H. is 25 m.s. behind the H.H.; therefore it has $(25-\frac{55}{12})$ m.s. $=20\frac{5}{12}$ m.s. to gain before it reaches the required position, but it gains 55 in every hour; therefore it gains $20\frac{5}{12}$ in $20\frac{5}{12}\div 55$ hours $=\frac{245}{11}$ minutes $=22\frac{3}{11}$ minutes. *Ans.* 5 hours $22\frac{3}{11}$ minutes.

Some complicated questions arise from considering when the hands *change positions*. A little consideration will show that in this case the H.H. is originally behind the M.H. and then moves up to take the position occupied by the M.H., while at the same time the M.H. has gone round to occupy the original H.H. position. Thus the *two together have described the whole circle* of 60 m.s., but the M.H. moves 12 times as fast as the H.H.; therefore they divide the journey of 60 m.s. in the ratio 12 : 1; therefore the H.H. moves through $\frac{60}{13}$ m.s., *i.e.* $4\frac{8}{13}$ m.s. originally separate the hands. Now at XII the hands are together, and, since the M.H. gains 55 m.s. per hour, it will gain $4\frac{8}{13}$ m.s. in $\frac{60}{55} \times 4\frac{8}{13}$ minutes $= 5\frac{5}{143}$ minutes Hence at 12 hours $5\frac{5}{143}$ minutes the hands are first in a position in which they can be interchanged. You should verify that when they are interchanged the time indicated is 1 hour $0\frac{60}{143}$ minute. Illustrations will occur in the examples which follow. It must be specially noted that in the case considered above the hands are not supposed to have crossed.

EXAMPLES CXLIII.

1. At what times between 7 and 8 are the hands of a clock at right angles to each other?

2. At what time between 3 and 4 o'clock are the hands separated by an interval of 25 minutes?

3. At what time between 3 and 4 o'clock are the hands (1) opposite to one another, (2) at right angles to one another?

4. If the dial of a clock has the hours marked from 1 to 24, at what time (1) between 19 and 20, (2) between 18 and 19, will the hands be together?

5. During an afternoon the hands of a clock are observed to be in the same straight line, but pointing in opposite directions. After an interval —less than half an hour—the minute hand is at the figure XII, and the two hands are observed to contain between them an angle of 120 degrees. Find, to the nearest second, the time of the first observation.

6. At what moment between 9 and 10 o'clock is the hour hand of a clock exactly $15\frac{2}{3}$ minute divisions in advance of the minute hand?

7. A watch set on Friday at 9 p.m. gains 45 seconds in 12 hours. What time does it show on the next Monday at 3 p.m.?

8. A watch which gains 5 seconds in every 3 minutes was set right at 6 a.m. What was the true time in the afternoon of the same day when the watch indicated a quarter past 3 o'clock?

9. At what time between 3 and 4 o'clock are the hands of a watch together?

10. At noon yesterday clock A was 5 minutes fast, and clock B was 2 minutes slow. A was right at 3 o'clock this morning, and when the two clocks last agreed they were both 2 minutes fast. When was B right?

11. A watch which gains uniformly is 150 seconds slow on Sunday at noon. On the following Friday at 12 minutes past 7 a.m. it is 210 seconds fast. When was it right?

12. If the hands of a clock indicate 3 o'clock when the proper time is 1 minute to 3, and 4 o'clock when the proper time is $\frac{1}{2}$ a minute past 4, what is the proper time when the hands coincide between 4 and 5 o'clock?

13. The minute hand of a clock is loose, and when descending it slips forward at a uniform rate through 1 minute in every 10 true minutes, and when ascending it slips backward through 1 minute in every 10 true minutes. If the clock is set right at noon, find when it marks 1 p.m.

14. Of two watches, one loses 5 seconds, and the other gains 4 seconds, in 12 hours; they are both right at noon on a certain day. When will one be 5 minutes before the other, and what time will each then show?

15. Three clocks, A, B, C, go at uniform speeds. At noon on Monday A is 1 minute slow, B is 2 minutes slow, and C is 1 minute fast. At 8 p.m. Tuesday B is 1 minute fast, and C is 3 minutes slow. At 8 a.m. Wednesday A is $2\frac{3}{8}$ minutes before B. When do A and C both indicate the same time?

16. A clock which gains was set 15 minutes slow at 6 p.m. on 6th May, and was found to be 7 minutes fast at 10 a.m. on 12th June. On what day and at what hour between these dates did it show correct time?

17. There are two clocks, one of which gains 3 minutes, while the other loses $2\frac{1}{2}$ minutes, each day. They are set right at 4 o'clock on Friday afternoon. What is the difference between them at noon on the following Wednesday? In how many days from the time they are set right will they both show the same time? What will that time be?

18. Two watches are together at 10 o'clock this morning (June 17th); one loses $8\frac{1}{2}$ seconds, and the other gains 9 seconds, in 24 hours. When (*i.e.* on what date and at what hour) will one be half-an-hour before the other, and what time will each show?

19. How many times in the course of a day do the hands of a watch cross each other?

20. (1) If a watch were constructed so that its minute hand moved backwards while its hour hand moved forward, how often in the course of a day would the hands cross?

(2) In a watch of this character find when the hands are first at right angles after XII.

(3) Find when the hands are first opposite after XII.

(4) Prove that in such a watch any relative position of the hands is repeated at intervals of $55\frac{5}{13}$ minutes.

(5)* Prove that, if the hands of such a watch change places, they must have crossed twice; and, if they have crossed only twice, they were originally separated by $5\frac{5}{11}$ minute-spaces.

(6)* Prove that, if the hands of such a watch change places after having crossed thrice, they were originally separated by $10\frac{10}{11}$ minute-spaces.

21. If the hands of an ordinary watch change places after having crossed once, they were originally separated by $9\frac{3}{13}$ minute-spaces.

22. If the hands of an ordinary watch change places after having crossed twice, they were originally separated by $18\frac{11}{13}$ minute-spaces.

23. I look at my watch between 3 and 4 o'clock. On looking again between 8 and 9 the hands have exactly changed places. When did I look first?

24. A watch is set correctly at noon, but gains at a uniform rate. It is noticed that the hands are together between the figures IV and V when it is really 4 o'clock. At what true time will the hands next be together, and what time will they then indicate?

25. Two clocks are set going at 4 o'clock, and then indicate the right time. One of them goes 10% faster, and the other $7\frac{1}{2}$% slower, than a clock which keeps the right time. What will the minute hand of the slow clock first be a right angle in advance of the hour hand of the other, and what time will be indicated by each clock?

26. Two watches provided with centre-second-hands, one of which gains 10 seconds (of its own time) per day of true time, while the other loses 10 seconds (of its own time) per day, are set together, with all their hands pointing to XII, at midday on January 1st, 1900. Prove that their hands can all point to the same figure only at 12 o'clock, and find the day of the week and the date on which this will next occur. When will the watches next indicate the *same time?* What will this time be?

27.* The minute-hand of a clock is so badly pivoted that in descending it slips down at a constant rate of 1 minute-space in 30 true minutes, and similarly in ascending it slips back 1 minute-space in 30 true minutes. It is set right at 12 noon. Find the next two and the last occasions when it represents true time, and prove that in $37\frac{1}{2}$ days it is 1 hour late for the first time.

28.* If the hands of a clock were exactly alike in size and shape, it would still be usually possible to read the time by noting their relative positions, but at about a quarter past 2, for instance, the time might equally well be about 11 minutes past 3. Find this instant exactly, and state what other time is simultaneously indicated.

29.* One of two clocks, A, gains steadily 3 minutes per true day, while the other, B, which is wound daily, gains 3 minutes in the first 12 true hours after winding, but loses 1 minute in the next 12 true hours. The clocks are started together and never corrected. At a certain moment on the first day I noted that A indicated a time a certain number of seconds ahead of B, and again, 3 hours 12 minutes true time later, that A was the *same* number of seconds ahead of B. Prove that A indicates this same number of seconds ahead of B five times in all, and find the true time of these occasions.

30.* On July 1st, at 8 a.m. by my watch, it was 8 hours 4 minutes by my clock. I took the watch to Greenwich, and, when it said noon, the true time was 12 hours 5 minutes. That evening when the watch said 6 hours the clock said 5 hours 59 minutes. On July 30th, at 9 a.m. by my watch, it was 8 hours 57 minutes by my clock. At Greenwich when the watch said 12 hours 10 minutes the true time was 12 hours 5 minutes. That evening when the watch said 7 hours the clock said 6 hours 58 minutes. My watch is only wound up for each journey and goes uniformly during any one day; the clock is always going, and goes uniformly. How am I to know when it is true noon on July 31st?

31.* Prove that there are 143 positions in which the hands of a clock may be interchanged, and *still indicate a possible time*, and that there are the same number of positions for a clock constructed as in Ex. 20.

32.* Looking at my watch at 4.36, I notice that the hands of my clock are equally distant from the half-hour mark on opposite sides, and I find that this is again the case with the clock when the watch indicates 5.32. When will both watch and clock first indicate the same time?

PROBLEMS ON THE CALENDAR. (See also § 55.)

242. **Historical.** — The exact time taken by the earth to make a complete revolution round the sun is 365 days 5 hours 48 minutes 46 seconds. If, therefore, we take the length of a year as 365 days, as we ordinarily do, we take it nearly $\frac{1}{4}$ day too short. If this were done for long, a serious difference would arise between the commencement of the seasons and the months which we associate with them. In the time of Julius Cæsar this difference was as much as 2 months. Cæsar added 2 months between November and December of the year 46 B.C., and a portion of a month in February of the same year, so that B.C. 45 commenced at a time approximately correct. To prevent the recurrence of the trouble he instituted the system of leap years, by which the year is of 366 days every 4 years. Thus, the average length of the year after the reform of Cæsar was $365\frac{1}{4}$ days. This is about $11\frac{1}{4}$ minutes too great, and in 1500 years the error had again become considerable. In 1582 it was rectified by Pope Gregory, who deducted 10 days from that year, and ordained that the centuries should be ordinary years except when their numbers were divisible by 400. It follows that in 400 years

there are 97 leap years, *i.e.* that 97 days are added every 400 years. On Gregory's computation the year is one of $365\frac{97}{400}$, *i.e.* 365·2425 days. As the actual length is 365 242218, the error is now only ·0002782 day per year, *i.e.* about 1 day in 4000 years. This reform of Gregory's was not adopted in England until 1752, by which time our calendar was 11 days wrong. In 1752 the days between September 2 and September 14 were omitted, and the "New Style" was adopted. Russia still keeps to the Old Style.

243. In problems on the calendar extending over a large number of years, it is important to bear in mind the facts thus stated. Thus, since in 400 years there are 97 leap years, there are 303 ordinary years, and therefore $303 \times 365 + 97 \times 366 = 146{,}097$ days $= 20{,}871$ weeks. Since there is an *exact number of weeks*, January 1st will be on the same day of the week 400 years hence as it is now.

Another important "cycle" is 28 years. 28 years (*not* including a century) contain 7 leap years; therefore $21 \times 365 + 7 \times 366$ days $= 1{,}461$ weeks; therefore, after 28 such years, the days of the week repeat themselves.

Ex.—*Required the day of the week of September 3rd*, 1861.

We find the number of days which have elapsed from September 3rd, 1861, to some date which we know in the present, say January 1st, 1901, which was a Tuesday. The reckoning is as follows:—

		Wks.	Days.
Sept. 3rd, 1861–Dec. 31st, 1861 = 27 + 31 + 30 + 31 days	=	17	0
1862–1900 inclusive = 30 ordinary + 9 leap years	=	2034	6
Jan. 1st, 1901	=		1
		2052	0

Hence an exact number of weeks intervenes; therefore, if January 1st, 1901, is a Tuesday, so also was September 3rd, 1861.

EXAMPLES CXLIV.

1. Fix the day of the week on which the following events happened:—

(1) Battle of Waterloo, 18th June, 1815.

(2) Accession of Queen Victoria, 20th June, 1837.

(3) The German Empire proclaimed at Versailles, 18th January, 1871.

2. Fix the day of the week on which the following events happened :—

(1) Declaration of Independence, U.S.A., 4th July, 1776.
(2) Massacre at Cawnpore, 28th June, 1857.
(3) Franco-Prussian War declared at Berlin, 19th July, 1870.

3. What date was 40,369 days before Sunday, January 1st, 1871?

4. Prove that the same day of the month is always the same day of the week in November as it is in March.

5. Prove that a particular day of the month can only be the same day of the week in any other month as it is in August when the year is leap year.

6. When will February 1st next fall on a Wednesday?

7. When will January 1st next fall on a Sunday?

8. When will Christmas Day next fall on a Sunday?

9. When will there next be five Sundays in February? When did this last happen?

10. Which months in 1887 had 5 Sundays?

11. Prove that calendars made for the year 1900 will also serve for the years 1906 and 1917. For what year after 1917 will they serve next?

***12.** Prove that the last day of a century cannot be either Tuesday, Thursday, or Saturday.

***13.** The first day of a century must be either Monday, Tuesday, Thursday, or Saturday. [N.B.—The first day of a century is 1st January, '01, except in Germany.]

***14.** Prove that Christmas Day falls least frequently on Monday or Saturday, and most frequently on Sunday, Tuesday, or Friday.

***15.** In a leap year selected at random, prove that the chance that February 29th falls on Monday is greater than its chance of falling on Sunday in the ratio of 15 to 13.

***16.** Find a formula for the years in which there are five Sundays in February.

***17.** Prove the following rule for finding the day of the week of the last day of March in any year of the twentieth century :—Take the last two figures of the year, add one-fourth of the number they form, reject fractions, divide the sum by 7. The remainder, if 1, indicates Sunday; if 2, Monday; &c.

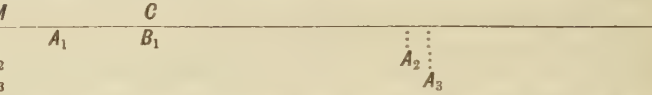
Miscellaneous.

244. Ex. 1.—*A thermometer stem is divided between the marks for freezing and boiling water into* 100 *equal parts on one side* (*Centigrade degrees*) *and* 180 *equal parts on the other* (*Fahrenheit degrees*). *Determine what graduations of the two scales exactly coincide, and also what graduations are less than one-eighth of a degree Centigrade apart.*

100 Centigrade degrees = 180 Fahrenheit; therefore 5° C. = 9° F.; therefore every fifth degree C. coincides with every ninth degree F., *i.e.* coincidences occur at 0° C., 5° C., 10° C., &c.

Between 0° and 5° C. the Fahrenheit divisions are at the following positions on the Centigrade scale:—$\frac{5}{9}$° C., $1\frac{1}{9}$, $1\frac{6}{9}$, $2\frac{2}{9}$, $2\frac{7}{9}$, $3\frac{3}{9}$, $3\frac{8}{9}$, $4\frac{4}{9}$, and 5. Of these the second is only $\frac{1}{9}$° division away from 1° C., and the seventh is only $\frac{1}{9}$° away from 4°. Hence the graduations which are less than $\frac{1}{8}$° C. apart are found at 1° C., 4° C., and then every 5° later than these points, *e.g.* 6° C., 9° C., &c.

Ex. 2.—*Two men,* A *and* B, *start to cover the distance between two towns,* M *and* N, A *on foot,* B *on a bicycle.* B *travels three times as fast as* A, *and they leave* M *at the same time.* B *breaks down at a point* C, *and walks his machine back to* M *at half the rate of* A's *walking. He stays there for repairs as many minutes as it took* A *hours to reach* C, *and they arrive at* N *simultaneously. What proportion does* MC *bear to the whole distance* MN?



A_1, B_1 give the positions when the breakdown occurs. B rides three times as fast as A; therefore $B_1M = 3A_1M$; therefore $A_1M = \frac{1}{3}MC$.

A_2, B_2 give the positions when B has reached M again. B was walking at one-half A's rate; therefore $A_1A_2 = 2B_1B_2 = 2MC$; therefore $A_2M = 2MC + \frac{1}{3}MC = \frac{7}{3}MC$.

A_3, B_3 give the positions when B is ready to start again. During the time B has been at M, A has walked a distance $= \frac{1}{60}MC$; for B has stayed as many minutes as A took hours to travel MC; therefore $A_3M = (\frac{7}{3} + \frac{1}{60})\,MC$.

But, since B and A now reach M simultaneously, and B goes three times as fast as A, $A_3N = \frac{1}{3}MN$; therefore $A_3M = \frac{2}{3}MN$; therefore $(\frac{7}{3} + \frac{1}{60})\,MC = \frac{2}{3}MN$; therefore $\dfrac{MC}{MN} = \dfrac{40}{141}$.

Ex. 3.—*The electrical resistance of wire of given material varies directly as the length and inversely as the sectional area.*

If the resistance of a wire 100 *ft. long, and weighing* 1 *oz. troy, be taken as unity, find that of a wire of the same material* 1 *metre long, weighing* 1 *grain* (1 *metre* = 39·37 *in.*)

One quantity varies *directly* as another when the one is multiplied (or divided) by 2, 3, &c., if the other is multiplied (or divided) by 2, 3, &c. One quantity varies *inversely* as another when the one is divided by 2, 3, &c., if the other is multiplied by 2, 3, &c.

The resistance of a wire varies as $\dfrac{\text{length}}{\text{area}}$ *i.e.* $\dfrac{\text{length} \times \text{length}}{\text{area} \times \text{length}}$. But the

weight is proportional to the area × length; therefore resistance varies as $\frac{\text{square of length}}{\text{weight}}$.

$$\therefore \frac{\text{resistance of 2nd wire}}{\text{resistance of 1st wire}} = \frac{(39\cdot37)^2}{1} \div \frac{(1200)^2}{20 \times 24}$$

$$= \frac{1549\cdot9969}{1} \times \frac{20 \times 24}{1200 \times 1200}$$

$$= \frac{1549\cdot9969}{3000}$$

$$= \cdot5166656.$$

But resistance of 1st wire is to be taken as unity; therefore resistance of 2nd wire = ·5166656.

Ex. 4.—*A leak is discovered in a ship, and a gang of 6 men make the pump suck after 6 hours. When the well is next sounded, the carpenter reports the same number of inches of water as before. A gang of 5 men now take 8 hours to free the ship. Supposing that the leak gains uniformly, and that the amount of water discharged by the pump varies directly with the number of men at work, how many men would have been needed to free the ship in 4 hours?*

In the first case 6 men work for 6 hours; ∴ 6 × 6 = 36 hours' work is given.

Hence 36 hours of work are needed to discharge original water + 6 hours' leakage;

so 40 ,, ,, ,, ,, + 8 hours' leakage;

∴ 4 ,, ,, ,, the leakage of 2 hours alone;

∴ 12 ,, ,, ,, ,, 6 hours alone;

∴ 24 ,, ,, ,, original water alone;

∴ 24 + 4 × 2 ,, ,, ,, + 4 hours' leakage.

Hence 32 hours of work are given. But the ship is cleared in 4 hours, and therefore 8 men must be at work.

EXAMPLES CXLV.

1. Find the smallest number which when divided by 25, by 30, or by 36, leaves in each case a remainder 17.

2. A boy was set to multiply 7,846,052 by 49,916; but, reading one of the figures in the question erroneously, he obtained 321,029,063,632 as his answer. Which figure did he mistake?

3. It is known that a certain part of the decimal ·324246 repeats. Calculate, as a vulgar fraction in its lowest terms, the difference between the greatest and least values that it can have.

4. Three numbers of three figures each are multiplied together. What is the greatest number of figures which may occur in the product, and what is the least?

5. A man rides on a bicycle for 50 miles without a rest. He rides at a uniform speed during each complete hour, reckoning from the start; but at the beginning of each hour he diminishes his speed by 2°/₀ of its previous value. His initial speed is 12 m.p.h. Find, correct to the tenth of a second, the time of his ride.

6. In a long division sum the dividend is 529,565, and the successive remainders from the first to the last are 246, 222, 542. Find the divisor and quotient.

7. Three numbers are known to end respectively with the figures 120, 125, 128. In each case state the greatest number which is certainly a factor.

8. All the numbers which can be formed with the digits 1 to 9, each digit occurring once, and once only, are added together. Prove that the sum is exactly divisible by the number 12,345,679.

9. A boy, on being asked what $\frac{12}{13}$ of a certain fraction was, made the mistake of dividing the fraction by $\frac{12}{13}$, and so got an answer which exceeded the correct answer by $\frac{25}{162}$. Required the correct answer.

10. Find three consecutive whole numbers such that one-half of the least, one-third of the next, and one-fourth of the greatest are together equal to 900.

11. In a certain experiment the mean of ten results gave 1·015102, that of the first six was 1·01267, that of the last five was 1·01688, while the seventh result was the square of the sixth. Find the mean of the last three.

12. A candle flame is placed between two parallel mirrors. The first, second, third, ... images are seen after the first, second, third, ... reflections. Supposing the brightness of the light to be halved after each reflection, how many images can be seen in each mirror if the image is no longer discernible after the reflection has only $\frac{1}{1000}$ part of its original brightness?

13. A bag contains sixpences, shillings, and half-crowns. The three sums of money expressed by the different coins are the same. If there are 102 coins in the bag, find its value.

14. A certain reef of quartz when crushed yields ·0011 °/₀ of gold. If the working expenses amount to 62·5 °/₀ of the gross receipts, and the net profit on each 100 tons crushed amounts to £52 10s., find the number of grains in a sovereign.

15. What is the smallest number of which the alternate figures are zero that is divisible both by 9 and by 11?

16. A clerk enters upon work at a yearly salary of £80, rising £15 a year every 3 years. What difference would have been made in his total income for 10 years if he had received an annual increase of £5 a year?

17. A certain number of tons, cwts., qrs. had to be multiplied by 10; but, in copying the sum down, tons were read as cwts., cwts. as qrs., and qrs. as lbs. The result obtained was 58 tons 4 cwts. 3 qrs. 18 lbs. too small. Find the correct number of tons, cwts., qrs. set.

18. The length of a line in miles is expressed by a number of two digits, which digits also express the respective rates of travelling in m.p.h. (supposed uniform) of two persons, A and B, who start simultaneously towards each other along the line from its two ends until they meet. If, at the same rates as before, they had similarly travelled along a line whose length was expressed by the same digits, but in the reverse order, they would have met one hour sooner. Find the length of the line.

19. When A and B commence walking towards each other they are $37\frac{1}{2}$ mls. apart. They walk uniformly at the respective rates of $3\frac{1}{2}$ and 4 mls. an hour until they meet. How long will this be after they set out, and how many miles will each have walked?

20. The weight of a body, calculated as the average of seven different experiments, is 53·735 grammes. The average of the first three is 54·005, the fourth was greater than the fifth by ·004, while the average of the sixth and seventh was ·010 less than the average of the first three. Find the weight of the body as obtained by the fourth experiment.

21. The price at which a publisher sells a book to a bookseller may be supposed to consist of the cost of the paper, the cost of printing and binding, general expenses, and the publisher's profit. If (the price of the book being the same) the cost of the paper be increased $33\frac{1}{3}$ %, and the cost of printing and binding be lessened 25 %, while the general expenses remain the same, the publisher's profit would be $\frac{1}{2}$d. more than before. If the cost of the paper and the cost of printing and binding are unaltered, while the general expenses are doubled, his profit is only half what it originally was; also, if the cost of the paper and the general expenses remain the same, while the cost of printing and binding be increased 10 %, his profit is still one-half of the original. The published price of such a book is 6s. If the retail bookseller allow a customer $1\frac{1}{2}$d. in the shilling discount on the published price, he makes a profit of 5 % on what he gave for it. Find the respective profits of the publisher and the bookseller.

22. In estimating the cost of 100 articles the mistake was made of reading pounds for shillings and shillings for pence in the price of each article, and thus the estimated cost was £212 18s. 4d. in excess of the true cost. Find the true price of each article.

23. Two sorts of coal at 21s. and 25s. per ton respectively are mixed together in a certain proportion. After an increase of price of 1s. per

ton on the cheaper sort and 2s. per ton on the dearer, it is found that, if the two sorts are mixed in the reverse proportion, the average price per ton is still the same. What was the proportion in which the two sorts were originally mixed?

24. If 3 men and 4 boys can do as much work as 2 men and 16 girls, and if 4 men and 2 boys can do as much as 12 boys and 12 girls, how much should a man who receives 44s. for a piece of work pay a boy and a girl who have been helping him all the time?

25. What is the present price of herrings per dozen when 5s. will purchase 2 dozen more, through a fall in price of 1s. per gross?

26. A reduction of 9s. 4d. per cwt. in the price of sugar enables a person to obtain 20 lbs. more for £1. Find the price of a ton at the cheaper rate.

27. A certain number of apples were distributed among the boys of two classes of a school. Each boy in the first class received 5 more than the number of boys in his class, and each boy in the second class received 4 more than the number in that class. In this way Class II. received altogether 63 more apples than Class I. If one boy had been transferred from Class II. to Class I., and the same plan of distribution followed, the total number of apples required would have been 1 less. Find the number of boys in each class and the number of apples distributed.

28. A man leaves by will the sum of £2325 to be divided among his three sons in the proportion of their ages at the time of his death. If the money had been at once divided in these proportions, the combined shares of the eldest and youngest sons would have been £1550. When the man's death occurs (at the age of fifty-five) the eldest son is entitled to £868. What was the youngest son's share?

29. The wages of 30 men and boys amount to £7 11s. a day, the men receiving 9s. and the boys 2s. each. What would be the amount of the daily wage-bill if each man received 6d. more and each boy 6d. less?

30. In a mixture of two ingredients the difference between the values per unit of the mixture and the dearer ingredient is to the difference between the values of the mixture and the cheaper ingredient in the same proportion as the quantity of the cheaper in the mixture is to the quantity of the dearer. Prove this by some perfectly general reasoning applied to some simple numerical example, *e.g.* price per lb. of a mixture of 3 lbs. of tea at 1s. 7d. per lb. and 5 lbs. at 1s. 11d. per lb. Show that the same reasoning is applicable to all cases.

31. In 1885 the Great Western Railway derived 10·5% of its passenger receipts from the first-class passengers, 17·7% from the second-class, and 71·8% from the third-class. In 1895 the receipts from the first and second classes showed decreases of 18 and 50% respectively on the corresponding amounts in 1885, while the receipts from the third class showed an increase of 38%. Find to two decimal places what percentage of the total receipts in 1895 was derived from each of the three classes.

32. A nugget consisting partly of gold and partly of quartz weighs 33·8 ozs.; a quantity of water having the same volume as that of the nugget weighs 12 ozs.; gold and quartz are respectively 19·35 and 2·65 times as heavy as equal volumes of water. Find the weight of the gold in the nugget.

33. If the diameters of a penny, halfpenny, and farthing were in the ratio of 6 : 5 : 4, and the weights were in the ratio 10 : 6 : 3, find the ratio of their thicknesses. Find the smallest number of pence which would form a column of equal height with one of half-pence and farthings, these latter being in equal numbers.

34. Out of a cask containing 360 qts. of wine a quantity of wine is drawn off and replaced by water; 80 qts. of mixture are withdrawn and replaced by water, with the result that wine and water are in the ratio of 91 to 89. What quantity of wine was first drawn off?

35. A vessel A contains 11 pts. of water and 7 pts. of wine, and a second vessel B contains 5 pts. of water and 13 pts. of wine. 2 pts. of the mixture in A are taken out and put into B. Afterwards 2 pts. of the mixture in B are taken out and put into A. By how much is the amount of each liquid in each vessel increased or diminished by the two operations?

36. A vessel trading between two ports left one of them under steam, but, 36 hours after starting, a fair wind springing up enabled her to go 3 mls. per hour faster during the remainder of her journey, and, in consequence, she reached her port of destination 21 hours sooner than if she had proceeded the whole way under steam only. Had she been able to utilize the wind during the whole journey, she would have arrived 6 hours before she actually did. Find her rate of steaming and the time required for the journey when she proceeds under steam only.

37. A triangle constructed of thin rods, two sides of which are of steel and one of copper, is supposed to be equilateral. Careful measurement, however, shows that the steel sides are 2·50000 units and the copper side 2·49986 units long. If steel and copper expand ·00001220 and ·00001718 of their respective lengths for an increase of 1 degree of temperature, find the increase to which the entire triangle must be subjected in order that it may be equilateral. (Answer to two decimal places.)

38. A and B row on a river from the same place, A down stream, B up stream. In $3\frac{3}{4}$ minutes they are 5 furls. apart; A then turns to follow B; and at the end of $12\frac{1}{2}$ minutes from that time the boats have together rowed 1 ml. 7 furls. in all. If the speeds of A, of B, and of the stream are constant, how many miles an hour does the river flow?

39. Two regiments commence their march together for a distant station at 6 a.m.; one arrives at 9.30 a.m., the other being then 160 yds. in the rear; at 9 o'clock this latter regiment was 3000 yds. from its destination. At what rate did each regiment march in yards per hour, neglecting fractions of a yard?

40. A cart-wheel, 4 ft. 6 ins. in diameter rolls, without slipping, through half a revolution. Find the horizontal distances traversed by the centre of the wheel, and by the two points of the rim upon a horizontal

diameter at the time when the wheel was set rolling. (The circumference of a wheel is $\frac{22}{7}$ times the diameter.)

41. Under the old tariff a telegram of 20 words, exclusive of the names and addresses of the sender and recipient, could be sent for 1s., and for each 5 words or fewer beyond 20 the charge was 3d. Under the new tariff the charge for each word, inclusive of names and addresses, is $\frac{1}{2}$d. A certain telegram would have cost 15d. under either tariff, but, if 9 words had been added to the body of the telegram (the addresses remaining the same), the cost under the new tariff would have been $1\frac{1}{2}$d. greater than under the old. Find, by a clear process of reasoning, what was the number of words in the body of the telegram.

42. A ship is steaming directly towards a headland from which a gun is fired. The interval between the sound and the flash is noted to be 10 seconds. Immediately that the sound is heard the engines are reversed, and the ship becomes stationary after travelling 1070 ft. from the time that the flash is seen. The gun is then fired again, and the interval between the sound and the flash is found to be $9\frac{3}{10}$ seconds. Find the velocity of of the ship at the time when the engines were reversed. (The velocity of sound is 1100 f.p.s.; the velocity of light is practically infinite.)

43. A marble floats in mercury with ·269 of its bulk below the plane of immersion. Water is poured on to the mercury until the marble is fully covered, and the volume now below the plane of immersion in the mercury is ·211 of the whole bulk. Compare the densities of water and mercury. (The weight of a floating body is equal to the weight of the liquid it displaces.)

44. The angle of inclination to the perpendicular of a bicycle moving round a circular track varies (approximately) as the square of the velocity and inversely as the circumference of the track. If the inclination is 6 degrees when the velocity is 20 m.p.h. and the circumference of the track $\frac{1}{3}$ of a mile, what is it when the velocity is 2 m.p.h. less and the circumference 440 ft. shorter?

45. A, B, C, D, E play at cards. A deals first and loses to B, C, D, E, respectively, as many counters as each of them had at the beginning of the deal; B deals second and loses to C, D, E, A, respectively, as many counters as each of them had at the beginning of the second deal; C, D, E deal in turn with a like result. They then find that each has 32 counters. How many had each at the beginning of the game?

46. In a hollow rectangular case of cast-iron, the external dimensions of which are length 14 ft. 4 ins., breadth 2 ft., height 2 ft., and the thickness of which is 2 inches, are placed 4 equal hollow closed cylinders of steel, each 14 ft. long and 1 in. thick, and having an internal radius of 4 ins. These cylinders are full of mercury, and the remaining spaces inside the case are full of water. If the area of a circle be $\frac{22}{7}$ multiplied by the square of its radius, and if cast-iron be 7 times, steel 8 times, and mercury $13\frac{1}{2}$ times as heavy as water, and if 1 cubic foot of water be taken to weigh $62\frac{1}{2}$ lbs., find the weights of the water, the mercury, the steel, and the iron respectively, and their sum.

47. When a leak was first discovered in a vessel the water had only 6 ft. more to rise to sink her. For the first 8 minutes 12 of the 18 men on board worked at the pumps, the water gaining 3 ft. against them; then 14 pumped for 6 minutes, but the water rose 2 ft. more; all hands then took to the pumps till the ship foundered. How long a time elapsed from the moment when the leak was discovered until the vessel sank?

48. A steamer, trading between a home port and a port distant 300 mls., realizes £400 on each to-and-fro voyage after paying for loading, discharging, and all dues. Expenses, amounting to £30 a day for wages, food, insurance, &c., are paid when the steamer is away from the home port. The steamer stays 6 days at the distant port, the actual time spent in sailing is 4 or $4\frac{1}{2}$ days, so that the length of the to-and-fro voyage is 10 or $10\frac{1}{2}$ days. The coal used costs 15s. per ton; the number of tons of coal used per hour varies as the cube of the speed, and a speed of 6 m.p.h. is got by burning 1 ton of coal per hour. Find for each length of voyage the profit per day.

49. A table is made of the mean monthly temperature for the first six months of the year. The temperature of January is $\frac{8}{9}$ that of April; that of February is $\frac{11}{10}$ that of January; March is the same as February; the average of April and June is 4 degrees above the average for the whole six months; the average of March and May is the same as the average for the six months; the average of January and May is equal to the temperature of March; and the temperature of May is 48 degrees. Find the monthly temperatures, and the average for six months.

50. A passenger steamer started from Liverpool for New York, and maintained a uniform speed for 40 hours, when her engines broke down, causing a delay of 6 hours. After temporary repairs she proceeded on her journey at a rate less than her former rate by $3\frac{1}{2}$ m.p.h., and, in consequence, arrived at New York 26 hours late. If she had left Liverpool at the reduced rate of speed and proceeded uniformly without mishap, she would have arrived 2 hours later than she did. Find the distance between the two ports.

51. A golfer has two clubs mended in London, a new head put on one, a leather face on the other. The former costs four times as much as the latter, the corresponding ratio in St. Andrews being 5; the leather face costs him twice the St. Andrews price. Including 1s. for a ball, he pays twice the St. Andrews charge for these repairs. What is the London charge for each?

52. Two towns, A and B, are connected by a straight line of railway, and the price of coal is 25s. per ton at A, and 20s. per ton at B. At a town on the line between A and B, which is 30 miles nearer to A than to B, the expense of getting coal is the same from each town. Find the cost of carriage of the coal per ton-mile.

53. Five years ago I invested a sum of money in the stock of a company which has ever since paid in half-yearly sums a return on my capital of 14·354 % per annum. At the end of each half-year I

re-invested in the same stock the interest which had accrued, together with a further sum of £75; and now at the beginning of the eleventh half-year I have the equivalent of £2075 invested in this stock. Calculate the sum originally invested, assuming that $(1{\cdot}07177)^{10}$ is equal to 2.

54. A thief steals a cycle and rides off on it at 9.30 a.m. at the rate of 16 m.p.h., followed 10 minutes later by the owner, who covers 15 mls. an hour while in the saddle, but spends 15 per cent. of his time after starting making inquiries. The thief, after riding 10 mls., transfers the stolen machine to an accomplice, who, after riding $4\frac{1}{2}$ mls. at 18 m.p.h., breaks down. After 15 minutes spent in repairing he remounts, but can progress at only 10 m.p.h. At what time to the nearest minute will he be overtaken by the owner?

55. The poor rate of a parish is spent partly in meeting expenses which do not vary from year to year, and partly in maintaining the poor. When there are 15 poor persons to be maintained the whole cost is £152 12s. 11d. a year; and when the number of the poor rises to 23 the whole yearly cost rises to £216 4s. 11d. What will it amount to in a year in which the number of poor is 12?

56. A father's present age is to his son's present age in the ratio of 7 to 3. Ten years ago the father was four times as old as the son. Find their present ages.

57. A testator bequeathed the sum of £110,223 in trust for his three daughters, A, B, C, whose ages at his death were respectively 11, 13, and 15 years, and he directed his trustees to divide the money into three parts, so that, at 3 % compound interest payable yearly, his daughters on respectively attaining their majority should become entitled to equal sums of money. Find the sums to be set aside for A, B, and C, respectively, to the nearest pound.

58. A rectangular box contains 1191 c.ins. of water, and the internal measurements of its sides each contain an exact number of inches. On the bottom of the box are placed as many spherical balls as it will contain; on the top of this layer is placed a similar layer, and so on, until the highest part of the top layer is just on a level with the surface of the water (which does not rise to the top of the box). If the balls be an inch in diameter, how many will there be? ($\pi = 3{\cdot}1416$.)

59. If the grass of a field grows uniformly and 15 cows consume it in 8 days, while 9 cows eat the grass of the same field in 16 days, how many cows will eat it in 12 days?

60. A meadow, in which the grass is supposed to grow uniformly and at a constant rate, is divided into two equal parts by a hurdle fence, and 50 oxen eat up all the grass in one part in 4 weeks. Four oxen are then slaughtered and the remainder are put into the second part. After 6 weeks 3 more oxen are slaughtered, and in another week, all the grass being eaten up, the whole of the oxen are sold. At the end of another week the hurdles are removed and 85 oxen are put into the field. How long will the grass last them?

SECTION X.

HIGHER PURE ARITHMETIC.

CHAPTER XXXII.

INVOLUTION.

[*This chapter should not be read until Algebraical Multiplication has been mastered and you are quite familiar with the process of combined multiplication and addition* (§ 30).]

245. The meaning of the phrases *power*, *square* or *second power*, *cube* or *third power*, has been given in § 31. It was there explained that **Involution** is the process of finding the various powers of a number, and that this might always be done by ordinary multiplication. But it will naturally be asked whether, having found a power of one number by multiplication, we can deduce others from it by some shorter method—whether, for instance, we cannot deduce 123^3 from 120^2, or 137^3 from 130^3, and the like. We proceed to explain the method of effecting this.

Algebraically the problem is solved by the formulæ

$$(a+b)^2 = a^2+2ab+b^2,$$
$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3,$$

&c. Thus,

$$(120+3)^2 = 120^2+2\,.\,120\,.\,3+3^2 = 14400+720+9 = 15129.$$

So that all that is needed is to discover the most convenient way of arranging this process. Now we note that

$$a^2+2ab+b^2 = a^2+(2a+b)b\,;$$

therefore we may put the rule thus: To deduce $(a+b)^2$ from a^2,

double a (getting $2a$), add b (getting $2a+b$), multiply by b (getting $2ab+b^2$), add to a^2, and we have $(a+b)^2$. This work may be arranged in columns—

240 $(2a)$	14400 (a^2)
3 (b)	729 $(2ab+b^2)$
243 $(2a+b)$	15129 $(a^2+2ab+b^2)$

or, if we omit unnecessary figures (printed in italics), and use the process of combined multiplication and addition,

240 14400 = 120^2,
243 15129 = 123^2.

Now we note that, to carry out this process, our columns must be headed by $2a$ and a^2, *i.e.* by twice the number already squared and by the square found. Similarly, in order to find another square from $(a+b)^2$, say $(a+b+c)^2$, we must start with two columns, one headed $2a+2b$, the other $(a+b)^2$. Our columns at present are $2a+b$, $(a+b)^2$. Hence we must add b once again to be ready for the next stage. We can now put the work for successively finding 120^2, 123^2, 127^2 thus—

240	14400 = 120^2
243	15129 = 123^2
246	16129 = 127^2
250	
254	

Since 127 is 4 more than 123, we add 4 to 246 (just as we added 3 to 240), obtaining 250, multiply this by 4, and add to 15129 (just as we multiplied the 243 by 3 and added to 14400). If we want to go on further, we add 4 once again to the 250, obtaining the figures printed in italics (*254*), which are not needed unless we want to proceed further.

246. Ex. 1.—*Calculate* 3^2, 30^2, 35^2, 350^2, 357^2, 3579^2, 35790^2, 35792^2.

6	9	= 3^2
60	**900**	= 30^2
65	1225	= 35^2
70	**122500**	= 350^2
700	127449	= 357^2
707	**12744900**	= 3570^2
714	12809241	= 3579^2
7140	**1280924100**	= 35790^2
7149	1281067264	= 35792^2
7158		
71580		
71582		

The first step is to make the columns $2a$, a^2, *i.e.* 6, 9. To deduce 30^2 we have clearly only to affix 00 to the square of 3, and, since twice 30 is 60, we affix *one zero in the first column and two zeros in the second column.* To make the steps clearer, this and corresponding steps are printed in darker type (**60**). We are now ready to proceed from 30^2 to 35^2. Having found 35^2, and completed the first column, we again affix one zero in the first and two in the second, reaching the **700**, **122500** stage and giving us 350^2, from which we proceed to 357^2, and so on. The steps printed in darker type may be, and should be, omitted, as in the next example.

Ex. 2.—Calculate 4^2, 47^2, 475^2, 4753^2.

80	*1600*
87	*220900*
940	*22562500*
945	*22591009*
9500	
9503	

The successive answers are printed in italics, *16*, *2209*, *225625*, *22591009*. Dotted lines indicate the division into the various steps.

Beginners should learn the first half, and advanced students the whole, of the following table :—

TABLE OF SQUARES.

1	**1**	5	**25**	9	**81**	13	169	17	289	21	441	
2	**4**	6	**36**	10	**100**	14	196	18	324	22	484	$25^2 = 625$
3	**9**	7	**49**	11	**121**	15	225	19	361	23	529	
4	**16**	8	**64**	12	**144**	16	256	20	400	24	576	

EXAMPLES CXLVI.

Calculate the squares of the following numbers :—

1. 3, 31, 312. **2.** 4, 41, 413. **3.** 6, 62, 623.

4. 5, 53, 532. **5.** 3, 39, 395, 3957. **6.** 4, 47, 473, 4735.

7. 71, 713, 7135. **8.** 82, 821, 8214. **9.** 93, 937, 9375, 93756.

10. 84, 846, 8464, 84643. **11.** 75, 757, 7575, 75754.

12. 66, 666, 6666, 66666. **13.** 12, 123, 1234, 12345, 123456.

14. 13, 134, 1345, 13456, 134567.

15. 55, 555, 5555, 55555, 555555, 5555555, 55555555.

16. 44, 444, 4444, 44444, 444444, 4444444, 44444444.

247. A very similar method can be applied to finding successive cubes. The formula upon which this depends is

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$
$$= a^3+\{3a^2+(3a+b)b\}b,$$

and the operation is worked in three columns instead of two, as in § 245, thus—

$$[1 \quad \frac{\begin{matrix}3a\\ b\end{matrix}}{3a+b} \quad \frac{\begin{matrix}3a^2\\ 3ab+b^2\end{matrix}}{3a^2+3ab+b^2} \quad \frac{\begin{matrix}a^3\\ 3a^2b+3ab^2+b^3\end{matrix}}{a^3+3a^2b+3ab^2+b^3}.$$

We may, if we please, introduce the figure 1 at the extreme left,† and then the operation is uniform—

Multiply first column by b and add to second.

Multiply sum in second column by b and add to third. (*A*)

Multiply sum in third column by b and add to fourth.

We shall speak of this as the "Operation A."

As a numerical instance let us find 14^3 from 12^3. We have $12^3 = 1728$, $12^2 = 144$, and $3 \times 12^2 = 432$, $3 \times 12 = 36$; so that the columns are

$$\begin{array}{r} 36 \\ 2 \\ \hline 38 \end{array} \quad \begin{array}{r} 432 \\ 76 \\ \hline 508 \end{array} \quad \begin{array}{r} 1728 \\ 1016 \\ \hline 2744 \end{array}, \quad \text{or} \quad \begin{array}{rrrr} & 36 & 432 & 1728 \\ 1 & 38 & 508 & 2744 \end{array}$$

performing the multiplication and addition in one operation.

Now we notice that, for the success of this method, it is essential that the columns should be headed [1], $3a$, $3a^2$, a^3. If we want to change $(a+b)^3$ into some other cube, we must have columns headed [1], $3(a+b)$, $3(a+b)^2$, $(a+b)^3$. At present they are headed [1], $3a+b$, $3a^2+3ab+b^2$, $(a+b)^3$. Thus the second and third of them need changing. This change can be made as follows :—

Use the operation A on the first three columns only; then use the operation A on the first two columns only,

and the columns will now be found ready for use in the next stage.

$$\begin{array}{rrrr} 1 & 3a & 3a^2 & a^3 \\ & b & 3ab+b^2 & 3a^2b+3ab^2+b^3 \\ & \hline 3a+b & 3a^2+3ab+b^2 & a^3+3a^2b+3ab^2+b^3 = (a+b)^3 \\ & b & 3ab+2b^2 & \\ & \hline 3a+2b & 3a^2+6ab+3b^2 & \\ & b & [=3(a+b)^2] & \\ & \hline 3a+3b = 3(a+b) & & \end{array}$$

† This 1 might have been introduced in § 245, and a similar uniformity obtained.

248. Ex. 3.—*Calculate* 7^3, 13^3, 130^3, 135^3.

1	21	147	343	$= 7^3$
	27	309	2197	$= 13^3$
	33	507	**2197000**	$= 130^3$
	39	**50700**	2460375	$= 135^3$
	390	52675		
	395			

Explanation.—We start with 1, 3×7, 3×7^2, 7^3. Then, since $13 = 7 + 6$, we "operate" (A) across with 6. The multiplication and addition are done together.

The 2197 in the last column is $(7 + 6)^3$, *i.e.* 13^3. To prepare for the next step "operate" (A) across, stopping at the last column but one; then "operate" across once more, stopping at the last column but two. The columns are now closed by 39 ($= 3 \times 13$), 507 ($= 3 \times 13^2$), and 2197. It is clear now that 130^3 will be 13^3 followed by 3 zeros, 3×130^2 will be 3×13^2 followed by 2 zeros, 3×130 will be 3×13 followed by zero. Hence *affix one zero to end of second column, two to third, three to next.* We reach the stage in blacker type. Since $135 = 130 + 5$, we now operate across with 5, and obtain our final result. To continue further we "operate" on three columns with 5, then on two columns.

The stage in blacker type may be omitted, as in the following example :—

Ex. 4.—*Calculate* 11^3, 112^3, 1123^3, *and* 1127^3.

1	330	36300	1331000	$= 110^3$
	332	36964	1404928000	$= 1120^3$, "operating" (A) with 2
	334	3763200	1416247867	$= 1123^3$, "operating" (A) with 3
	3360	3773289	1431435383	$= 1127^3$, "operating" (A) with 4
	3363	3783387		
	3366	3796879		
	3369			
	3373			

The following table should be accurately known :—

TABLE OF CUBES.

1	**1**	3	**27**	5	**125**	7	**343**	9	**729**	11	**1331**
2	**8**	4	**64**	6	**216**	8	**512**	10	**1000**	12	**1728**

249. An exactly similar process applies to fourth and higher powers, provided that for fourth powers we start with the columns 1, $4a$, $6a^2$, $4a^3$, a^4; for fifth powers we start with the columns 1, $5a$, $10a^2$, $10a^3$, $5a^4$, a^5; &c.

Ex. 5.—*Calculate* 6^4, 61^4, 611^4.

1	240	21600	864000	12960000	$= 60^4$
	241	21841	885841	13845841	$= 61^4$
	242	22083	907924	**138458410000**	$= 610^4$
	243	22326	**907924000**	139368569041	$= 611^4$
	244	**2232600**	910159041		
	2440	2235041			
	2441				

Ex. 6.—*Find the least multiplier which will make* 1026 *into* (1) *a perfect square*, (2) *a perfect cube.*

Splitting 1026 into factors, we have $1026 = 2 \times 3^3 \times 19$. To make it into a perfect square we must multiply by $2 \times 3 \times 19$ ($= 114$), thus turning it into $2^2 \times 3^4 \times 19^2$. To make it into a perfect cube we must multiply by $2^2 \times 19^2 = 1444$.

EXAMPLES CXLVII.

Calculate the cubes of the following :—

1. 50, 53, 530, 531.

2. 40, 45, 451, 4510.

3. 41, 413, 415, 418.

4. 52, 521, 523, 525.

5. 61, 612, 6121, 6123.

6. 82, 821, 8211, 8213.

7. 72, 721, 7215, 7218.

8. 93, 931, 9315, 9317.

9. 21, 213, 218, 2187, 2189.

10. 31, 312, 317, 3178, 3179.

11. 13, 1301, 1306, 13061.

12. 81, 8103, 8107, 81071.

13. 17, 190, 190010.

14. 19, 210, 210010.

15. 12, 123, 1234, 12345, 123456.

16. 21, 213, 2134, 21345, 213456.

Calculate the fourth powers of the following :—

17. 11, 12, 121. **18.** 20, 21, 211. **19.** 111, 113. **20.** 101, 107.

Calculate the fifth powers of—

21. 2, 21, 211.

22. 3, 31, 311.

Find the least multipliers which will make the following numbers (1) perfect squares, (2) perfect cubes :—

23. 1422, 12825, 29040.

24. 1917, 90250 18360.

CHAPTER XXXIII.

EVOLUTION.

250. Involution and **Evolution** are **inverse** operations: *i.e.* the one undoes the effect of the other.

Similarly addition and subtraction, multiplication and division, are inverse operations.

By Involution we find the powers of a number. By Evolution, when the power is given, we discover the number which gave rise to it, which is called the **root.**

Thus 49 is the square of 7; 7 is the *square root* of 49, denoted $\sqrt[2]{49}$.

So 64 is the cube of 4; 4 is the *cube root of* 64, denoted $\sqrt[3]{64}$.

Similarly a fourth root is denoted $\sqrt[4]{}$. The 2 in $\sqrt[2]{}$ is generally omitted.

The tables in §§ 246, 248 supply, therefore, not merely squares and cubes, but also square roots and cube roots. You should practise the use of the tables in both ways, and thus recognize at once not merely that 8^3 is 512, but also that $\sqrt[3]{512}$ is 8.

Evolution by Factorization.

251. If the prime factors of a number can be determined, all the roots which it possesses can be seen at once. Thus, $1296 = 2^4 \times 3^4$, and therefore $\sqrt[2]{1296} = 2^2 \times 3^2 = 36$. Similarly $35937 = 3^3 \times 11^3$, and therefore $\sqrt[3]{35937} = 3 \times 11$. Obviously it is necessary for the success of this method in square root that all the indices should be even; in cube root that all the indices should be divisible by 3.

EXAMPLES CXLVIII.

Find, by factorization, the square root of—

1. 784, 1024, 1764. **2.** 1089, 1225, 2304. **3.** 4096, 9216, 9801.

4. 7056, 3025, 7744. **5.** 15625, 17424. **6.** 20736, 15876.

7. 527076, 893025. **8.** 194481, 777924. **9.** 1285956, 700131600.

10. 3175524, 952956900. **11.** 46294416. **12.** 76527504.

Find, by factorization, the cube root of—

13. 2000376 and 2299968. **14.** 2097152 and 2460375.

15. 209584584. **16.** 203297472. **17.** 10720765125.

18. 136193423625. **19.** 311114456576. **20.** 362162161152.

Find, by factorization, the fourth root of—

21. 50625 and 104976. **22.** 65536 and 194481.

23. 3111696. **24.** 3748096

Find, by factorization, the fifth root of—

25. 243 and 3125. **26.** 1024 and 7776.

27. 130691232. **28.** 164916224.

Square Root.

252. The process by which square (and all higher roots) are obtained is one known as *Successive Approximation.* Thus, if the actual square root were 7532, we should obtain successively for the answer 7000, 7500, 7530, and finally 7532. These answers are tested by squaring them one after the other by the methods just described, and noting that the squares are closer and closer to the number under consideration.

There remain two questions to be considered before we can give examples of the practical application of the method—

(1) How are we to determine the *first approximation?*

(2) How can the first approximation be made to suggest the second, the second the third, and so on?

253. *Determination of the First Approximation.*

A number of 1 digit is less than 10.

∴ The square of a number of 1 digit is less than 100 (10^2), *i.e.* has either 1 or 2 digits.

A number of 2 digits is not less than 10, but is less than 100.

∴ The square of a number of 2 digits is not less than 100 (10^2), but is less than 10,000 (100^2), *i.e.* has either 3 or 4 digits.

A number of 3 digits is not less than 100, but is less than 1000.

∴ The square of a number of 3 digits is not less than 10,000, but is less than 1,000,000, *i.e.* has either 5 or 6 digits; and so on.

Putting these results together, we deduce the following table:—

Number.		Square.
1 digit	.....	1 or 2 digits.
2 digits	.....	3 or 4 digits.
3 digits	.....	5 or 6 digits.

&c.

Hence to determine the number of digits in the *square root* of a square, we mark the digits of the square off into pairs, say by dashes above the number, beginning from the unit's figure, and count the number of pairs, counting one digit over as being a complete pair.

Thus, 17′79′99′61 has 4 pairs; therefore there are 4 digits in the square root.

So 8′41′58′01 has 3 pairs and 1 digit (8) over, which counts as a pair; so that there are again 4 digits in the square root.

We can now at once determine the first approximation.

Ex. 1. *Determine the first approximation to the square root of* 1024.

Mark into pairs 10′24. There are 2 pairs; therefore 2 figures in the square root. Now $30^2 = 900$, $40^2 = 1600$, and, since 1024 lies between 900 and 1600, $\sqrt{1024}$ must lie between $\sqrt{900}$ and $\sqrt{1600}$, *i.e.* between 30 and 40. Hence the first figure of the square root is 3, and the first approximation is 30.

The argument of this example shows the truth of the following rule:—

> *After marking the digits of the number into pairs from the right, take the left pair (or single digit). If this is not an exact square, think of the square next lower than this pair by table* (§ 246). *The square root of this is the first figure of the required square root, and the first approximation may be obtained by affixing 0's until the required number of digits is obtained.*

Thus, 15′21. The square next below 15 is 9. The square root of this is 3; therefore 3 is the first figure of, and 30 is the first approximation to $\sqrt{1521}$.

5′19′84. The square next below 5 is 4. The square root of this is 2; therefore 2 is the first figure of, and 200 is the first approximation to $\sqrt{51984}$.

254. *Determination of Second Approximation.*

We must next inquire how, when one approximation has been obtained, the next approximation is suggested.

Suppose that N is the number whose square root we are finding and that a is the part already found ("first approximation"), while x is the part which remains to be found; so that $a+x$ is the complete square root and $\sqrt{N} = a+x$.

Since $\sqrt{N} = a+x$, $(\sqrt{N})^2 = (a+x)^2 = a^2+2ax+x^2$;

$$\therefore \quad N-a^2 = 2ax+x^2.$$

Now x is less than a; for it is always a number of at least one digit less than a; therefore $x \times x$ is less than $a \times x$, *i.e.* x^2 is less than ax.

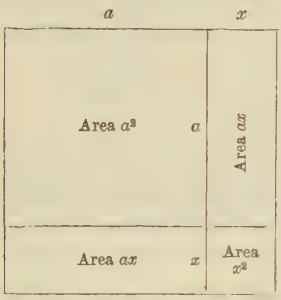
$$\therefore \quad 2ax \text{ is the greater part of } N-a^2.$$

$$\therefore \quad x \text{ is the greater part of } (N-a^2) \div 2a.$$

Hence, if we *square the part already found* (a^2), *subtract from the complete square* (N), *and divide the difference by twice the part already found, the integer next below the quotient will always be near*† *to the part which still remains to be found* (x).

† See Exx. cxlix., 27, 28.

A geometrical illustration may help to make the matter clearer.



The square on the line $a+x$ (of which $a>x$) can easily be cut into four pieces by lines drawn as in the figure. These four pieces are

A square whose area is seen to be $a \times a$, *i.e.* a^2.

Two rectangles the area of each of which is ax.

A small square whose area is x^2.

When we cut away from the large outer square the square whose area is a^2, it is at once clear that the rectangles ax form the greater part of what is left, *i.e.* that $2ax$ is the greater part of $(a+x)^2-a^2$ or $N-a^2$.

255. We can now show how these principles may be applied to finding square roots accurately.

Ex. 2.—*Find the square root of* 961.

		a x
	9′61 (N)	(30 + 1
60 ($2a$)	*900* (a^2)	
61	61 ($N-a^2$)	
	61	

Explanation.—We first pair the digits off. The left pair is 9. The nearest square to this is 9. The square root of this is 3; therefore 3 is the first figure and 30 is the first approximation. We find the square of 30 (as in § 246, *q.v.*), *i.e.* 900, and subtract this from 961, getting 61. Now the very process of § 246 has provided us with the $2a$ referred to in the last §. If we divide this $2a$ (*i.e.* 60) into the $N-a^2$ (*i.e.* 61), we shall get a quotient nearly equal to the part (x) which remains to be found. But 60 will divide into 61 *once;* therefore 1 is suggested for the figure of the square root still to be found. We find the square of 31 by the process of § 246. In this process we should have added together the figures (*900, 61*) printed in *italics*. Instead of adding these and subtracting the sum from 961 to see if anything remains, as we have already subtracted *900*, it is sufficient to subtract the new part *61*. Nothing remains, and therefore we see that 30 + 1, *i.e.* 31, is really the number which, when squared, produces 961, *i.e.* $31 = \sqrt{961}$.

Ex. 3.—*Find the square root of* 169744.

	16′97′44	(400 + 10 + 2
800	16 00 00	
810	97 44	
820	81 00	
822	16 44	
	16 44	

Explanation.—After pairing we find 16 in the pair at the left; therefore the first figure is 4, and the first approximation 400. Squaring this, as in § 246, we get 160000 ($= a^2$) and 800 ($= 2a$). Thus, 9744 is what is left when we subtract the square of the first approximation from the given square. But 800 will go into 9744

rather more than 12 times; 12 is suggested for the missing part. We might have tried 12 and finished the question at once, as in the work on the right; but it is usually safer to proceed one figure at a time and try 10, giving 1 for the second digit. We add this 10, getting 810, and multiply by it, getting 8100 (as in § 246), and subtract from 9744, thus getting the remainder 1644 left after taking the square of 410 away. As we are proceeding further we add 10 once more in the first column (§ 246), getting 820, which is twice the part already found. But, applying once more the principle of § 254, we see that 820 divided into 1644 must give us a close guess at the figure still left. But 1644 ÷ 820 is 2 and a fraction; therefore we try 2, and we find that when we have subtracted the square of 412 we have nothing left; therefore 412 is the square root.

	16′97′44 (400 + 12
800	16 00 00
812	97 44
	97 44

	16′97′44 (412
	16
81	97
	81
822	16 44
	16 44

Abbreviation.—When you are once familiar with the meaning of the various steps a large number of the figures employed above may be omitted. We write 16 instead of 160000, leaving the position of the 16 to explain its value. We bring down two figures at a time: thus, 97 instead of 9744. Instead of writing 800, and then adding the 10, we write 8–, leaving the figure to follow the 8 vacant until we have settled what it is to be, and then writing it in after the 8. Also we put 81 instead of 810. If we do the final multiplications and subtractions in one operation, it assumes the concise form on the right.

	16′97′44 (412
81	97
822	16 44
	0

Ex. 4.—*Find the square root of* 3744225.

	3′74′42′25 (1935
	1
29	2 74
	2 61
383	13 42
	11 49
3865	1 93 25
	1 93 25

When we have found the first figure 1 we double it for the first trial divisor (2*a*). Now 20 will divide into 274 13 times. This seems to give 13 for the next *digit*. This is absurd, The digit cannot be greater than 9. Try 9. taking a digit as near 13 as we can. Write the 9 after the 2, and divide 29 (really 2900) into 274 (really 2744225). It goes 9 times, and gives the 261, which is written and subtracted from 274. The result of the subtraction is 13, and two more figures are brought down, making 1342. For the next trial divisor we add the figure 9 just obtained to the 29, getting 38–, and this goes 3 times into 1342; so we write the 3 in the vacant place after the 38– and perform the division by 383, and so on. *If the second figure* 9 *had proved too big, as will sometimes happen, we should have tried* 8. A little experience will show you the right figure. If we combine the multiplication and subtraction, we get the result shown on the right, which gives the work in its most concise form.

	3′74′42′25 (1935
29	2 74
383	13 42
3865	1 93 25

EXAMPLES CXLIX.

Find the square roots of the following numbers :—

1. 2704, 5329, 6561.
2. 3844, 5041, 6889.
3. 3364, 6084, 7921.
4. 3481, 5929, 9604.
5. 17161, 15129, 53824.
6. 19881, 45369, 97969.
7. 42025, 106276, 368449.
8. 50625, 93636, 826281.
9. 247009, 151321, 488601.
10. 356409, 158404, 485809.
11. 17230801, 26656569, 38576521.
12. 17363889, 27019204, 39828721.
13. 49266361, 65658609, 36108081.
14. 49463089, 82973881, 49098049.
15. 152399025.
16. 175668516.
17. 395055376.
18. 892037689.
19. 45563463936.
20. 97628751936.
21. 39955212544.
22. 35720622001.
23. 1526900119684.
24. 1759833068569.
25. 560673398602849.
26. 695684619195556.

27.* Show that the figure obtained by using $2a$ as the trial divisor into $N-a^2$ after the *first* approximation cannot be more than $\frac{5}{3}$ of the correct figure.

28.* Show that the digit obtained by using $2a$ as the trial divisor into $N-a^2$ after the *second* approximation can never be more than 1 in excess of the correct digit.

Cube Root.

256. *Determination of the First Approximation.*

A number of 1 digit is less than 10.

∴ The cube of a number of 1 digit is less than 1000 (10^3), *i.e.* has either 1, 2, or 3 digits.

A number of 2 digits is not less than 10, but is less than 100.

∴ The cube of a number of 2 digits is not less than 1000 (10^3), but is less than 1000000 (100^3), *i.e.* has either 4, 5, or 6 digits ; and so on.

Putting these results together, we deduce the following table :—

Number.		Cube.
1 digit	.....	1, 2, or 3 digits.
2 digits	.....	4, 5, or 6 digits.
3 digits	.....	7, 8, or 9 digits.

&c.

Hence, to determine the number of digits in the *cube root* of a cube, we mark off the digits of the cube into *threes*, beginning from the unit's figure, and counting the number of threes, one or two digits over being reckoned as a complete three. The number of such groups is the number of digits in the cube root.

Thus, 8'120'601 has 3 groups (one incomplete); therefore there are 3 digits in the cube root.

By an argument exactly similar to that used in the Example in § 253 we can now deduce the following rule :—

> *After marking the digits of the number from the right into* THREES, *take the left group. If this is not an exact cube, think of the cube next lower than this group by table* (§ 248). *The cube root of this is the first figure of the required cube root, and the first approximation may be obtained by affixing 0's until the required number of digits is obtained.*

Thus, 32'768. The cube next below 32 is 27. The cube root of this is 3: therefore 3 is the first figure and 30 is the first approximation to $\sqrt[3]{32768}$.

257. *Determination of the Second Approximation.*

Suppose that N is the number whose cube root we are finding and that a is the part already found ("first approximation"), while x is the part which remains to be found, so that $a+x$ is the complete cube root and $\sqrt[3]{N} = a+x$;

$$\therefore \quad N = (a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3.$$

$$\therefore \quad N - a^3 = 3a^2x + 3ax^2 + x^3.$$

Now x is less than a; therefore $ax \times x$ is less than $ax \times a$, *i.e.* ax^2 is less than a^2x, and similarly x^3 is less than a^2x. Hence

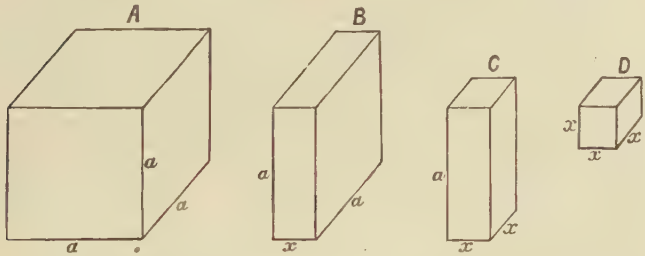
$3a^2x$ is usually the greater part of $N - a^3$.

$\therefore$ x is usually the greater part of $(N - a^3) \div 3a^2$.

Hence, if we *cube the part already found* (a^3), *subtract from the complete cube* (N), *and divide the difference by three times the square of the part already found, the integer next below the quotient will usually be near*† *to the part which still remains to be found.*

† See Exx. cl., 25, 26.

A geometrical illustration similar to that in § 254 can easily be given, and the teacher should provide himself with the so-called *Binomial Cube* for the purpose.



The private student should satisfy himself that a cube such as A (a^3), three blocks like B (a^2x), three like C (ax^2), and a cube like D (x^3) can be fitted together to form a perfect cube ($= \overline{a+x}^3$), and that when A (a^3) is taken away the three blocks B form, usually, the greater part of what is left.

258. We can now show how the principle may be applied to finding cube roots accurately.

Ex. 1.—*Find the cube root of* 29791.

				a	x
			29'791	(30	+ 1
1	90 ($3a$)	2700 ($3a^2$)	*27 000*	(a^3)	
	91	2791	2 791	($N-a^3$)	
			2 791		

Explanation.—We first group the digits off into threes. The left-hand incomplete group is 29. The nearest cube below this is 27. The cube root of this is 3; therefore 3 is the first figure and 30 is the first approximation. We first find the cube of 30 (as in § 248), arranging altogether 4 columns 1 90 ($= 3 \times 30$) 2700 ($= 3 \times 30^2$) 27000 ($= 30^3$). Subtracting the 27000 from the given cube 29791, we have 2791 left. But we have seen that $3a^2$ (*i.e.* 2700) divided into $N-a^3$ (*i.e.* 2791) usually gives a close approximation to the part of the cube root left. 2700 divides into 2791 once; therefore 1 is suggested as the next figure. Calculate the cube of $30+1$ by the method of § 248. To do this we should have to add the *2791* to the *27000* printed in italics. Instead of adding these and subtracting from 29791 to see if anything remains, as we have already subtracted *27000*, it is sufficient to subtract the new part *2791*. Nothing remains, and we see that $30+1$, *i.e.* 31, is really the number which when cubed produces 29791, *i.e.* $= \sqrt[3]{29791}$.

Ex. 2.—*Find the cube root of* 69934528.

			69′934′528 (400 + 10 + 2
1	1200	480000	64 000 000
	1210	492100	5 934 528
	1220	504300	4 921 000
	1230	506764	1 013 528
	1232		1 013 528

Explanation.—After grouping we find 69 in the group at the left. 64 is the nearest cube below this; therefore the first figure is 4 and the first approximation is 400. Treating this as in § 247 we get the columns 1, 1200 ($3a$), 480000 ($3a^2$), 64000000 (a^3). Thus, 5934528 is what is left when we subtract the cube of the first approximation from the given cube. But 480000 ($3a^2$) will divide into 5934528 ($N-a^3$) rather more than 12 times; therefore 12 is suggested for the missing part. We might have tried 12 and finished the work at once as in the work on the right; but it is usually safer to proceed one figure at a time, and try 10, giving 1 for the second digit. We change the cube of 400 into that of 410 by the method of § 248 and subtract the additional part (4921000) from what was left before (5934528). *As we propose to proceed further, we must carry out the operation A of* § 247 across the columns once more, missing out the last, then once again, missing out the last but one. We are now ready to proceed, and the third column (from the left) will again give the $3a^2$ (where a represents the part already found, now 410) which we need for a trial divisor. This $3a^2$ (now 504300) divided into the new $N-a^3$ (1013528) suggests 2 for the figure remaining. Calculate 412^3 from 410^3, and we find that when we have subtracted this from the original we have nothing left; therefore 412 is the cube root.

1	1200	480000	69′934′528 (400 + 12
	1212	494544	64 000 000
			5 934 528
			5 934 528

			69′934′528 (412
1	120	4800	64
	121	4921	5934
	122	504300	4921
	1230	506764	1013528
	1232		1013528

Abbreviation.—When you are once familiar with the meaning of the various steps a number of the figures employed may be omitted. Thus we can write 64 instead of 64000000, leaving the position of the 64 to explain its value. Also we bring down only 3 figures at a time, and thus write 5934 instead of 5934528. The result is the omission of figures printed in italics, and the work proceeds as follows:—After deciding that 4 is the first figure, we write on the line below the number whose cube root we are finding 1, 12 (3×4), 48 (3×4^2), 64; then subtract 64 from 69. This completes the first step. To prepare for the second *affix one* 0 *in second column, two in third, and bring down three figures* (934) *in the fourth.* We are now ready to proceed. 4800 is the trial divisor into 5934 and gives the figure 1. Write 1 after the 4 in the answer, and "operate" (A, § 247) across with the 1 all the way, then all the way missing the last, and so on. When

this is done the columns are closed by the Subtract in the last column. We are now ready for the next stage. Affix one 0 in second column, 2 in third, and bring down three figures (528) in the fourth. The rest of the work proceeds as before. The 1 at the left may be omitted, and the subtraction worked simultaneously with the multiplication, and then the work assumes its most concise form shown on the right.

figures 123, 5043, 4921.

		69934528 (412
120	4800	5934
121	4921	1013528
122	504300	
1230	506764	
1232		

Ex. 3.—*Find the cube root of* 7245075375.

		7′245′075′375 (1935
1 **3**0	**3**00	**1**
39	651	6245
48	**1083**00	**5859**
570	110019	386075
573	**11174700**	**330057**
576	11203675	5601 8375
5790		5601 8375
5795		

Here the various steps of the process are distinguished by the printing in darker figures (**57**) of the last figures in each step, and also by lines separating one step from the next.

The first trial divisor, 300, when divided into 6245 suggests 21 for the next digit. This is absurd. The *digit* cannot be more than 9. We try the digit 9 as being the highest possible. It might have been found that this was too high. Nothing but practice can settle this question; but you are advised to *calculate roughly, mentally, what the third column will come to when the new operator* (9 *in this case*) *has been used upon it* before writing any figure down. The difficulty will rarely occur after the second figure of the root has been found. (See Ex. cl., 26.)

We add another example in its most abbreviated form—

Ex. 4.—*Find the cube root of* 12189231526744.

		12′189′231′526′744 (23014
60	**12**00	4189
63	1389	22 231 526
66	**1587**0000	6 354 625 744
6900	15876901	
6901	**1588380**300	
6902	15886 56436	
69030		
69034		

One difficulty arises here. On finding the third figure of the root the trial divisor is 158700 and the dividend is 22231. The divisor will not divide at all into the dividend, and the figure is 0. We write this figure, and then *once more put one* 0 *at the end of our first column* (we have omitted the 1 which formed the first column in the earlier work), *two* 0*'s at the end of the second column, and bring down three figures in the third column.* The trial divisor is now 15870000, the dividend 22231526, and the quotient is clearly 1.

EXAMPLES CL.

Find the cube root of—

1. 9261, 74088, 148877.
2. 32768, 68921, 226981.
3. 389017, 571787, 614125.
4. 373248, 531441, 857375.
5. 6859, 21952, 117649.
6. 24389, 19683, 110592.
7. 8365427, 68417929.
8. 8615125, 127263527.
9. 26463592, 487443403.
10. 26730899, 674526133.
11. 726572699, 929714176.
12. 967361669, 904231063.
13. 1018108216, 27189441343.
14. 8060150125, 64144108027.
15. 67568337017.
16. 127795585653.
17. 268086587392.
18. 320575259584.
19. 963259373376.
20. 906477133752.
21. 1107111813625.
22. 1171905573007.
23. 15034474613241.
24. 2503821503539.

25.* Prove that the figure obtained by using $3a^2$ as the trial divisor into $N-a^3$ after the *first approximation* cannot be more than the integer next below $2\frac{16}{27}$ of the correct figure.

26.* Show that the digit obtained by using $3a^2$ as the trial divisor into $N-a^3$ after the *second* approximation can never be more than 2 in excess of the correct digit.

Roots of Fractions.

259. If we multiply a fraction by itself, we obtain a new fraction whose numerator is the square of the given numerator and whose denominator is the square of the given denominator. Thus $\frac{3}{4}\times\frac{3}{4}=\frac{9}{16}$. It follows that whenever the numerator and denominator of a fraction are both squares the square root of the fraction is a new fraction whose numerator is the square root of the original numerator and denominator the square root of the original denominator.

An exactly similar method applies to cube roots.

The fractions should first be reduced to their lowest terms, and a mixed number should usually be expressed as an improper fraction; though it is sometimes better to express the fractional part as a decimal, and proceed by the method of the following paragraphs (*e.g.* Exx. cli., 3, 6, 13, 14):—

Ex. 1.—*Find the square root of* $\frac{121}{169}$.

$$\sqrt{\tfrac{121}{169}}=\frac{\sqrt{121}}{\sqrt{169}}=\tfrac{11}{13}. \quad \textit{Ans.}$$

Ex. 2.—*Find the cube root of* $15\frac{5}{8}$.

$$\sqrt[3]{15\tfrac{5}{8}}=\sqrt[3]{\tfrac{125}{8}}=\frac{\sqrt[3]{125}}{\sqrt[3]{8}}=\tfrac{5}{2}=2\tfrac{1}{2}. \quad \textit{Ans.}$$

EXAMPLES CLI.

Find the square roots of the following fractions:—

1. $\frac{784}{841}, \frac{1024}{1225}, \frac{1681}{2116}$. **2.** $\frac{529}{1156}, \frac{1849}{1936}, \frac{441}{2401}$. **3.** $210\frac{1}{4}$. **4.** $25\frac{121}{144}$.
5. $145\frac{144}{169}$. **6.** $410\frac{1}{16}$. **7.** $101\frac{92}{169}$. **8.** $37\frac{212}{841}$.
9. $9\frac{67}{121}, 2\frac{23}{49}$. **10.** $1\frac{122984}{38025}$. **11.** $5346\frac{55}{289}$.
12. $2517\frac{196}{289}$. **13.** $41943\frac{1}{25}$. **14.** $301675\frac{9}{16}$.
15. $1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6\frac{7}{47}}}}$.

Find the cube roots of the following fractions:—

16. $\frac{64}{729}, \frac{2744}{9261}, \frac{33}{8}$. **17.** $\frac{27}{1331}, \frac{1331}{10648}, 1\frac{91}{125}$. **18.** $\frac{140608}{205379}$.
19. $\frac{132651}{226981}$. **20.** $397\frac{2684}{4913}$. **21.** $353273\frac{733}{4096}$. **22.** $1277289\frac{27}{216}$.

Roots of Decimals.

260. Since decimals are like ordinary numbers, in that each place has a value one-tenth of that to its left, it follows that their roots may be found by exactly the same method as for ordinary numbers. The following rules are important:—

(1) Mark off the groups (pairs of digits for square root, threes for cube root) *from the decimal point* both ways, right and left, and *not* from the end of the decimal.

Thus, 123·43216 is marked 1′23·43′21′6 for square root. This point is of great importance in dealing with *Surds* (§ 262).

(2) Put a decimal point in the root when you have finished dealing with the integral part of the power, and before you bring down any figures from the decimal part of the power.

(3) Circulating decimals should be converted into vulgar fractions, or are usually better dealt with by the methods of § 262.

261. Ex. 1.—*Find the square roots of* 841·5801 *and of* ·0008555625.

(A)

```
          8′41·58′01 (29·01
          4
49        4 41
          4 41
5801          58 01
              58 01
```

(B)

```
          ·00′08′55′56′25′ (·02925
          4
49        4 55
          4 41
582        14 56
           11 64
5845        2 92 25
            2 92 25
```

The work might, of course, have been contracted by the omission of the partial products in both cases. In (A) you should notice carefully the treatment of 0. In (B), since there is no integral part in the power, we put a decimal in the answer at once. The first period is 00. The nearest square to this is 0; therefore we commence with a 0 in the answer. The rest of the work is regular. Having fixed the position of the decimal point, &c., we find the root just as if there were no decimal point there at all.

Ex. 2.—*Find the cube roots of* ·039304 *and of* 9393·931.

(A)

			·039′304 (·34
1	9	27	27
	90	2700	12 304
	94	3076	12 304

(B)

		9′393·931 (21·1
60	1200	1 393
61	1261	132 931
62	132300	
630	132931	
631		

We mark off the digits in threes, beginning at the point, and moving right and left. The work in (A) has been done in full, that in (B) in its most contracted form. In both cases the work begins by the formation of the columns 1, $3a$, $3a^2$ (and a^3). Thus, 1, 9, 27, 27 represent 1, 3×3, 3×3^2, and 3^3.

EXAMPLES CLII.

Find the square roots of—

1. 2·89, 5·76, 127·69. **2.** 7·29, 12·96, 153·76.

3. ·0289, ·000576, 1·2769. **4.** ·0729, ·1296, ·00015376.

5. 30·3601, ·303601, 3036·01. **6.** ·327184, ·0000327184, 32·7184.

7. 341·1409. **8.** ·00868624. **9.** 1383·0961. **10.** 1618·4529.

11. 50·253921. **12.** ·0002679769. **13.** 2704·416016.

14. 968·641129. **15.** 676·208016. **16.** 557·196025.

17. ·02$\dot{7}$. **18.** 4·69$\dot{4}$.

Find the cube roots of—

19. ·027, ·000343, 19·683. **20.** ·064, ·000729, 24·389.

21. ·166375. **22.** ·287496. **23.** ·008869743. **24.** 10793·861.

25. 2571·353. **26.** 440711·081. **27.** ·002202073901.

28. 4921·675101. **29.** 2·197507039001.

30. ·002744588042001.

Approximate Evolution. Surds.

262. At present we have applied the process of evolution only to numbers which are exact squares or exact cubes. But the majority of numbers are neither. Among the numbers from 1 to 1000 there are only 31 which are squares and only 10 which are cubes. We are naturally led to ask the meaning of such quantities as $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{4}$, $\sqrt[3]{5}$, &c. Let us apply the evolution process to $\sqrt{2}$ and study the results.

```
        2·00′00′00′00′00′00 (1·41421
        1
24      1·00
         96
281        4 00
           2 81
2824       1 19 00
           1 12 96
28282        6 04 00
             5 65 64
282841         38 36 00
               28 28 41
               10 07 59
                 &c.
```

Now the process of finding the square root shows us that the successive remainders represent the result of subtracting from the number with which we started the square of the part already found, *i.e.* inserting the decimal point in its right place in the remainders.

$$2-1^2 = 1{\cdot}00$$
$$2-(1{\cdot}4)^2 = {\cdot}04$$
$$2-(1{\cdot}41)^2 = {\cdot}0119$$
$$2-(1{\cdot}414)^2 = {\cdot}000604$$
$$2-(1{\cdot}4142)^2 = {\cdot}00003836$$
$$2-(1{\cdot}41421)^2 = {\cdot}0000100759$$

&c.

As it is clear that the process will make the remainders, whatever they are, come further and further to the right of the decimal point, it follows that the evolution process applied to a number (2) which is not an exact square gives us a series of numbers (1, 1·4, 1·41, 1·414, 1·4142, 1·41421, ...) *whose squares become closer and closer to the number with which we started, and may be made as close as we please by continuing the process far enough.* The *limit* (see § 137) to which these numbers approach is called the square root of the number, and is denoted in the present case by $\sqrt{2}$.† Similar considerations apply to $\sqrt[3]{4}$, $\sqrt[3]{5}$, $\sqrt[4]{6}$, &c., and such quantities are called **Surds** or **Irrational Numbers,** which may be defined as "quantities involving roots which are not capable of being expressed either in integers or as the ratio of two integers." We may evaluate a surd to any number of decimal places; but we cannot find its value *exactly.*

† This important question is more fully treated in Mackay's *Arithmetic,* p. 359. Mackay points out that "the usual definition that $\sqrt{2}$ is a number which when multiplied by itself produces 2 is open to the objection that we cannot speak of multiplying by $\sqrt{2}$ until we have defined what $\sqrt{2}$ is."

263. In the evaluation of surds, there are two rules which save an immense amount of labour.

(1) When n figures of a square root have been found by evolution, the next $n-1$ figures, but *not* usually more, can be found by dividing the next divisor into the last remainder.

Thus, in the example worked above, we have found 6 figures of $\sqrt{2}$ by evolution, and 5 more figures may be found by simply dividing 282842 the *next* divisor into 100759(0), which operation should be performed by contracted division. The (0) is added as in the process of converting a fraction into a decimal.

(2) When n figures of a cube root have been found by evolution, the next $n-2$ figures, but not usually more, can be found by dividing the next divisor into the last remainder.

Thus, if we had found 4 figures of $\sqrt[3]{2}$, by evolution, we should be able to find $4-2$ more figures by (contracted) division.

For the proof of these rules we must refer to th ordinary Algebras.

Caution.—The rule for pointing the decimal [Rule (1), § 260] is of great importance with surds. Thus, $\sqrt{\cdot 4}$ is *not* $\cdot 2$; for $\cdot 4$ should be pointed $\cdot 40'00'00'\ldots$, and the first period is *40*, not 4. Verify: $\sqrt{\cdot 4} = \cdot 63245553$, $\sqrt{\cdot 04} = \cdot 2$, $\sqrt{\cdot 004} = \cdot 06324555$, $\sqrt{\cdot 0004} = \cdot 02$, &c.

Hints.—(1) For circulating decimals write the decimal in full, and bring down figures as required, *e.g.* $1\cdot\dot{4} = 1\cdot 444444\ldots$, and we bring down 4's as required.

(2) Some labour is saved in evaluating $\sqrt{2}$, for instance, if we write it as $\sqrt{1\cdot\dot{9}}$, *i.e.* $\sqrt{1\cdot 99'99'99'\ldots}$. There will then be much less "carrying" in the subtractions.

264. Ex. 1.—*Calculate* $\sqrt{2}$ *to* 14 *figures, using contracted methods.*

	2 (**1·4142135**623731
24	100
281	400
2824	11900
28282	60400
282841	383600
2828423	10075900
28284265	159063100
	17641775
	671216
	105531
	20678
	879
	30
	2

The process is the ordinary one of square root until 8 figures have been found. After this contracted division is used to produce $8-1$, *i.e.* 7 more figures. As, however, only 1 figure is left at the end, the seventh figure is uncertain, and, if we had wanted this figure, it would have been necessary not to commence contracting the division so early, but to add 0's in the ordinary way to 17641775 for one step more. The actual value of $\sqrt{2}$ is $1\cdot 41421356237309504880168872420 9\ldots$. (See also Hint (2) above.)

EXAMPLES CLIII.

[The answers to the following questions are given to six significant figures, except where otherwise stated, and the *nearest* integer is chosen for the sixth figure. The advanced student should apply the rules of approximation to *all* of them.]

Find the value of the square roots of—

1. 2·5. **2.** 3·6. **3.** ·9. **4.** 1·6. **5.** 123. **6.** 231.
7. 1234. **8.** 2314. **9.** 17·453. **10.** 235·6. **11.** 6·4. **12.** 28·9.
13. 9592·3504. **14.** 66135·3171845. **15.** $1\cdot\dot{6}$. **16.** $2\cdot\dot{5}$.
17. $\cdot\dot{1}4285\dot{7}$. **18.** $\cdot\dot{4}2857\dot{1}$.

Find the value of the cube roots of—

19. ·8. **20.** ·08. **21.** 2·7. **22.** ·27. **23.** 4. **24.** 5.
25. 6. **26.** 61. **27.** 81. **28.** 812. **29.** 182. **30.** 1234.
31. 1342. **32.** 3·00415. **33.** ·0565. **34.** $\cdot\dot{0}7692\dot{3}$.
35. $\cdot\dot{8}$. **36.** $\cdot 9\dot{1}$. **37.** $\cdot\dot{9}2307\dot{6}$. **38.** $\cdot 1\dot{8}6\dot{0}$.

Calculate to 15 places of decimals the square roots of—

39. 3. **40.** 5. **41.** 6. **42.** 7.

Calculate to 7 places of decimals the cube roots of—

43. 2. **44.** 3. **45.** 4. **46.** 5.

Calculate the value of—

47. $8-(\sqrt{8}$ to 4 decimal places$)^2$. **48.** $11-(\sqrt{11}$ to 4 decimal places$)^2$.

49.* Prove that the operation of finding the square root of 2 can never terminate.

50.* Prove that the operation of finding the cube root of 2 can never terminate.

265. We have already proved (§ 125) that fractions are subject to the Fundamental Laws of Arithmetic (§ 22). It may be proved that surds are also subject to these laws. We shall assume this to be the case, and shall state, without proof, the following results as true, whatever numbers the letters stand for:—

(1) $\sqrt{a}\times\sqrt{a}=a$. (2) $\sqrt{a}+\sqrt{b}=\sqrt{b}+\sqrt{a}$.

(3) $\sqrt{a}\times\sqrt{b}=\sqrt{ab}=\sqrt{b}\times\sqrt{a}$. (Commutative Law.)

(4) $\sqrt{\dfrac{a}{b}}=\dfrac{\sqrt{a}}{\sqrt{b}}$.

(5) $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=a-b$.

(6) $(\sqrt{a}+\sqrt{b})(\sqrt{c}+\sqrt{d})=\sqrt{ac}+\sqrt{ad}+\sqrt{bc}+\sqrt{bd}$.

266. Ex. 2. — *Evaluate* $\frac{2}{\sqrt{3}}$ *to five decimal places.*

Method I. $\frac{2}{\sqrt{3}} = \frac{\sqrt{4}}{\sqrt{3}} = \sqrt{\frac{4}{3}} = \sqrt{1\cdot\dot{3}} = \sqrt{1\cdot3333\ldots} = 1\cdot15470\ldots.$

Here we bring down 3's as required in the course of the work.

Method II. $\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{3}\times\sqrt{3}} = \frac{2\sqrt{3}}{3}$. Here we may either evaluate $\sqrt{3} = 1\cdot73205\ldots$, then multiply by 2 and divide by 3; or, usually better, we may proceed $\frac{2\sqrt{3}}{3} = \frac{\sqrt{4}\sqrt{3}}{3} = \frac{\sqrt{12}}{3} = \frac{3\cdot46410\ldots}{3} = 1\cdot15470\ldots.$

Caution.—*Never* work thus: $\frac{2}{\sqrt{3}} = \frac{2}{1\cdot73205\ldots}$; for this leads to a division sum, which will be very much longer, and involves more chance of error.

The process just exhibited, by which we changed $\frac{2}{\sqrt{3}}$ into $\frac{2\sqrt{3}}{3}$, and thus turned the denominator from an irrational into a rational quantity, is called *rationalizing the denominator.*

Ex. 3. — *Evaluate* $\frac{1}{\sqrt{3}-\sqrt{2}}$.

If we multiply both numerator and denominator by $\sqrt{3}+\sqrt{2}$, we get

$$\frac{\sqrt{3}+\sqrt{2}}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})} = \frac{\sqrt{3}+\sqrt{2}}{3-2}\text{[by (5) above]} = \sqrt{3}+\sqrt{2}$$

$$= 1\cdot73205\ldots + 1\cdot41421\ldots = 3\cdot14626\ldots.$$

Ex. 4. — *Evaluate* $\frac{\sqrt{2}+1}{2+\sqrt{3}}$.

$$\frac{\sqrt{2}+1}{2+\sqrt{3}} = \frac{(\sqrt{2}+1)(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})} = \frac{2\sqrt{2}+2-\sqrt{6}-\sqrt{3}}{4-3}$$

$$= 2\sqrt{2}+2-\sqrt{6}-\sqrt{3} = \cdot646886\ldots.$$

For further information on surds reference may be made to Chap. ii. of the *Middle Algebra* or the *Tutorial Algebra.*

EXAMPLES CLIV.

[The answers are given to the nearest six significant figures.]

Find the value of—

1. $\frac{15}{\sqrt{3}}$. **2.** $\frac{6}{\sqrt{5}}$. **3.** $\frac{\sqrt{2}}{\sqrt{3}}$. **4.** $\frac{\sqrt{3}}{\sqrt{2}}$. **5.** $\frac{1}{\sqrt{24}}$.

6. $\frac{1}{\sqrt{300}}$. **7.** $\frac{1}{\sqrt[3]{2}}$. **8.** $\frac{1}{\sqrt[3]{3}}$. **9.** $\frac{1}{\sqrt{2}-1}$. **10.** $\frac{1}{\sqrt{2}+1}$.

11. $\frac{1}{2-\sqrt{3}}$. **12.** $\frac{1}{3-\sqrt{5}}$. **13.** $\frac{1}{\sqrt{3}+\sqrt{2}}$. **14.** $\frac{1}{\sqrt{5}-\sqrt{3}}$.

15. $\frac{\sqrt{2}}{\sqrt{6}-1}$. **16.** $\frac{\sqrt{3}}{\sqrt{6}+1}$. **17.** $\frac{\sqrt{2}}{\sqrt{2}+1}$. **18.** $\frac{\sqrt{3}}{\sqrt{3}-1}$.

19. $\frac{\sqrt{2}+1}{\sqrt{2}-1}$. **20.** $\frac{\sqrt{2}-1}{\sqrt{2}+1}$. **21.** $\frac{\sqrt{3}-1}{\sqrt{3}+1}$. **22.** $\frac{\sqrt{3}+1}{\sqrt{3}-1}$.

23. $\frac{\sqrt{3}}{2+\sqrt{3}}$. **24.** $\frac{\sqrt{5}}{\sqrt{5}-2}$. **25.** $\sqrt{7}+\frac{1}{\sqrt{7}}$. **26.** $\frac{3}{\sqrt{5}}+\frac{\sqrt{5}}{3}$.

27. $\frac{3}{\sqrt{7}-1}$. **28.** $\frac{2}{\sqrt{7}-1}$. **29.** $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{5}-2}$. **30.** $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{5}+2}$.

31. $\frac{\sqrt{9}-\sqrt{\cdot 9}}{\sqrt{4}+\sqrt{\cdot 4}}$. **32.** $\frac{\sqrt{\cdot 9}+\sqrt{\cdot 09}}{\sqrt{\cdot 4}-\sqrt{\cdot 04}}$. **33.** $\frac{\sqrt{5}+\sqrt{2}}{\sqrt{3}+1}$.

34. $\frac{\sqrt{3}+\sqrt{5}}{\sqrt{5}-\sqrt{2}}$. **35.** $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}-\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$.

36. $\frac{\sqrt{7}+2}{\sqrt{7}-2}-\frac{\sqrt{7}-2}{\sqrt{7}+2}$. **37.** $\frac{3\sqrt{7}-4}{\sqrt{7}+3}$. **38.** $\frac{2\sqrt{5}-3}{\sqrt{5}+4}$.

39. $\frac{3}{\sqrt[3]{4}}+\frac{4}{\sqrt{5}}+\frac{5}{\sqrt{6}-1}$. **40.** $\frac{3}{\sqrt[3]{5}}+\frac{4}{\sqrt{6}}+\frac{5}{\sqrt{7}-\sqrt{2}}$.

EVOLUTION: MISCELLANEOUS NOTES.

267. *Fourth roots* should be found by taking the square root of the square root.

For $a^4 = a.a.a.a = (a.a)^2$; $\therefore \sqrt{a^4} = a.a$; $\therefore \sqrt{\sqrt{a^4}} = a$.

Sixth roots should be found by taking the *cube root of the square root.* The square root should be taken first.

For $a^6 = a.a.a.a.a.a = (a.a.a)^2$; $\therefore \sqrt{a^6} = a.a.a$; $\therefore \sqrt[3]{\sqrt{a^6}} = a$.

Fifth roots can be found by an extension of the process given for cube roots. The headings of the columns are 1, $5a$, $10a^2$, $10a^3$, $5a^4$, and a^5.

268. Ex. 1.—*Find* $\sqrt[5]{55}$ *to two decimal places.*

1	100	4000	80000	800000
	102	4204	88408	976816
	104	4412	97232	11712800000
	106	4624	106480000	11927704816
	108	484000	107452408	
	1100	486204		
	1102			

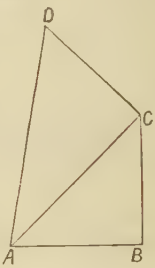
Ans. 2·22...

```
55 ( 2·22
32
2300000
1953632
 34636800000
 23855409632
 10781390368
```

N.B.—Fourth roots can also be found by a similar process; but it is not recommended. The column headings are 1, $4a$, $6a^2$, $4a^3$, a^4.

269. Square roots may be found with fair accuracy by means of a ruler divided into inches and eighths (or, better, tenths) and a T-square. Such methods are called *graphic.*

Ex. 2.—*Determine* $\sqrt{3}$ *graphically.*



Let AB be a line exactly 1 in. long,† and BC a line at right angles to it, also exactly 1 in. long. Then, by Euc. I. 47, $AC^2 = AB^2 + BC^2 = 1^2 + 1^2 = 2$; therefore $AC = \sqrt{2}$. Let CD be 1 in. long and at right angles to AC; then $AD^2 = AC^2 + CD^2 = (\sqrt{2})^2 + 1^2 = 3$; therefore $AD = \sqrt{3}$. Hence by measuring AC, AD we get $\sqrt{2}$ and $\sqrt{3}$. With care it is easy to get three significant figures of any square root this way. The figure shows (using a common penny ruler) that $\sqrt{3}$ is slightly less than $1\frac{3}{4}$, *i.e.* than 1·75.

270. Ex. 3.—*A rectangular field which is twice as long as it is broad contains* 115 *acs.* 32 *pls. What is its perimeter, and how far is it from corner to corner?*

If a plan of the field is drawn, it will be clear that it divides into two squares the area of each of which is $\frac{1}{2}$(115 acs. 32 pls.), *i.e.* is 57 acs. 2 rs. 16 pls., *i.e.* $9216 \times \frac{121}{4}$ sq.yds., *i.e.* 2304×121 sq.yds. Hence each side of one of these small squares is $\sqrt{2304 \times 121}$ yds., *i.e.* is $48 \times 11 = 528$ yds. Hence the sides of the original field are 528 and $2 \times 528 = 1056$ yds.; therefore the perimeter is $528 + 1056 + 528 + 1056 = 3168$ yds. *Ans.* The distance from corner to corner ("diagonal") $= \sqrt{528^2 + 1056^2}$ yds. (Euc. I. 47) $= \sqrt{1393920}$ yds. $= 1180{\cdot}6$ yds. nearly. It is sometimes better to treat this surd as $528\sqrt{5}$.

† The figure has been reduced so that AB in the text is only $\frac{2}{3}$ of an inch.

EXAMPLES CLV.

Find the fourth roots of—

1. 221533456. **2.** 187388721. **3.** 2058346161. **4.** 2300257521.

[See also Exx. cxlvii., 17–20, and Exx. cxlviii., 21–24.]

Find the fifth roots of—

5.* 28153056843. **6.*** 438427732293. **7.*** 48261724457.

8.* 73439775749. [See also Exx. cxlvii., 21–22, and Exx. cxlviii., 25–28.]

Find the sixth roots of—

9. 17596287801. **10.** 139314069504. **11.** 3814697265625.

12. 12332795428864.

Give a graphical construction for—

13. $\sqrt{17}$, $\sqrt{18}$, $\sqrt{19}$. **14.** $\sqrt{26}$, $\sqrt{27}$, $\sqrt{31}$; and, by means of your diagrams, calculate these roots to the nearest third significant figure.

15. A square field contains 6 acs. 2 rs. 19 pls. $6\frac{1}{4}$ sq.yds. Find the length of one side in yards.

16. A square field of $2\frac{1}{2}$ rs. is to be enclosed with iron railings. What length of railings will be required?

17. A square field contains 3 acs. 1 r. 38 pls. $20\frac{1}{2}$ sq.yds. Find in acres, &c., the area of a rectangular field whose sides are respectively 75 and 65 yds. longer than those of the square field.

18. In measuring a square field, by accident I enter in my note-book the length of one of the sides 1 pl. too great. As a result the area found is 1 r. 19 pls. too great. What is the true area?

19. The cost of levelling and turfing a square cricket field at £175 9s. 4d. per acre is £987. Find the cost of surrounding it with a railing at 3s. 2d. per yard.

20. A square field contains 22 acs. 2 rs. and is increased by 9 acs. so as to form a rectangle whose shorter side is the former side of the square. At what rate (in m.p.h.) does a man run who runs round it in 1 minute 39 seconds longer than would be occupied in running round the square field at the rate of 12 miles per hour?

21. The area of a square field is 4·9 acs. How long will it take a man to walk round the outside of it at the rate of 3 miles an hour?

22. Find in yards the length of fence required to enclose a square field containing 7 acs. 2 rs. 18 pls. $19\frac{1}{2}$ sq.yds.

23. A railing encloses a rectangular field of 15 acs. The length of the field is to its breadth as 3 : 2. What is the whole length of the railing? (Divide it into six squares.)

24. The area of a rectangular field is 15 acs. 37 pls. $8\frac{3}{4}$ sq.yds. Its length is twice its breadth. Find to the nearest yard the distance between opposite corners.

25. Express $\sqrt[3]{44 \cdot 6}$ as a decimal of $\sqrt{167 \cdot 9619}$ correct to two decimal places.

26. A cube of iron weighs 7 cwts. 101 lbs. 13·6 oz., and is 7·82 times as heavy as the water which would fill a cubical vessel of equal volume. Find to the nearest tenth of an inch a side of the cube of iron, given that a cubic foot of water weighs 1000 oz.

27. Place in ascending order of magnitude $\sqrt[3]{16}$, 2·5, and $\sqrt{\frac{19}{3}}$.

28. On the large Ordnance maps a square inch represents an acre. Find to two places of decimals the number of yards represented by a linear inch.

29. Find the square root of the sum of the squares of 3·9, 5·2, and 15·6.

30. Calculate $\frac{1}{\sqrt{\cdot 4}}$ correct to four places of decimals.

31. Calculate $\frac{1}{\sqrt{8}}$ to seven places. **32.** Calculate $\sqrt{\frac{7}{5}}$ to seven places.

33. Calculate $2\sqrt{3} - \frac{1}{2}\sqrt{12} + \sqrt{27}$ to seven places.

34. Prove that $\sqrt{3}$ lies between $\frac{1351}{780}$ and $\frac{265}{153}$. (Archimedes, B.C. 250.)

35. Prove that $\sqrt{4500} = 67 + \frac{4}{60} + \frac{55}{60^2}$ nearly. (Theon, A.D. 350.)

36.* Evaluate $\sqrt{5658\,(17081120)^2 + 1}$.

37.* The product of two integers is 15375; their difference is 2. Find them.

38.* A number of four figures is a perfect square. Find these four figures knowing that the first two are equal and also the last two.

39.* Prove that the square of an integer ending in 5 cannot end in 125 nor in 215.

40.* The difference of the squares of two consecutive integers is 28731. Find them.

41.* The product of three consecutive integers is 2985840. Find them.

42.* Without using evolution, find a number of three figures, knowing that the sum of the figures is 14 and that the cube of the number is 83453453.

43.* The square of an integer ends in 9. What are the only figures possible for the tens' place of this square?

44.* Show that all the numbers in the series 49, 4489, 444889, .. obtained by inserting 48 in the middle of each number to form the next are perfect squares.

45.* I wish to arrange some counters in a perfect square. On the first attempt there are 130 counters over. Putting 3 more counters to each side of the square I find I have still 31 over. How many counters had I?

46.* What is the units' figure in 7^{458} and in 87^{45}?

47.* Find all the perfect squares formed by four even figures.

48.* In 1696 De Lagny, a member of the Academy of Sciences, and an able mathematician, said that the most skilful computer could not in less than a month find within a unit the cube root of 6965364833186400350736410 37. Time yourself to do it in less than a quarter of an hour.

CHAPTER XXXIV.

SCALES OF NOTATION.

271. There is no doubt that the human race has learnt to count in tens because man has 10 digits on his hands. But there is no reason why Arithmetic should be confined to numbers reckoned in tens, hundreds (10^2), thousands (10^3), &c. We might have agreed, for instance, to count by 7's, in which case the first place on the right of a number would have denoted units, the place on the left of this *sevens*, the place on the left of this *seven-squareds* (7^2), and so on, so that 123, instead of meaning 3 units, 2 tens, 1 hundred, would mean 3 units, 2 sevens, 1 seven-squared, *i.e.* $3+2\times7+1\times49$, *i.e.* 66. The number which is at the base of our system of Notation (*Scale of Notation*) is called the **radix,** or the **scale.** In this chapter we shall denote the radix by Roman letters placed after the numbers.

Thus	123 (x.)	means	$1.100+2.10+3$;
	123 (vii.)	,,	$1.49\ +2.7\ +3=66$ (x.);
	123 (iv.)	,,	$1.16\ +2.4\ +3=27$ (x.).

It must be noted that, when a number is properly expressed, no digit as great or greater than the radix can appear in it.

Thus 29 (vii.) is not properly expressed, for 2 sevens and a 9 ought to be written as 3 sevens and a 2, *i.e.* 32 (vii.).

In reading a number be careful to avoid the names of scale x. Thus 11 (viii.) is read "one one," *not* "eleven."

The Four Rules.

272. The processes of the "Four Rules" are exactly those of the "Compound Rules" in a system of weights or measures in which each denomination is the same multiple of the one below it as all the others are of those below them.

Ex. 1.—*Add* 1035, 3176, 1234, 176 *in scale* ix.

```
1035
3176
1234
 176
5643 Ans.
```

The working is as follows:—

Units 6, 10, 16, 21 = 2 nines and a **3**. Write the **3** and carry the 2.

Nines 2, 9, 12, 19, 22 = 2 nines and a **4**. Write the **4** and carry the 2; &c.

Ex. 2.—*Multiply* 1234 *by* 765 (viii.).

```
   1234
    765
  11104
   7650
   6414
1215514 Ans.
```

Ex. 3.—*Divide* 1234567 *by* 147 (ix.).

```
        7386 Quot.
147 ) 1234567
      1164
       605
       453
       1426
       1322
        1047
        1016
          31 Rem.
```

In Division sums it is difficult to see how often the divisor "will go," because we have only learnt our tables in scale x. To overcome this difficulty, we may form a table of multiples of the divisor beforehand (see § 39, Note 2), or we may mentally convert the first few figures into scale x. Thus we should assume in the example given that 147 would go into 1234 about as often as 14 (or 15) would go into 123. But 14 (ix.) = $1 \times 9 + 4 = 13$ (x.) and 123 (ix.) = $1 \times 81 + 2 \times 9 + 3 = 102$ (x.), and 13 goes 7 times into 102. Therefore we try 7 as the multiplier.

In scales higher than x. some symbol is needed for ten, eleven, twelve, &c. In scale xi., for instance, 10 would not mean "ten," but one eleven and a zero, *i.e.* eleven. It is usual in such cases to let *t* stand for 10 (x.), *e* for 11 (x.).

If symbols are needed for twelve, &c., 2′, $\underline{2}$, or ɔ may be used.

EXAMPLES CLVI.

1. Add 17736, 61354, 12345, 10007 (viii.). Subtract 44444 from the result (viii.).

2. Add 28847, 72465, 23456, 20008 (ix.). Subtract 56555 from the result (ix.).

3. Make a table of the squares and cubes of the "digits" in xi.

4. Make a table of the squares and cubes of the "digits" in xii.

5. 42404 ÷ 871 (xi.). **6.** 142772 ÷ 482 (xii.). **7.** 65502 ÷ 45 (vii.).

8. Find a number which, when divided by 3, gives the quotient 133 (iv.).

9. $9e0t \times 4e3$ (xii.). **10.** 2302×213 (iv.).

11. Prove that 12345654321 is divisible by 12321 in any scale greater than 6.

12. Prove that in scale xiv. the square of any number ending in 7 also ends in 7.

13. Prove that in scale xiv. any power of a number ending in 8 also ends in 8.

14. Prove that all perfect cubes in ix. end in either 0, 1, or 8.

15.* What is the condition among the digits in a scale of notation whose radix is an odd number that a given number is (1) odd, (2) even?

16.* Prove that, in order that a number in vii. should be divisible by 15 (vii.), it is necessary (1) that the sum of its digits should be divisible by 3; (2) that the difference between the sum of the figures in the odd places and the sum of the figures in the even places should be divisible by 4.

17.* Multiply $24e17$ by 247417 (xii.) in three lines, excluding the answer-line.

18.* Multiply 247417 by $24e17$ (xii.) in three lines, excluding the answer-line.

273. The change from one scale to another is simply a process of "reduction." The methods are identical.

Ex. 1.—*Convert* 134598 (x.) *into* vi.

6) 134598	
6) 22433	0
6) 3738	5
6) 623	0
6) 103	5
6) 17	1
2	5

∴ 134598 (x.) = 2515050 (vi.).

Ex. 2.—*Convert* 1234543 (vi.) *into* x.

1234543 (vi.)
8
51
310
1865
11194
67167 (x.)
Ans.

The process (compare that of "Reduction") is one of successive multiplication by 6. Thus $6 \times 1 + 2$ gives the 8, $6 \times 8 + 3$ gives the 51, &c.

Or thus:

t) 1234543 (vi.)	
t) 51032	7
t) 3035	6
t) 151	1
t) 10	7
t) 0	6

Ans. 67167 (x.).

The divisions must take place in scale vi.

This method is the shortest in turning from any scale to a scale other than x., but is not recommended for x.

274. The processes of H.C.F., L.C.M., square and cube root are identical with those in x. In L.C.M., unless the numbers may be converted into x., a continual use of the H.C.F. process is necessary to determine the common factor. In square and cube root the hint appended to § 272, Ex. 3 will be found useful in regard to the only difficulty, that of the trial figure.

Ex. 3.—*Find the square root of* 1563052 (vii.).

```
        1′56′30′52 ) 1234
        1
22      56
        44
243     1230
        1062
2464     13552
         13552
```

Ex. 4.—*Find the cube root of* 2502043 (vi.).

```
                     2′502′043 ( 123
1    30    300       1
     32    404       1502
     34    52000     1212
     400   54013     250043
     403             250043
```

275. Ex. 5.—*A box of weights is given you containing weights of* 1 *g.*, 2 *g.*, 4 *g.*, 8 *g.*, 16 *g.*, *and so on, there being only one of each kind. What weights would you employ to weigh an object whose weight is known to be* 153 *g.?*

```
2 ) 153
 2 ) 76   1
 2 ) 38   0
 2 ) 19   0
  2 ) 9   1
  2 ) 4   1
  2 ) 2   0
    1     0
```

Converting the number 153 into scale ii., it becomes 10011001, *i.e.* $1 + 1 . 2^3 + 1 . 2^4 + 1 . 2^7$. Hence it follows that the weights to be taken are the 1 g., the 8 g., the 16 g., and the 128 g.

EXAMPLES CLVII.

1. Convert 16784 (x.) to xii., also to vii. and to vi.

2. Convert 17648 (x.) to xi., also to viii. and to ix.

3. Convert 1885 (ix.) to x., 20305 (vii.) to x., and 11000000 (iii.) to x.

4. Express in x. the greatest and least numbers that can be written with four digits in vi.

5. Convert 37214 (viii.) into ix., and 801706 (ix.) into v.

6. Transform 34402 (v.) into iv., and 8978 (xi.) into xii.

7. Transform 12345 (xi.) into ix., then change the result into viii., then that result into v., and verify by retransforming to xi.

8. Transform 13254 (ix.) into xi., then change the result into v., then that result into vi., and verify by retransforming to ix.

9. Find the H.C.F. of 6411 and 11414 (vii.); also of 46061 and 75073 (viii.).

10. Find the H.C.F. of 3183 and 16166 (ix.); also of 15868 and 23*tt*7 (xi.).

11. Without converting the numbers into x., find the L.C.M. of 264, 1560, and 3069 (xii.).

12. Without converting the numbers into x., find the L.C.M. of 22*tt*, 3630, and 5698 (xi.).

13. Simplify $\frac{13}{1t} \times \frac{104}{47} \div \frac{31}{t1}$ (xii.). **14.** Simplify $\frac{23}{55} \times \frac{9}{15} \div \frac{33}{2t}$ (xi.).

15. Find the square root of 373444 (ix.) and of 32*e*75721 (xii.).

16. Find the square root of 360784 (ix.) and of 252872469 (xii.).

17. Find the cube root of 113140331 (vii.).

18. Change the numbers 180 and 150 from x. to vii.; find their product in vii. and its cube root.

19. A number 12551 (x.) becomes in another scale 30407. What is the new radix?

20. A number 357234 (x.) becomes in another scale 3015333. What is the new radix?

21. Prove that, in any system of numeration whatever, the numbers 121, 12321, 1234321, 123454321, ... are perfect squares.

22. Prove that, in any system of numeration whatever, 1367631 is a perfect cube.

23. Find which of a series of weights of 1 lb., 2 lbs., 4 lbs., 8 lbs., 16 lbs., &c., must be used in order to weigh 233 lbs., if only one weight of each kind be used.

24. If we have one each of a series of weights marked 1 lb., 3 lbs., 9 lbs., 27 lbs., &c., how can we weigh an object weighing 322 lbs.?

25. Show how to form all weights from 1 to 312 lbs. by the use only of the 8 weights: 2 of 1 lb., 2 of 5 lbs., 2 of 25 lbs., 2 of 125 lbs.

26.* If a three-figure number in scale r have its figures written in the reverse order, the difference between these two numbers has $2r-2$ for the sum of its digits.

27.* A familiar method of telling the age of a person younger than 32 is to hand to him five cards on which the following numbers are written:—

Card i.: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31.
Card ii.: 2, 3, 6, 7, 10, 11, 14, 15, 18, 19, 22, 23, 26, 27, 30, 31.
Card iii.: 4, 5, 6, 7, 12, 13, 14, 15, 20, 21, 22, 23, 28, 29, 30, 31.
Card iv.: 8, 9, 10, 11, 12, 13, 14, 15, 24, 25, 26, 27, 28, 29, 30, 31.
Card v.: 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31.

He is then asked to return to you those cards on which his age occurs. Thus, if his age is 21, he returns Cards i., iii., and v. Adding together the first numbers on each card, you give the age at once. Thus, for 21, $1+4+16=21$. Explain this.

Radix Fractions.

276. Just as in scale 10 we were led to extend our system of numeration so as to include decimal fractions where ·123 meant $\frac{1}{10}+\frac{2}{10^2}+\frac{3}{10^3}$, so, in any scale whatever, we may have **radix-fractions**, where each figure has a value bearing the same relation to the value of the figure on its left as it does in a number in the same scale.

Thus, in vii., ·123 (*read* "point 1, 2, 3") means $\frac{1}{7}+\frac{2}{7^2}+\frac{3}{7^3}$.

The rules for decimal fractions were deduced wholly from the place-values of the figures, and consequently the same rules apply to radix-fractions.

Similarly, we have recurring radix-fractions, and almost exactly the same rules for their evaluation as in scale x.

Thus, in viii., $\cdot\dot{2}\dot{3}$ means ·232323..., *i.e.* $\frac{2}{8}+\frac{3}{8^2}+\frac{2}{8^3}+\frac{3}{8^4}+\ldots$.

277. Ex. 1.—*Convert* $\frac{1}{27}$ (x.) *into a radix-fraction in* vi.

$$\tfrac{1}{27}\,(\text{x.}) = \frac{1}{3\times3\times3}\,(\text{x. or vi.})$$

```
3 ) 1·000  (vi.)
3 ) ·200
3 ) ·040
    ·012   Ans.
    ·012   Ans.
```

Or thus: $\frac{1}{27}$ (x.) = $\frac{1}{43}$ (vi.)

```
43 ) 1000
      43
      130
      130
```

Or as in Ex. 3.

Ex. 2—*Convert* $\frac{1}{13}$ (x.) *into a radix-fraction in* xi.

```
                ·093425t
1.  12     12 ) 100...
2.  24          t7
3.  36           40
4.  48           36
5.  5t            50
6.  71            48
7.  83             30
8.  95             24
9.  t7              70
t.  109             5t
                    110
                    109
                      2
```

```
·093425t
  17685
  093...
```

We first convert $\frac{1}{13}$ (x.) into $\frac{1}{12}$ (xi.). Then form a table of multiples of 12 (xi.) by means of which the conversion of $\frac{1}{12}$ into a radix-fraction is rapidly accomplished in the ordinary way. Directly we reach the remainder *2*, we note that it is $\frac{1}{2}$ of a previous remainder. The work can therefore be finished by division, as shown. (See § 140, Ex. 1.)

Ans. $\cdot\dot{0}93425t1768\dot{5}$.

Ex. 3.—*Convert* ·1254 (viii.) *into a radix-fraction in* xii.

·1254
2·0020
0·030
0·44
6·60
9·0 *Ans.* ·20069 (xii.)

The process is one of continual multiplication in viii. by 12 (x.). The multipliers are not shown, and the multiplication extends only to the figures to the *right* of the point in each line. The figures to the left of the point, printed dark (**2**, &c.) give the figures of the answer. The process is again one of reduction, and is exactly parallel with that employed in § 159 and in Ex. 1, § 163.

Since ·1254 (viii.) × 12 (x.) = 2 units + ·0020,

$\therefore$ ·1254 (viii.) = $\frac{2}{12}$ units + $\frac{1}{12}$ (x.) of ·0020 (viii.), &c.

The methods of the last example might have been employed, but would have been longer.

Ex. 4.—*Convert* ·25 (viii.) *and* $\cdot 5\dot{5}\dot{9}$ (xii.) *into vulgar fractions in* x.

(1) ·25 (viii.) = $\frac{25}{100}$ (viii.) = $\frac{21}{64}$ (x.) *Ans.* Beware of cancelling 25 into 100 in viii.

(2) $\cdot 5\dot{5}\dot{9} = \dfrac{559-5}{ee0}$ (xii.) $= \dfrac{554}{ee0}$ (xii.) $= \dfrac{784}{1716}$ (x.) $= \dfrac{196}{429}$. *Ans.*

Here we subtract the non-recurring part 5 from the whole 559, place it over as many *e*'s as there are recurring figures, and as many 0's as non-recurring. We use *e*'s in scale xii. in this rule, just as we used 9's in scale x., in all cases the digit next below the scale.

N.B.—In the case of a mixed number, the two parts must be reduced separately. Thus, to change 57·76 (viii.) to ix., we firstly treat 57 by the method of § 273, and then treat ·76 by the method of Ex. 3 above.

278. *Laws of radix-fractions.* (*Cf.* § 146.)

The following laws may be established for radix-fractions by methods exactly similar to those used in § 146.

I. If N/D be a proper fraction in its lowest terms, and D be a prime number which is not a factor of the radix, then N/D gives rise to a pure circulator (radix fraction), and the number of places in the recurring period is either $D-1$ or one of the factors of $D-1$.

Thus, in Ex. 2 above, $\frac{1}{13}$ (x.) had 12 (= 13 − 1) places in xi.

II. is exactly similar to Law II. in § 146.

III. If D be a prime number, which is not a factor of the radix, and $1/D$ is a pure circulator of p figures, then $1/D^2$ will *generally* have a period of pD figures, $1/D^3$ of pD^2, &c. [See p. 479.]

IV. If D be a prime number, which is not a factor of the radix, and such that $1/D$ has p recurring figures, and if d be a number containing no factors prime to the radix, then $1/Dd$ is a mixed circulator with p recurring figures and m non-recurring figures, where m is the highest index in d expressed as the product of prime factors, unless the radix contains any factor more than once.

V. *The Cyclic Law* of § 146 holds unchanged.

Thus $\frac{1}{7} = \cdot\dot{1}86t3\dot{5}$ (xii.), and we can at once deduce

$$\tfrac{2}{7} = \cdot\dot{3}5186\dot{t}\ (\text{xii.}),\quad \tfrac{3}{7} = \cdot\dot{5}186t\dot{3}\ (\text{xii.}),\quad \&c.$$

VI. *The Complementary Law* of § 146 holds if for 9 we substitute $r-1$, where r is the radix, and for (3) we read "the product of such primes with any numbers *not* prime to the radix."

Thus $\frac{1}{7} = \cdot\dot{1}86t3\dot{5}$ (xii.) and $1+t = 8+3 = 6+5 = e$.

So $\frac{1}{13} = \cdot\dot{0}93425t17685$ (xi.) and $0+t = 9+1 = 3+7 = \ldots = t$.

EXAMPLES CLVIII.

1. $4\cdot302 \times 2\cdot104$ (v.).

2. $3\cdot621 \times 4\cdot561$ (vii.).

3. $786\cdot548 \div \cdot04572$ (ix.).

4. $107e\cdot8028 \div 98\cdot74$ (xii.).

5. $15\cdot1\dot{3}\dot{2} + \cdot0\dot{1}6\dot{7} + 7\cdot00\dot{5}$ (viii.).

6. $26\cdot1\dot{2}\dot{3} + \cdot01\dot{6}\dot{7} + 7\cdot\dot{0}0\dot{5}$ (viii.).

7. Reduce $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{8}$, and $\frac{1}{9}$ (x.) into radix fractions in vii.

8. Reduce $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, and $\frac{1}{9}$ (x.) into radix fractions in viii.

9. Reduce $67\cdot75$ from x. to v.

10. Reduce $37\cdot2291\dot{6}$ from x. to xii.

11. Transfer $355\cdot41$ from vi. to x.

12. Transfer $2102\cdot21$ from iii. to viii.

13. Express $\frac{139}{192}$ (x.) as a radix-fraction in xii.

14. Express $\frac{1}{7}$, $\frac{1}{19}$ (x.), and $\frac{22}{e1}$ (xii.) in radix-fractions in xi.

15. A block of silver, 13 ins. long, 8 ins. broad, and 6 ins. deep, is beaten into a bar which has a uniform section of $1\cdot01$ sq.ins. in v. Find its length in feet in v.

16. If the area of a rectangle is $333\cdot06$ sq.ft. in xii., and the length of one side is $22\cdot e$ ft., what is the length of the other side in feet and inches?

17.* Find in scale x. all possible denominators for fractions which in scale v. give pure circulators with five-figure periods.

18.* Find in scale x. all possible denominators for fractions which in scale vii. give pure circulators with five-figure periods.

19.* Without finding more than five figures of the period of $\frac{1}{41}$ (x.) expressed in vii., prove that the sum of the digits of the period is a factor of 120.

20.* Assuming that $\frac{1}{97}$ (x.) has 48 (x.) figures in its period in xi., find the 1st, 2nd, 23rd to 26th, 47th, and 48th figures *without finding any others.*

21.* In converting $\frac{1}{829}$ (x.) into xii., the figures 333 occur among the period at a certain stage of the work. Find the next four figures.

22.* Use scale xx. in order to convert $\frac{1}{10}$ (x.) into a decimal in x.

Duodecimals.

279. It is customary to refer in Arithmetic to a measurement often adopted by builders and called the Method of Duodecimals. In reality this is only the expression of the foot and its fractions in radix-fractions in scale xii., and any question in "Duodecimals" should be worked by immediately converting the data into xii. Names are assigned as follows:—

1 *foot* = 12 *primes* (*i.e.* inches), 1 *prime* = 12 *seconds*,
1 *second* = 12 *thirds*;

1 *cubic foot* = 12 *solid primes*, 1 *solid prime* = 12 *solid seconds*,
1 *solid second* = 12 *solid thirds* (= cubic inches).

A length of 3 ft. 10 primes 9 seconds 9 thirds is usually written 3 ft. 10′ 9″ 9‴, but you are advised to write it at once as 3·*t*99 (xii.) ft., for in xii. ·*t* means $\frac{10}{12}$, *i.e.* 10 primes, ·*t*9 means $\frac{10}{12}+\frac{9}{12^2}$, *i.e.* 10 primes + 9 seconds.

It may be remarked that no builder possesses any measure *capable of measuring* a "third," *i.e.* $\frac{1}{144}$th of an inch; so that the use of "thirds" is a manifest absurdity. It should also be noticed that 3·*t*99 (xii.) ft. is often written 3′ 10″ 9‴ 9⁗.

280. Ex. 1.—*Express* 7 *ft.* 8½ *ins. in duodecimals.*

This is 7 ft. 8′ + ½ in. But the ½ in. = $\frac{6}{12}$ in. Therefore the length can be written 7 ft. 8′ 6″ or 7·86 (xii.).

Ex. 2.—*Express* 17 *ft.* 8½ *ins. in duodecimals.*

This can clearly be written 17 ft. 8′ 6″ or 15·86 (xii.), for in xii. 17 becomes 15.

Ex. 3. —*Express* 7 *c.ft.* 3′ 4″ 5‴ *in cubic feet and cubic inches.*

This is 7·345 c.ft. (xii.), *i.e.* is 7 c.ft. and 345 (xii.) solid thirds. But a solid third is a cubic inch. Therefore there are 345 (xii.) c. ins. But 345 (xii.) = 485 (x.). *Ans.* 7 c.ft. 485 c.ins.

Ex. 4.—*Find the area of a room of length* 22 *ft.* 6 *ins., breadth* 10 *ft.* 3 *ins.*

1*t*·6	Converting the numbers into xii., we have for length
t·3	1*t*·6 ft., breadth *t*·3 ft., and hence the area is 172·76 (xii.)
———	sq.ft., *i.e.* 172 (xii.) sq.ft. and 76 (xii.) "superficial seconds"
169·0	or square inches. Converting *Working.*
5·76	172 (xii.) and 76 (xii.) into x., we 172 76
———	get at once for the answer: 19 *90*
172·76	230 sq.ft. 90 sq.ins. *230*

N.B.—A builder does *not* reduce the integral feet to xii. His work is really a mixture of scales x. and xii. The method is sometimes perversely named "Cross-Multiplication."

EXAMPLES CLIX.

Multiply the following, expressing the result in *builders'* duodecimals, and also in square feet and square inches :—

1. 4 ft. 7 ins. × 9 ft. 6 ins.

2. 2 ft. 8 ins. × 2 ft. 8 ins.

3. 18 ft. 9 ins. × 14 ft. 7 ins.

4. 25 ft. 8 ins. × 16 ft. 9 ins.

5. 45 ft. $5\frac{1}{2}$ ins. × 8 ft. $3\frac{3}{4}$ ins.

6. 37 ft. $9\frac{2}{3}$ ins. × 76 ft. $6\frac{1}{2}$ ins.

7. Find the number of cubic feet and cubic inches in a room 23 ft. 10 ins. long, 18 ft. 4 ins. broad, and 11 ft. 3 ins. high.

8. Find the volume of a cube whose edge is 13 ft. 7′ 7″, correct to the nearest cubic foot.

9. Find the area of the floor of a hall 60 ft. 11 ins. long and 37 ft. 7 ins. broad, correct to the nearest square foot.

10. A pile of stones is 17 ft. 3 ins. long, 5 ft. 4 ins. broad, and 4 ft. high. Find its volume, by duodecimals.

CHAPTER XXXV.

CONGRUENCES.

[*A knowledge of the meaning of "negative number" is desirable before reading this chapter. It is not, however, essential, and the first part of the chapter will be readily understood by any one who does not possess it.*]

281. If the difference of two numbers is divisible by a third number, these two numbers are said to be **congruent** with regard to the third number, which is called the **modulus** of the congruence. The symbol of congruence is $\equiv$, which must be most carefully distinguished from $=$.

Thus 33 and 19 are congruent with regard to 7, because $33-19$, *i.e.* 14, is divisible by 7. This is expressed $33 \equiv 19 \pmod 7$, and is usually read "33 is congruent with 19 to modulus 7." The modulus, *i.e.* the divisor, frequently remains unaltered throughout a piece of work, and in that case it is not usual to state it more than once. Such a statement as $33 \equiv 19$ is called a *congruence*. The symbol $\equiv$ is occasionally used in algebra to denote "is identically equal to."

The essential property of congruent numbers is the following:—

Two numbers are congruent with regard to a given modulus if they leave the same remainder when divided by the modulus, and conversely.

Or, differently stated:—

Two numbers are congruent with regard to a given modulus if they have the same figure in the units' place when expressed in the scale of notation whose radix is the modulus, and conversely.

These statements are almost obvious. If $M \equiv N \pmod r$, $M-N =$ multiple of r; therefore $M = N+$multiple of r; but the remainder on dividing ($N+$a multiple of r) by r must be the same as that on dividing N by r; therefore M and N

leave the same remainder when divided by r. The second statement is another form of the first; for we have seen that the units' place of M expressed in scale r is the remainder on dividing M by r. Or it is otherwise clear that in scale r the difference of two numbers is divisible by r only when the units figures agree.

For example, 33 and 19 when divided by 7 leave remainder 5; expressed in scale vii. they are 45 and 25.

282. Laws of Congruences.

I. *If* $M \equiv N$, $M-N \equiv 0$.

This is a restatement of the definition of congruence, for the second congruence $M-N \equiv 0$ means $M-N-0$ is divisible by r, *i.e.* $M-N$ is divisible by r, and that is the same thing as $M \equiv N$.

E.g.: $123 \equiv 35 \pmod{11}$; therefore $123-35$, *i.e.* $88 \equiv 0 \pmod{11}$.

II. *If* $M \equiv N$, $aM \equiv aN$, *a being any integer.*

For M and N end in the same figure when expressed in scale r;
therefore aM and aN ,, ,, ,, ,, ,, ,, r;
therefore $aM \equiv aN$.

III. *If* $M \equiv N$ *and* $P \equiv Q$, $M+P \equiv N+Q$ *and* $M-P \equiv N-Q$.

For the last figure of $M+P$ expressed in scale r depends *only* on the last figures of M and P (and not at all on the other figures). But the last figure of M is the same as that of N, and the last figure of P is the same as that of Q. Therefore the last figure of $M+P$ (scale r) is the same as the last figure of $N+Q$. Therefore $M+P \equiv N+Q$.

Similarly, $M-P \equiv N-Q$.

E.g.: 1000 and 100 leave rem. 1 when divided by 9, *i.e.* $1000 \equiv 100\ (9)$,
236 ,, 632 ,, ,, 2 ,, ,, ,, 9, ,, $236 \equiv 632$
therefore 1236 ,, 732 ,, ,, 3 ,, ,, ,, 9, ,, $1236 \equiv 732$.

IV. *If* $M \equiv N$ *and* $P \equiv Q$, $MP \equiv NQ$.

The argument of III. applies unchanged, for the last figure of a product MP depends *only* on the last figures of M and P, and these are the same, respectively, as the last figures of N and Q. Therefore the last figure of MP (scale r) is the same as the last figure of NQ (scale r); therefore $MP \equiv NQ$.

V. *If* $M \equiv N$, $M^n \equiv N^n$.

For, by IV., if $M \equiv N$, $MM \equiv NN$, *i.e.* $M^2 \equiv N^2$. Therefore, again by IV., $MM^2 \equiv NN^2$, *i.e.* $M^3 \equiv N^3$, and so on. Or, independently, the last figure of any power whatever depends only on the last figure of the root. Thus the last figure of 1374^5 (x.) is the same as that of 4^5. But M and N have the same last figure in scale r. Therefore M^n and N^n have the same last figure.

Caution.—The symbol $\equiv$ has so many of the properties of $=$ that some caution is needed,

Thus, if $M \equiv N$, it is NOT always true that $M \div a \equiv N \div a$.†

E.g.: $60 \equiv 12 \pmod 8$; but 15 is *not* $\equiv 3 \pmod 8$.

Similarly, if $M \equiv N$ and $P \equiv Q$, it is *not* usually true that $M/P \equiv N/Q$.

283. Ex. 1.—*Find the remainder when* 356^{16} *is divided by* 7.

$$356 \equiv 6 \pmod 7; \quad \therefore \quad 356^2 \equiv 6^2 \text{ (Law V.)} \equiv 36 \equiv 1;$$
$$\therefore \quad 356^4 \equiv 1^2 \equiv 1; \quad \therefore \quad 356^8 \equiv 1; \quad \therefore \quad 356^{16} \equiv 1.$$

Hence the remainder required is 1.

Ex. 2.—*Prove that any square number when divided by* 5 *leaves either* 0, 1, *or* 4 *as remainder.*

Any number divided by 5 must leave either 0, 1, 2, 3, or 4 as remainder,

i.e. any number $\equiv 0, 1, 2, 3,$ or $4 \pmod 5$;

$\therefore$ (any number)$^2 \equiv 0, 1, 4, 9,$ or 16, these being the squares of 0, 1, 2, 3, and 4,

$\equiv 0, 1, 4, 4,$ or 1, since $9 \equiv 4$ and $16 \equiv 1 \pmod 5$,

$\equiv 0, 1,$ or 4.

Ex. 3.—*Prove that, if the sum of any two square numbers is equal to a third square number, then one of the three numbers must be divisible by* 5.

Let the square numbers be A, B, and C, and let $A + B = C$; then we have A, B, and C all $\equiv 0, 1,$ or $4 \pmod 5$ by Ex. 2. If A or $B \equiv 0$, it is divisible by 5, and what was required is proved. If not, one of the following must happen:—

$A \equiv 1$, $B \equiv 1$; $\therefore$ $C = A + B \equiv 2$. But this is impossible, for $C \equiv 0, 1,$ *or* 4.

$\left.\begin{array}{l} A \equiv 1, \ B \equiv 4 \\ A \equiv 4, \ B \equiv 1 \end{array}\right\}$; $\therefore$ $C = A + B \equiv 5$, *i.e.* C is divisible by 5.

$A \equiv 4$, $B \equiv 4$; $\therefore$ $C = A + B \equiv 8 \equiv 3$. But this is impossible.

Hence, unless A or B is divisible by 5, C is so divisible.

---

† On the other hand, the advanced student may note that, if $M \equiv N$, $\sqrt{M}$ is $\equiv$ *either* $+\sqrt{N}$, *or* $-\sqrt{N}$. It is not generally possible to foretell which of these congruences is true. Thus $25 \equiv 4 \pmod 7$ and $5 \equiv -2$. But $81 \equiv 4 \pmod 7$ and $9 \equiv +2$. It is assumed that M and N are square numbers, and that the modulus is a prime number.

Ex. 4.—*In converting* $\frac{1}{401}$ *into a decimal, the first three remainders are* 10, 100, *and* 198. *From these data alone determine the* 100*th remainder, and prove that recurrence takes place with the* 201*st figure.*

It is clear from the process that 198 is the remainder on dividing 1000 by 401; that the next remainder will be the remainder on dividing 10000 by 401, and so on, *i.e.* that the first remainder is the remainder of 10^1, the second of 10^2, the third of 10^3, and the nth of 10^n.

Now $10^3 \equiv 198 \pmod{401}$; $\therefore\ 10^6 \equiv 198^2 \equiv 39204 \equiv 307$;

$$\therefore\ 10^{12} \equiv 307^2 \equiv 94249 \equiv 14\ ;\quad \therefore\ 10^{24} \equiv 14^2 \equiv 196\ ;$$

$$\therefore\ 10^{48} \equiv 196^2 \equiv 38416 \equiv 321\ ;\quad \therefore\ 10^{96} \equiv 321^2 \equiv 103041 \equiv 385\ ;$$

$$\therefore\ 10^{99} \equiv 10^{96} \times 10^3 \equiv 385 \times 198 \equiv 76230 \equiv 40\ ;$$

$$\therefore\ 10^{100} \equiv 10^{99} \times 10 \equiv 40 \times 10 \equiv 400.$$

Hence the 100th remainder is 400;

$$\therefore\ 10^{200} \equiv 400^2 \equiv 160000 \equiv 1\ ;$$

and, since the 200th remainder is 1, recurrence occurs at the 201st figure.

EXAMPLES CLX.

1. Find the remainders of—

(1) $21^{21} \div 19$; (2) $19^{19} \div 21$; (3) $1256^{732} \div 11$;
(4) $41^{41} \div 83$; (5) $82^{82} \div 83$; (6) $5^{99} . 8^{154} \div 13$.

2. Prove, by congruences, that $2^{120} - 1$ is divisible by 9.

3. Find the possible remainders on dividing a square number (1) by 3, (2) by 4, (3) by 6, (4) by 7, (5) by 8, (6) by 11, (7) by 19.

4. Find the possible remainders on dividing a cube number (1) by 3, (2) by 4, (3) by 5, (4) by 6, (5) by 7, (6) by 8, (7) by 9.

5. Prove that every square number is either a multiple of 5 or differs from a multiple of 5 by 1.

6. Prove that, if the sixth power of an integer be divided by 9, the remainder is either 0 or 1.

7. Prove that the fifth power of a number ends with the same figure as the number itself.

8. Prove that all fourth powers (1) end in either 0, 1, 5, or 6; (2) when divided by 3, 4, 8, or 16, leave for remainder either 0 or 1.

9. If A, B, C be square numbers such that $A + B = C$, prove (1) either A or B is a multiple of 9, (2) either A or B is a multiple of 16, (3) ABC is divisible by 3600.

10. Show that the remainders, on dividing N and $10000N$ by 101, are identical, whatever N is.

11. Show that $10^{6n+4} + 5 . 10^{3n+2} + 8$ is divisible by 37.

12. Find the remainder on dividing $10^{3n}(10^{3n} + 1)$ by 111.

13. The difference of two numbers is divisible by 13. Prove that, when their product is divided by 13, the remainder cannot be either 2, 5, 6, 7, 8, or 11.

14. The sum of two numbers is divisible by 11. Prove that, when their product is divided by 11, the remainder cannot be either 1, 3, 4, 5, or 9.

15. If the sum of two numbers is a multiple of 10, prove that the units-figures of their squares are the same.

16. Show that, if unity be added to a power of 3, the result cannot be divisible by 13.

17. Find the lowest power of 3 to which, if 1 is added, it becomes a multiple of 29.

18. If $3n + 1$ ciphers be placed between the digits 7 and 3, prove that the number thus formed is divisible by 37.

19. If $3n$ ciphers be placed between the digits 3 and 7, prove that the number thus formed is divisible by 37.

20. Prove that the difference between the cubes of two consecutive integers terminates in one of the digits 1, 7, 9.

21. If the fourth power of any whole number be not exactly divisible by 5, show that it will be so on subtracting unity.

22. If 3^{60} be divided by 7, 8, or 13, the remainder in each case is 1. Prove this, and find the remainder when 3^{60} is divided by the product of these numbers.

23. Prove that, for all values of n, either $3^{3n} + 2^{3n}$ or $3^{3n} - 2^{3n}$ is divisible by 35.

24. Prove that $2^{155} - 1$ is divisible by 31^2.

25.* It is a general theorem that, if p be a prime number > 31, $1^{p-3} + 2^{p-3} + \ldots + \left(\frac{p-1}{2}\right)^{p-3}$ is divisible by p. Verify this theorem in the case when $p = 31$.

26.* Prove, by the method of congruences, that $\frac{1}{211}$ has a recurring period of 30 places, and $\frac{1}{701}$ of 700 places.

27.* Show that, in scale xii., $\frac{1}{5e}$ (xii.) has a recurring period of 35 (x.) places.

28.* Find the 2nd, 4th, 8th, 16th, 32nd, 64th, and 128th figures of the period of $\frac{1}{701}$, *without finding any intermediate figures.*

29.* Find the same figures (2nd, 4th, &c.) as in Ex. 28 for the period of $\frac{1}{131}$.

30.* Prove that the product of three consecutive integers is always divisible by 504 if the middle integer is a perfect cube.

We proceed to examine a few subjects in the Theory of Arithmetic which we have up to this point left incomplete.

284. *Remainder on division by* 9 *and* 11.

Suppose a number N in scale x. be expressed $a+10b+10^2c+\ldots$ where $a, b, c, \ldots$ are the digits of the number commencing with the units-digit; then

(i.) Since $10 \equiv 1 \pmod 9$, $10^2, 10^3 \ldots, 10^n$ all $\equiv 1 \pmod 9$ (Law V.).

$\therefore \quad a+10b+10^2c+\ldots \equiv a+b+c+\ldots$ (Laws III. and IV.).

$\therefore$ N and $a+b+c+\ldots$ have the same remainder when divided by 9.

This is the ordinary rule for "Casting out Nines."

A precisely similar proof applies for "Casting out Eights" in scale ix., and so on.

(ii.) Since $10 \equiv -1 \pmod{11}$, $10^2 \equiv +1$, $10^3 \equiv -1$, $10^4 \equiv +1$, and so on.

$\therefore \quad a+10b+10^2c+\ldots \equiv a-b+c-d+\ldots.$

$\therefore$ N and $a-b+c-d+\ldots$ have the same remainder when divided by 11.

This is the ordinary rule for "Casting out Elevens."

A precisely similar proof applies for "Casting out Tens" in scale ix., and so on.

285. *Proving by Nines or Elevens.*

Suppose $M \equiv m \pmod 9$, m being the remainder when M is divided by 9;

$N \equiv n \pmod 9$, n being the remainder when N is divided by 9.

Then $MN \equiv mn$ (Law IV.); therefore the remainder, after casting out 9's from MN, must be the same as the remainder after casting out 9's from mn. This is the law for "proving" multiplications "by Nines."

An identical proof applies to "Elevens."

Fermat's Theorem.

286. *If r be a prime number and N be a number prime to r, then $N^{r-1}-1$ is divisible by r.*

Consider the series of numbers N, $2N$, $3N$, ... $(r-1)N$. Each number in this series is made up of two factors, one of which is less than r, and therefore cannot contain r, and the other of which is N, which is prime to r and cannot therefore be divisible by r. It follows that no number in this series can be divisible by r; therefore it is not possible for the difference of two numbers from this series to be divisible by r; *for the difference of two numbers in the series is itself one of the numbers of the series* (*e.g.* the difference of $2N$ and $5N$ is $3N$, a member of the series). Now, since the difference of two numbers of the series *cannot* $\equiv 0 \pmod r$, it follows that no two numbers in the series can give the same remainder when divided by r.

Hence, N, $2N$, ..., $(r-1)N$, when divided by r, all give different remainders. Since none of them is divisible by r, these remainders must be 1, 2, ..., $(r-1)$, though *not* usually in this order.

$\therefore\ N, 2N, \ldots, (r-1)N \equiv 1, 2, \ldots,$ or $(r-1) \pmod r$.

$\therefore\ N \times 2N \times \ldots \times (r-1)N \equiv 1 \times 2 \times \ldots \times (r-1)$ by Law IV.

$\therefore\ 1 \times 2 \times \ldots \times (r-1)\, N^{r-1} - 1 \times 2 \times \ldots \times (r-1) \equiv 0.$

$\therefore\ 1 \times 2 \times \ldots \times (r-1)\,(N^{r-1}-1) \equiv 0$, *i.e.* is divisible by r.

But r is a prime number and cannot be split into factors:

therefore it must divide completely into one of the factors. But it cannot divide into the factors 1, 2, ..., $(r-1)$, which are all less than itself; therefore it must divide into $N^{r-1}-1$. This is Fermat's Theorem.

Examples:—$36^{36}-1$ is divisible by 37. Put $N = 36$, $r = 37$.

$14^{72}-1$ is divisible by 73.

These should be verified by the method of Ex. 1, § 283.

287. *Laws of Recurring Decimals.*—We can now complete the theory of recurring decimals and give proofs of the laws stated in § 146, to which reference should be made.

Law I.

Since D is a prime number other than 2 and 5, it is prime to 10; therefore $10^{D-1}-1$ is divisible by D, *i.e.*

$$10^{D-1} \equiv 1 \pmod{D};$$

therefore (see Ex. 4, § 283) the $(D-1)$-th remainder in the conversion of $1/D$ into a decimal is 1, *i.e.* is the same as the figure we started with; therefore after $D-1$ figures the decimal certainly recurs.

If it recurs before, then by repeating the period it must be possible to bring up the number of figures to $D-1$; therefore the number of figures in the period must be a factor of $D-1$.

A similar proof applies to N/D, which we leave as an exercise.

There has not yet been discovered any method of generally foretelling *which* factor of $D-1$ gives the length of the period. It can be proved, however, that if when D is divided by 40 the remainder is 7, 11, 17, 19, 21, 23, 29, or 33, then $1/D$ either runs to its full period of $D-1$ figures or to $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, or some odd sub-multiple of $D-1$ figures; if, however, the remainder is 1, 3, 9, 13, 27, 31, 37, or 39, then $1/D$ runs to $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, or some even sub-multiple of $D-1$ figures. The converse of these statements is true.

Of the 1227 primes less than 10000 (excluding 2 and 5) 467 run to the full period, 360 to the half-period, 84 to one-third, 84 to one-quarter, 20 to one-fifth, and 61 to one-sixth.

Law III.

We present the proof of this (1) by numerical illustration; (2) generally.

Since $\frac{1}{7}$ has a period of 6 figures, 7 divides 999999;

$$\therefore \quad 1000000 \equiv 1 \pmod 7; \qquad \ldots\ldots (1)$$

Add 1 to both sides;

$$\therefore \quad 1000001 \equiv 2.$$

Square the congruence (1);

$$\therefore \quad 1000000000000 \equiv 1.$$

Add to the last.

$$\therefore \quad 1000001000001 \equiv 3.$$

Similarly cubing (1) and adding, we get

$$1000001000001000001 \equiv 4;$$

and so on, noting that the figure on the right-hand side shows the number of 1's on the left. Clearly we shall get a number divisible by 7 when we have 1 followed by 6 groups 000001. If we multiply by 999999, which is itself divisible by 7, we must obtain a number divisible by 49. But this number consists of 7 groups of 6 nines; therefore 1/49 has a period of $7 \times 6 = 42$ figures.

Since $1/D$ has a period of p figures, D divides a group of p nines;

$$\therefore \quad 10^p \equiv 1 \pmod D; \qquad \ldots\ldots (1)$$

$$\therefore \quad 10^p + 1 \equiv 2.$$

This is the number made by writing 1, $(p-1)$ 0's, and 1.

Square the congruence (1);

$$\therefore \quad 10^{2p} \equiv 1.$$

Add to the last;

$$\therefore \quad 10^{2p} + 10^p + 1 \equiv 3.$$

This is the number made by writing

$$1 \ldots (p-1) \text{ 0's} \ldots 1 \ldots (p-1) \text{ 0's } 1;$$

and if we continue this process, noting that the figure on the right-hand side shows the number of 1's on the left, we shall clearly get a number divisible by D when we have D 1's, *i.e.* 1 followed by $(D-1)$ groups of $00\ldots1$. If we multiply by p-nines, which is itself divisible by D, we must obtain a number divisible by D^2. This number consists of D groups of p-nines; therefore $1/D^2$ has pD figures in its period.

The proof just given shows that $1/D^2$ can certainly be expressed with a period of pD figures, and, therefore, must have a period either of p or of

pD figures. Only two cases ($D = 3$, $D = 487$) are known where $1/D$ and $1/D^2$ have both the same period. We have no means at present of knowing whether there are other cases.

The question has been investigated by Lt.-Col. Cunningham [*Messenger of Mathematics*, Vol. xxix., 1900, pp. 145-179; see also *Mathematical Gazette*, Jan. 1908, p. 209], who finds altogether 31 instances in various scales. For instance, $1/D$ and $1/D^2$ ($D = 487$) have the same length of period not merely in scale x., but also in scales 100, 175, and 307.

The following proof of Law iii. is shorter, but involves the Binomial Theorem:—

$$10^p \equiv 1 \pmod{D}.$$

Let $$10^p \equiv x \pmod{D^2}, \quad \therefore x = yD + 1,$$

$$\therefore 10^{pD} \equiv (yD+1)^D \pmod{D^2}$$
$$\equiv 1 \pmod{D^2}.$$

The extension to mod D^3 may be made on these lines.

Law VI.—The Complementary Law.

If $1/D$ have $2n$ figures in its period, $10^{2n} \equiv 1 \pmod{D}$; therefore $10^{2n} - 1 \equiv 0 \pmod{D}$; therefore $(10^n - 1)(10^n + 1)$ is divisible by D Now, if D contains no factor which is capable of giving a period of n figures, it contains no factor which divides $10^n - 1$; therefore $10^n + 1$ must be divisible by D;

$$\therefore 10^n \equiv -1 \equiv D - 1 \pmod{D};$$

$\therefore$ The n-th remainder is $D - 1$;

$\therefore$ The second half of the period of $1/D^2$ gives the period of $(D-1)/D$. But $1/D + (D-1)/D = 1 = \cdot 999 \ldots$. Hence, if the original decimal be added to the decimal beginning with the second half of the period (and continuing for ever), the result is $\cdot 999 \ldots$. But when this is done the figures of the first half will come underneath those of the second half, and *vice versa*. Hence the first half and the second half when added produce a number consisting only of nines.

EXAMPLES CLXI.

1. If the digits forming a number be rearranged in any manner, prove that the difference between the new number and the original is divisible by 9.

2. If one figure be omitted in the difference of two numbers the sum of whose digits is the same, show that it is generally possible to determine which figure has been omitted by an examination of the figures remaining. What are the exceptions?

3. If the digits of any number composed of an even number of figures be reversed, prove that the sum of the new and the original number is divisible by 11.

4. Prove that, if $1/D$ has p figures in its period, $1/D^3$ will generally have pD^2 figures.

SECTION XI.

HARDER PROBLEMS.

1. Determine the missing figures in the following division sum :—

$$\begin{array}{r} 7..\,)\,51..1\,(\,.2 \\ \underline{..1.} \\ 1.1. \\ \underline{....} \\ 1.9 \end{array}$$

2. A boy multiplied a number by 467 and obtained the product 1925817. The figures 9 and 7 were wrong. What ought they to have been?

3. The number $1a5b1c2$ (where a, b, and c denote digits) can be divided exactly by 198, and the units digit in the quotient is 4. Prove (1) $a+b+c=9$; (2) c is odd. Also (3), if c is 7 and the thousands digit in the quotient is 6, find the dividend and the quotient.

4. Working throughout in scale xx., in which $t = 10$, e or $\text{ᴛ} = 11$, $\text{ᘓ} = 12$, $\text{ᘔ} = 13$, &c., effect the following operations—

(1) Add together $\cdot\dot{\text{ᘔ}}\,\text{ᘓ}\,1\dot{2}$, $\cdot\dot{\text{ᘔ}}\,6\,\dot{\text{ᴛ}}$, $\cdot\dot{\text{ᘔ}}\,\dot{9}$, and $\cdot\dot{t}$.

(2) Subtract $\cdot\dot{\text{ᴛ}}\,\text{ᘓ}\,\text{ᘔ}\,4\,5\,\dot{6}$ from $\cdot\dot{\text{ᴛ}}\,\text{ᘔ}\,\text{ᘔ}\,t\,e\,\dot{2}$.

(3) Convert $1\,2\,3\,4\,5\,6\,7\,8\,9\,t\,e\,\text{ᘓ}\,\text{ᘔ}\,\text{ᘓ}\,\text{ᘓ}\,\text{ᘓ}\,\text{ᘓ}$ into scale 10.

(4) Find the square root of $1\,2\,\text{ᘓ}\,\text{ᴛ}\,\text{ᘓ}\,2\,\text{ᴛ}\,7\,4\,4$.

(5) Find the cube root of $4\,\text{ᘓ}\,1\,7\,6\,7\,\text{ᘓ}\,3\,\text{ᘓ}$.

(6) Multiply $2\,1\,\text{ᴛ}\,e\,0\,\text{ᴛ}$ by $1\,\text{ᘓ}\,1$.

(7) Divide $e\,1\,\text{ᘓ}\,1\,e\,0\,t$ by 5555.

(8) Find the fourth root of $\dfrac{5\,9\,\text{ᘔ}\,3}{t\,1\,6\,t\,\text{ᘓ}\,\text{ᘓ}\,\text{ᘓ}\,8}$.

(9) State which of the following fractions give pure recurring vicesimals, and give the number of non-recurring figures in the others :— $\dfrac{1}{5}$, $\dfrac{1}{\text{ᘓ}}$, $\dfrac{1}{15}$, $\dfrac{1}{7}$, $\dfrac{1}{\text{ᴛ}}$, $\dfrac{1}{17}$.

(10) Convert $\dfrac{1}{9}$, $\dfrac{1}{\text{ᘔ}}$, and $\dfrac{1}{19}$ into vicesimal radix fractions.

5. Multiply 1234567 by 89*teT* (T = twelve) in scale xiii, and prove your result by "casting out fourteens."

6. Simplify $\dfrac{2340136986301369863013698630136962 9}{3450072992700729927007299270072958 2}$.

7. Simplify $\dfrac{\frac{1}{198}-\frac{4}{199}+\frac{6}{200}-\frac{4}{201}+\frac{1}{202}}{\frac{1}{197}-\frac{4}{198}+\frac{6}{199}-\frac{4}{200}+\frac{1}{201}}$.

8. Simplify $\dfrac{\frac{1}{196}-\frac{2}{99}+\frac{3}{100}-\frac{2}{101}+\frac{1}{204}}{\frac{1}{194}-\frac{2}{98}+\frac{3}{99}-\frac{2}{100}+\frac{1}{202}}$.

9. Simplify $\dfrac{\frac{1}{2}-\frac{1}{3}-\frac{1}{7}-\frac{1}{43}-\frac{1}{1807}-\frac{1}{3263443}}{\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{7}\cdot\frac{1}{43}\cdot\frac{1}{1807}\cdot\frac{1}{3263443}}$.

10. Simplify $\dfrac{3+1\frac{1}{2}+1\frac{2}{7}+1\frac{14}{67}+1\frac{938}{5623}+1\frac{5274374}{37772347}}{3\,.\,1\frac{1}{2}\,.\,1\frac{2}{7}\,.\,1\frac{14}{67}\cdot 1\frac{938}{5623}\cdot 1\frac{5274374}{37772347}}$.

11. Simplify $\dfrac{10+1\frac{1}{9}+1\frac{9}{91}+1\frac{819}{9181}+1\frac{7519239}{92480761}}{10\times 1\frac{1}{9}\times 1\frac{9}{91}\times 1\frac{819}{9181}\times 1\frac{7519239}{92480761}}$.

12. Evaluate (1) $\sqrt[3]{341,574,893,363}$. (Scale xi.)

(2) $\sqrt[5]{t,639,012,218,57e}$. (Scale xii.)

13. Evaluate (1) $\sqrt[2]{999,993\times 999,994\times 999,995\times 999,996+1}$.

(2) $\sqrt[2]{999,993\times 999,995\times 999,997\times 999,999+16}$.

14. Find the fifth root of 30517578126 to 30 places of decimals.

15. Find the seventh root of the duodecimal 107192e2e199701.

16. Prove that $9x^2+1,481,481,451,851,856$ is an exact square when $x = 123,456,787,654,321$, and find its square root.

17. Find the last 13 digits in the product of the numbers from 1 up to 52 inclusive.

18. Prove that the product of the first thousand numbers is terminated by 249 zeros; the product of the first million by 249998; the product of the first billion by 249,999,999,997; and so on.

19. If $A = 50^3-12^3-14^3-24^3$, $B = 50^3-13^3-18^3-19^3$, $C = 823^3-139^3-329^3-355^3$, $D = 823^3-234^3-247^3-342^3$, find the value of

(1) $\dfrac{AD}{BC}$; (2) $\sqrt{\dfrac{CD}{AB}}$; (3) $\sqrt[3]{24AC}$; (4) $\sqrt[4]{\dfrac{36ACD}{B}}$.

20. Arrange the nine digits 1 to 9 in such order that addition will make 100.

21. Prove that the square of a number consisting of n 6's is found by writing $(n-1)$ 4's, a 3, $(n-1)$ 5's, and a 6.

22. Show that the middle digit in the square of the number 12345678987654321 is 2.

23. Prove that the two middle figures in the product of 1234567887654321 by 8642753113572468 are 6 and 7.

24. Every number formed by writing 111... 3^n times is a multiple of 3^n.

25. Prove $\left(\frac{59347}{18162}\right)^3 + \left(\frac{8693}{18162}\right)^3 = 35$.

26. Find a sum of money which can be correctly reduced to farthings by simply removing the dots which separate pounds, shillings, and pence.

27. Prove that $10^5 + 10$ is divisible by $9^3 + 1$.

28. Find the number of integers between 1000 and 2000 which are not divisible by 2, 3, or 5.

29. Five ladies, accompanied each by her daughter, purchased cloth at the same shop. Each of the ten bought as many feet of cloth as she paid farthings per foot. Each mother spent 8s. $5\frac{1}{4}$d. more than her daughter. Mrs. Robinson spent 6s. more than Mrs. Evans, who only spent about a quarter of what Mrs. Jones did, while Mrs. Smith spent most of all. Mrs. Brown bought 21 yds. more than did Bessie, one of the girls, while of the other girls Annie bought 16 yds. more than Mary and spent £3 0s. 8d. more than Emily. The other girl's Christian name was Ada. Give the surnames of all the girls.

30. Express 75075 as (1) the product of two integral factors such that their difference is as small as possible, (2) the difference of two squares.

31. The number 114438555 can be obtained by multiplying together four numbers the difference of the greatest and least of which is 10. What are they?

32. The number 976484376000 is the product of 5 consecutive integers. What are they?

33. The ancient Egyptians could only deal with fractions with unit numerators, and decomposed any fractions whose numerators differed from 1 into the sum of unit fractions. Develop a process for effecting this decomposition, which provides that each denominator shall be an exact multiple of the last, and apply your methods to prove

$$(1)\quad \frac{11}{13} = \frac{1}{2} + \frac{1}{2.2} + \frac{1}{2.2.3} + \frac{1}{2.2.3.7} + \frac{1}{2.2.3.7.13};$$

$$(2)\quad \frac{113}{244} = \frac{1}{2} - \frac{1}{2.13} + \frac{1}{2.13.24} - \frac{1}{2.13.24.61};$$

$$(3)\quad \frac{355}{113} = 3 + \frac{1}{8} + \frac{1}{8.8} + \frac{1}{8.8.17} + \frac{1}{8.8.17.19} + \frac{1}{8.8.17.19.113}$$

34. A number is known to consist of 4 digits whose sum is 10. Show that the odds are 154 : 65 in favour of the sum of the digits of twice the number being equal to 11.

35. The sums of the digits in the successive products in the multiplication table are connected by certain laws, *e.g.*:

Multiples of 8: 8, 16, 24, 32, 40, 48, 56, ...;

Sums of digits: 8, 7, 6, 5, 4, 12, 11, ...

Examine the case suggested, and enunciate the laws involved.

36. Show that the number 900008000070000600005000040000 2 is divisible by 41, even though all the figures except the zeros be interchanged.

37. Verify that the numerator of $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{22}$ is divisible by 23^2.

38. Find all unequal proper fractions with denominators 100 and 240 which differ by less than ·001.

39. Find all unequal proper fractions with denominators 120 and 400 which differ by less than ·001.

40. The first 8 figures of the cube root of an integer are 9·9966655. What are the next 8?

41. In which decimal place does the cube root of $10^{15} + 2$ differ from 10^5?

42. The reciprocal of an integer is ·00013906.... What are the next 8 figures?

43. A number is known to be divisible by 383. In finding its cube root a figure in the answer was accidentally erased, and the answer then read 7283648. What was the missing figure, and where did it stand?

44. Indicate a short method for dividing by 999,999. Apply your method to find the *remainder* after division of 111,111,111,111,111,111,111 by 9999 × 9999.

45. Indicate a process by means of which the decimal value of $\frac{1}{199}$, $\frac{1}{1999}$, $\frac{1}{19999}$, &c., may be rapidly calculated *mentally*, and apply your process to find the first 30 figures of each of the fractions named.

46. Calculate to 30 places of decimals the value of

$$\frac{1}{10000} + \frac{1}{9999} + \frac{1}{9998} + \frac{1}{9997} + \frac{1}{9996}.$$

47. Find correct to 30 places of duodecimals the value of

$$\frac{eee,eee,ee2}{eee,eee,eee} \times \frac{eee,eee,ee4}{eee,eee,ee9} \times \frac{eee,eee,ee6}{eee,eee,ee7} \times \frac{eee,eee,ee8}{eee,eee,ee5} \times \frac{eee,eee,ee\text{t}}{eee,eee,ee3}$$

48. Calculate the value to 30 places of duodecimals of

$$\frac{\dfrac{eee,eee,ee4}{eee,eee,ee5} - \dfrac{eee,eee,ee2}{eee,eee,ee3}}{\dfrac{eee,eee,ee8}{eee,eee,ee9} - \dfrac{eee,eee,ee6}{eee,eee,ee7}}.$$

49. Factorize— (1) 100,000,004 ; (2) 400,000,001.

50. Find, by inspection, a factor of 6000004049.

Factorize, as far as possible—

51. (1) $10^{20}-1$; (2) $2^{50}-1$; (3) $16^{45}-1$.

52. (1) $3^{51}+1$; (2) $2^{120}-1$; (3) $3^{27}+1$.

53. (1) $3^{45}+1$; (2) $2^{54}+1$; (3) $2^{58}+1$.

54. Factorize— (1) 1,000,000,000,000,000,027 ;
(2) 27,000,000,000,000,000,001 ;
(3) 64,000,000,000,729 ;
(4) 729,000,000,000,064.

55. Factorize— (1) 31,249,999,999,999 ;
(2) 31,999,999,999,999,999.

56. If 142,857 be multiplied by 326,451, all the figures in the same vertical column in the partial products are equal. Find other numbers which possess the same property.

57. Show that the numbers 10101, 101010101, 1010101010101, ... cannot be prime in any scale of notation.

58. If the number formed by writing down in ascending order, beginning with 1, the first n digits in scale r be multiplied by $r-2$, and n be added to the product, the result is the number formed by writing down in descending order the last n digits of the scale, beginning with the last.

59. Prove that the sum of the H.C.F. and the L.C.M. of two numbers is greater than the sum of the numbers.

60. How many pairs of four-figure numbers are there which have the same H.C.F. as 7833973 and 7524949?

61. Prove that we can generally find two numbers smaller than either of two proposed numbers, so that all four have the same H.C.F. Mention any exceptions, and find two numbers of 6 figures each which have the same H.C.F. as 7833973 and 7524949.

62. There are four numbers, A, B, C, D, which have no factor common to all, and no factor of one which is not a factor of one or more of the others, and such that the L.C.M. of A, C is 2800733 ; of A, D is 3774901 ; of B, C is 5107219 ; of B, D is 2993887 ; H.C.F. of B, C is 5681 ; of A, C is 551. What are the numbers?

63. A and B are two integers, A consisting of $2m$ figures, each being 1, and B consisting of m figures, each being 4. Prove that $A+B+1$ is a square.

64. Show that the product of 2 consecutive numbers cannot be a perfect square, and find a series of products of two consecutive numbers which are doubles of squares.

65. Prove that the sum of 3 or 4 consecutive squares can never be a square.

Prove that the sum of 5 or 7 consecutive squares can never be a square.

66. Show that the product of 3 consecutive numbers cannot be a perfect square, nor the double of a square.

67. Prove that the product of 4 consecutive numbers cannot be a perfect square.

68. Show that the product of 5 consecutive numbers cannot be a perfect square.

69. Prove that all powers of numbers which terminate with 12890625 will terminate with 12890625.

70. Find a series of numbers of 1, 2, 3, ... figures such that, if each be multiplied by the number next above it, the product ends in at least as many zeros as the number had figures. Show that there are only two such series.

71. Find a series of numbers of 1, 2, 3, ... figures such that, if each be multiplied by the number next below it, the product ends in at least as many zeros as the number had figures. Show that there are only two such series.

72. A number consists of n digits; its square ends with the same n digits in the same order. Show that (1) for any given value of n there are 2, and only 2, such numbers in each of the scales of notation whose radices are 6, 10, 12, respectively; and (2) in any other scale (radix not greater than 12) no number but 1 or 0 exists having this property.

73. Find, in scale x., two numbers of 10 figures each such that all powers of numbers which end in either of these groups of 10 figures will also end in the same.

74. Find, in scale vi., two numbers of 10 figures each such that all powers of numbers which end in either of these groups of 10 figures will also end in the same.

75. Find, in scale xii., two numbers of 10 figures each such that all powers of numbers which end in either of these groups of 10 figures will also end in the same.

76. Prove that, if 76*e*45 (scale xii.) be multiplied by any number (< 31), the sum of the digits of the product is a multiple of 33 (x.). Determine the laws which decide which multiple of 33 (x.) is produced.

77. Prove that, if 8579214*e*36429*t*7 (scale xii.) be multiplied by any number, the sum of the digits is 88 (< 29) or a multiple of 88 (x.).

78. Show that the sum of the digits of every multiple of 2739726 up to the 72nd must be 36.

79. Find the remainder of the division of $(265429)^{347486}$ by 977.

80. Find how many numbers of 4 digits exist such that, if the digit to be placed to the right of the digit in the units place, the number of 5 digits so formed is divisible by 11. Show that 14 of these numbers are also divisible by 61.

81. Prove that, if $a_n a_{n-1} \ldots a_2 a_1$ be the digits of a number in scale 9, and $S_1 = a_1 + a_4 + a_7 + \ldots$, $S_2 = a_2 + a_5 + \ldots$, $S_3 = a_3 + a_6 + \ldots$, the number is divisible by 13 (scale 10) if $S_1 - 4S_2 + 3S_3$ is so divisible, by 7 if $S_1 + 2S_2 - 3S_3$ is so divisible, by 8 if $S_1 + S_2 + S_3$ is so divisible, and by 56 (scale 10) if $S_2 - 3S_3$ and $S_1 - 4S_3$ are both so divisible.

82. Required, short rules for finding the remainder when a very long number is divided by (1) 73, (2) 47.

83. If a number consists of 7 figures whose sum is 59, prove that the odds are 17 to 4 against it being exactly divisible by 11.

84. Show how to place 6 other figures on the right hand of 77777 so that the whole 11 figures may form a perfect square.

85. Prove that neither 88888 . . . . . 5 nor 88888 . . . . . 6 nor 88888 . . . . . 9 can be square numbers.

86. Prove that a perfect cube of 12 figures cannot have the first 5 figures identical unless they are all 8's.

87. Prove that, if the first 4 figures of a perfect square of 8 figures are identical, they cannot be either 5 or 9.

88. 4327 . . . . . is a perfect cube. What are the missing figures?

89. Prove that 405 . . . . . 5 cannot be a perfect cube.

90. Prove that there is no square number of 10 figures which commences at the left with 77777.

91. The decimal part of the square root of an integer of 4 figures commences ·4381903.... What is the integer?

92. The decimal part of the square root of an integer of not more than 2 figures commences ·1414284.... What is the integer?

93. The square root of an integer of not more than 4 figures commences in its decimal part with ·1211010.... What is the integer?

94. Prove that no square root of an integer of 4 figures can have a decimal part commencing ·1111111....

95. Show that the figure 9 cannot occur in the decimal value of a fraction whose denominator is < 10. In the scale whose radix is r, what figures, if any, cannot occur in the decimal part of a fraction whose denominator is $< r$?

96. Express $(\cdot\dot{0}00\ldots\dot{1})^2$ in any scale as a recurring decimal, stating the law of the sequence of the figures and explaining the exception.

97. Show that, in any scale of notation, it is possible to convert a fraction into a radix fraction by finding first the *last figure* in the radix fraction, or in the circulating period, if it have one, and then the last but one, and so on, finding all the figures in the reverse order.

98. Explain, and prove, the following method of decimalizing $\frac{7}{39}$:—

$\frac{7}{39} = 4\,\underline{)\,7\cdot\quad}$

$\cdot\overset{3}{1}$, the remainder 3 being written small, above the 1;

$\cdot 1\overset{3}{7}$, the 1 being repeated, the 7 obtained by division into 31 by 4;

$\cdot 17\overset{1}{9}$, the 9 being obtained by division of 37 by 4;

$\cdot 179\overset{3}{4}$, &c., &c.

Show that a similar method can be applied to all fractions ending in 9, and apply the method to find the first 10 figures of $\frac{7}{89}$.

99. Explain, and prove, the following method of decimalizing $\frac{3}{19}$, which gives the last figure of the recurring period first, and the others by successive multiplication by 2:—

$$\frac{3}{19} = \cdot\dot{\ }\ldots\ldots\dot{3}$$
$$_{1}63$$
$$263$$
$$5263$$
$$\dot{0}5263$$
$$105263$$
$$2105263$$
$$42105263, \text{ \&c., \&c.}$$

Show that a similar method can be applied to all fractions whose denominators end in 9, whatever their numerators, and apply such a method to find the last 10 figures of the period of $\frac{7}{89}$.

100. If n be prime to r and greater than p, and $\frac{p}{n}$, $\frac{n-p}{n}$ be converted into recurring radix fractions in scale r, what relation holds between the periods of these two fractions?

101. Prove that the odds are exactly 6 to 1 that a fraction written down at random will in scale 7 give rise to a pure circulator.

102. Prove that, if N be an integer, the last figure of the recurring period of $\frac{1}{10N-1}$ is 1, the last figure of the recurring period of $\frac{1}{10N-3}$ is 7, and of $\frac{1}{10N-7}$ is 3.

103. Prove that, under certain restrictions, if $\frac{1}{N}$ contain $2n$ places in its recurring period, the nth figure is 0, 6, 2, or 8, according as N ends in 1, 3, 7, or 9. State the restrictions.

104. Show that there are only 2 vulgar fractions which, in their lowest terms, have 148 for denominator, each one of which is equal to a decimal consisting of 2 non-recurring figures followed by $\dot{5}\dot{6}\dot{7}$.

105. Prove, by the Theory of Circulating Decimals, that $10^{189}+1$ is divisible by 379, and therefore also by 4169, 379379, and 3790000379.

106. Prove that, if $\frac{1}{\lfloor 100}$ (scale x.) be converted into a recurring decimal, there are 97 non-recurring figures; as a recurring undecimal, there are 9 such; as a recurring duodecimal, 49.

107. Show that the 49,513th figure of $\frac{1}{6841t}$ (scale xi.) is 5.

[N.B.—49,499 and 98,999 are both primes.]

108. Convert $\frac{1}{239}$ into a recurring decimal, and prove that there is only one other fraction with prime denominator and unit numerator with 7 figures in the recurring period.

109. The fraction $\frac{1}{N}$ gives rise to a pure circulator of 7 figures. Show that N has 9 possible values, and find them. (See Prob. 108.)

110. The denominator of $\frac{1}{108808}$ (scale ix.) is a prime number. Prove that this fraction gives rise to a radix fraction with a circulating period of which the number of figures must be a power of 2.

111. It being given that $\frac{1}{709}$ is a pure circulator of 708 figures, and that $\frac{1}{827}$ is a pure circulator of 413 figures, prove that $\frac{1}{506600352}$ commences to recur at the 4962nd figure.

112. Prove that the figures of the circulating period of $\frac{39et}{18237}$ (scale xii.), beginning with the 3951st and ending with the 3960th (and last), are 538343*e*6*e*2.

113. Prove that $\frac{1}{184,759,871,889,775}$ is a mixed circulator with one non-recurring figure, the recurrence of which commences with the 1563rd figure.

114. Show that there are only two scales with radix less than 40 in which $\frac{1}{45}$ (x.) and $\frac{1}{56}$ (x.), when reduced to radix fractions, have 2 and 3 non-recurring figures respectively, and find the corresponding radix fractions in both scales. [Write "digits" greater than 9 in brackets, thus:—(12), (21).]

115. Prove that, in scale xi. the number of vulgar fractions with unit numerator which give rise to undecimals having not more than 100 non-recuring figures and not more than 7 recurring figures is 23735, and that the non-recurring figures must all be zero if there are any recurring figures.

116. It being given that 16002 is the product of 2 consecutive integers, prove that $\frac{1}{16002}$ is a mixed circulator with 1 non-recurring figure, and a period of 42 recurring figures, of which the last is 5.

117. Prove that $\frac{1}{2111109}$ has 18 places in its period as a circulating decimal.

118. (1) Prove that, in scale r, $\left(\frac{1}{r-1}\right)^2 = \cdot\dot{0}123\ldots\overline{r-3}\,\dot{\overline{r-1}}$.

(2) Deduce that $\frac{1}{88209}$ has 1782 figures in its period.

(3) Deduce also that

$$\frac{1}{1{,}002{,}003{,}004\ldots997{,}999} = \cdot\dot{0}00\ldots(2991 \text{ zeros altogether})\ldots 99800\dot{1}.$$

119. If r be of the form $4n$, $4n+1$, or $4n+2$, prove that the number of digits in the recurring period of $\left(\frac{1}{r-1}\right)^3$, in scale r, is $(r-1)^2$; but, if it be of the form $4n-1$, only half this number.

120. Prove that the number of fractions of the form $\frac{1}{N}$ which have 6 places in their recurring period is 53.

121. Show that the number of places in the recurring period of $\frac{1}{9{,}200{,}575{,}223{,}941}$ is either 2,300,142,289,363 or twice that number, and that, in either case, from the 2,300,142,289,363rd to 2,300,142,289,375th figures are all 0 or all 9, and the next figure and the next but two together make 9. [N.B.—1,516,583 ; 1,516,661 ; 3,033,167 ; 3,033,323 are all primes.]

122. The 75th, 76th, and 77th figures in the period of $\frac{1}{701}$ are 716. Find the 150th to 160th figures *without finding any intermediate ones.*

123. The 50th and two following figures in the period of $\frac{1}{701}$ are 897. Find the 101st and the 152nd figures.

124. The figures 142857, being arranged at random as the period of a circulating decimal, are then reduced to a vulgar fraction in lowest terms. Show that the odds are 119 : 1 against the denominator being 7.

125. If $\frac{1}{P}$ be converted into a radix-fraction in scale r (prime to and less than P), and if the successive remainders be R_1, R_2, ..., then, if each remainder be subtracted from the one that follows (increased, if necessary, by P), the series of numbers thus formed reproduces in cyclical order the original series of remainders, provided that among the remainders be found the number $r-1$.

126. If $\frac{1}{P}$, when converted into a radix fraction in scale r (prime to P) have p places in its period, prove that, if 1 be subtracted from the pth power of any one of the "remainders" found during the process, the difference is divisible by P.

127. If N is a prime number and $\frac{1}{N}$ contain $2n$ places in its recurring period, of which the first n form the number M, prove that $\frac{1}{M+1}$ will also be a pure circulator with $2n$ places in its recurring period, of which the first n form the number $N-1$.

128. Prove that, if, in any scale, $\frac{1}{N}$ (N prime) contain 6 figures in its recurring period, another scale can always be found in which $\frac{1}{N}$ also contains 6 figures such that the sum of the two radices diminished by 1 is a multiple of N.

129. Prove that, in scale 202447984713, $\frac{1}{601}$ (scale x.) and $\frac{1}{421}$ (scale x.) will each contain 6 places.

130. Find all the scales less than 1000 in which $\frac{1}{13}$ (scale x.) and $\frac{1}{17}$ (scale x.) both have 4 figures in their recurring period.

131. If 7 be divided by 5, giving 1 and 2 over, and then, successively, 21 by 5, giving 4 and 1 over; 14 by 5, giving 2 and 4 over; 42 by 5, giving 8 and 2 over; and so on until the figures recur, the result of placing all the quotients after the 7 and adding a decimal point suitably is $\cdot\dot{7}1428\dot{5}$, which is equal to $\frac{5}{7}$. Investigate the general form of the fractions $\frac{a}{b}$ which can be expressed as circulating decimals by dividing b by a in the manner explained with reference to the case $\frac{5}{7}$.

132. In a certain decimal, if the first recurring dot is shifted one place to the right (the other remaining fixed), the value will be altered by $\frac{1}{1425}$. Find the decimal.

133. A, B, and $\frac{1}{21649}$ are circulating decimals with recurring periods of the same length and such that $B \times (A+2) = 1$. Find A and B.

134. Find a fraction with prime denominator which, when expressed decimally, has a recurring period consisting of the 10 digits. Show that there is only one possible denominator and 240 different numerators.

135. In the process of converting $\frac{1}{100103}$ into a recurring decimal, the 273rd remainder is 100067, the 5005th is 23447, the 6126th is 14, the 6290th is 26789, and the 7587th is 100091. Find where the remainders 1, 2, 3, ..., 10 will occur.

APPENDIX I.

LOGARITHMS.

288. **Fractional and Negative Indices.**—If we consider all numbers from one upwards, we see that only such numbers as 10, 100, 1000, etc., are exact powers of 10, or writing them as 10^1, 10^2, 10^3, etc., we see that they are the only ones in which the indices of the various powers of 10 are positive whole numbers. Consider any other number, say 68. This lies between 10^1 and 10^2, and it is clear that we cannot multiply the factor (or, as it is generally called, the **base**) 10 by itself an exact number of times in order to get 68. The necessary number of times, *i.e.* the necessary index, would lie between 1 and 2, and so would be partly fractional. We may, therefore, conveniently make use of such expressions as $10^{\frac{1}{2}}$, $10^{\frac{1}{3}}$, $10^{1\cdot5}$, etc.

Again, we may continue the above series in the reverse order, thus:— . . . . 1000, 100, 10, 1, $\frac{1}{10}$, $\frac{1}{100}$, etc., or . . . . $10^3, 10^2, 10^1, 1, \frac{1}{10}, \frac{1}{10^2}$, etc., the indices following the descending series . . . . 3, 2, 1, etc. If we continue this last series, we get a succession of negative numbers for our indices, ... 3, 2, 1, 0, -1, -2, etc. Thus the numbers

$$\ldots\ldots 10^3,\ 10^2,\ 10^1,\quad 1,\quad \frac{1}{10},\quad \frac{1}{10^2},\quad \frac{1}{10^3}\ldots\ldots$$

may conveniently be represented by

$$\ldots\ldots 10^3,\ 10^2,\ 10^1,\ 10^0,\ 10^{-1},\quad 10^{-2},\ 10^{-3}\ldots\ldots$$

289. If we wish to find the value of (1) $10^3 \times 10^2$, (2) $7^5 \times 7^3$, we may reason thus:—

Since $10^3 = 10 \times 10 \times 10$ and $10^2 = 10 \times 10$,

$\therefore$ $10^3 \times 10^2 = 10 \times 10 \times 10 \times 10 \times 10 = 10^5$, *i.e.* the final index 5 is obtained by *adding* together the two indices 3 and 2.

So $$7^5 \times 7^3 = 7^{5+3} = 7^8.$$

By similar reasoning we can show that
$10^3 \div 10^2 = 10^{3-2} = 10^1$; $7^5 \div 7^3 = 7^{5-3} = 7^2$; $(10^3)^2 = 10^3 \times 10^3 = 10^6$, and so on.

These are simple illustrations of a general law, known as the **Index-law**, by which the various powers of a common base may be combined in the processes of multiplication and division.

290. **Index-Law.** $a^m \times a^n = a^{m+n}$.—Proof for the case when m and n are positive whole numbers.

Since $a^m = a \times a \ldots.$ to m factors, and $a^n = a \times a \ldots.$ to n factors,

$\therefore\ a^m \times a^n = (a \times a \ldots. \text{to } m \text{ factors}) \times (a \times a \ldots. \text{to } n \text{ factors})$

$$= a \times a \ldots.. \text{ to } (m+n) \text{ factors}$$
$$= a^{m+n}.$$

So $a^m \div a^n = a^{m-n}\ (m > n)$ and $(a^m)^n = a^{mn}$.

291. The proof of the law just given is valid only for positive whole numbers. It will be assumed here that the law is true also for fractional and negative indices, provided we give a suitable interpretation to our symbols.* Thus

$a^{\frac{1}{2}}$, when raised to the 2nd power,

$$= a^{\frac{1}{2}+\frac{1}{2}} = a, \quad \therefore\ a^{\frac{1}{2}} = \sqrt{a};$$

$a^{\frac{1}{3}}$, when raised to the 3rd power,

$$= a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a, \qquad \therefore\ a^{\frac{1}{3}} = \sqrt[3]{a};$$

$a^{\frac{2}{3}}$, when raised to the 3rd power,

$$= a^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}} = a^2, \qquad \therefore\ a^{\frac{2}{3}} = \sqrt[3]{a^2}, \text{ and so on.}$$

The index 0 and negative indices are interpreted as follows:—

$$a^0 = 1, \quad a^{-1} = \frac{1}{a}, \quad a^{-2} = \frac{1}{a^2}, \quad a^{-n} = \frac{1}{a^n},$$

$$a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}} = \frac{1}{\sqrt[n]{a^m}}, \text{ and so on.} \qquad \text{(Cf. § 288.)}$$

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* For a full discussion of these cases see *Tutorial Algebra (Advanced Course)*, Chapter I.

Ex.—Find the values of

$$49^{\frac{1}{2}},\quad 6^{-3},\quad (a^{\frac{1}{2}})^4,\quad 125^{-\frac{2}{3}}. \qquad 49^{\frac{1}{2}} = \sqrt{49} = 7;$$

$$6^{-3} = \frac{1}{6^3} = \frac{1}{216};\quad (a^{\frac{1}{2}})^4 = a^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}} = a^2;$$

$$125^{-\frac{2}{3}} = \frac{1}{125^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{125^2}} = \frac{1}{5^2} = \frac{1}{25}.$$

(Note the order of the steps; do not work out 125^2 and then take the cube root. The result would be the same, but the work very long.)

EXAMPLES CLXII.

Simplify:—

1. $2^2 \times 2^3$. **2.** $2^3 + 2^2$. **3.** $2^3 \div 2^2$. **4.** $x^5 \times x^7$.

5. $(x^5)^2$. **6.** $a^p \times a^2$. **7.** $10^{10} \div 10^6$. **8.** $(10^2)^3$.

9. $(10^3)^2$. **10.** $100^{\frac{1}{2}}$. **11.** $1000^{\frac{1}{3}}$. **12.** $(10^3)^{\frac{1}{3}}$.

13. $(x^3)^{\frac{1}{3}}$. **14.** $(x^{\frac{1}{3}})^3$. **15.** $10^{\frac{1}{2}} \times 10^{\frac{1}{2}}$. **16.** $100^{-\frac{1}{2}}$.

17. $4^{\frac{1}{2}}$. **18.** $16^{\frac{1}{4}}$. **19.** $16^{-\frac{1}{4}}$. **20.** $8^{\frac{2}{3}}$.

21. $8^{-\frac{2}{3}}$. **22.** $\sqrt{x^{2m}}$. **23.** $27^{\frac{1}{3}}$. **24.** $625^{\frac{3}{4}}$.

25. $\sqrt[4]{81}$. **26.** $25^{\frac{3}{2}}$. **27.** $\sqrt[3]{y^3}$. **28.** $32^{\frac{2}{5}}$.

29. $(27^2)^{\frac{1}{3}}$. **30.** $32^{1\cdot 2}$. **31.** $\sqrt[4]{y^8}$. **32.** $y^{32} \div y^{30}$.

33. $y^{30} \div y^{32}$. **34.** $1000^{-\frac{1}{3}}$. **35.** $36^{-\frac{1}{2}}$. **36.** $27^{1\frac{1}{3}}$.

37. $100^2 \div 10^3$. **38.** $8^3 \div 64$. **39.** $7^4 \div 49$. **40.** $7^4 \times 7^{-2}$.

41. $125^{-\frac{1}{3}}$. **42.** $16^{-\frac{3}{4}}$.

Find values for x in the following:—

43. $2^x = 4$. **44.** $8^x = 2$. **45.** $1000 = 10^x$. **46.** $7^x = \frac{1}{49}$.

47. $\sqrt{16} = 2$. **48.** $\sqrt{x} = 10$. **49.** $27^{\frac{1}{3}} = x$. **50.** $2^x = 1$.

51. $1000^x = 10$. **52.** $2^{-x} = \frac{1}{4}$.

292. Definition.—**The logarithm of a number to a given base** is the *index* of the power to which the base must be raised to produce that number. Thus, if $10^m = x$, then m is called the logarithm of x to the base 10, and is usually written $\log_{10} x$.

Ex.— $\log_3 81 = 4$, since $3^4 = 81$.

Similarly $\log_{10} 100 = 2$, since $10^2 = 100$;

$\log_2 8 = 3$, since $2^3 = 8$.

Any base might be employed, but for the practical purposes of calculation the base 10 is always used, as our numbers are formed on the decimal system. The base 10 is not generally inserted, so that $\log_{10} 100 = 2$ is shortened into $\log 100 = 2$.

Logarithms calculated to base 10 are called **common logarithms.** These only are used throughout this chapter.

N.B.—Since $10^0 = 1$, $\therefore$ $\log 1 = 0$; also $\log 10 = 1$, $\log 100 = 2$, $\log 1000 = 3$, and so on.

293. Since logarithms are the same as indices, any properties of the latter will hold equally for the former. Also since we may conceive of any number being expressed as some power of 10 (integral, fractional, or negative), from the index-law (§ 290) we see that the index of the power for the product of two numbers is found by *adding* the separate indices for the two numbers themselves;

i.e. the logarithm of a product is equal to the sum of the logarithms of its factors.

Thus, let M and N be any two numbers; let

$$M = 10^m,\ N = 10^n, \quad \therefore\ M \times N = 10^{m+n}.$$

By definition

$$\log_{10} M = m,\ \log_{10} N = n,\ \log_{10}(M \times N) = m + n.$$

Hence $\quad \log_{10}(M \times N) = \log_{10} M + \log_{10} N.$

So

$$\log_{10}(M \times N \times P \times \ldots) = \log_{10} M + \log_{10} N + \log_{10} P + \ldots$$

or omitting the 10,

$$\log(M \times N \times P \times \ldots) = \log M + \log N + \log P + \ldots$$

294. *The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.*

As in § 293, let $\quad M = 10^m,\ N = 10^n.$

Then $$M \div N \text{ or } \frac{M}{N} = 10^{m-n}.$$

By definition $\log_{10} M = m$, $\log_{10} N = n$, $\log_{10} \frac{M}{N} = m - n$,

$$\therefore \log_{10} \frac{M}{N} = \log_{10} M - \log_{10} N.$$

295. *The logarithm of any power of a number is equal to the product of the logarithm of the number and the index of the power.*

For if $M = 10^m$, then $M^r = (10^m)^r = 10^{mr}$, where r is positive or negative, integral or fractional.

By definition $\log_{10} M = m$, $\log_{10} (M^r) = mr$,

$$\therefore \log_{10} M^r = r \log_{10} M.$$

If the number be in the form of a root, *e.g.* $\sqrt{M}$, $\sqrt[3]{M}$, etc., we should have $\log \sqrt{M} = \log (M^{\frac{1}{2}}) = \frac{1}{2} \log M$; so $\log \sqrt[3]{M} = \frac{1}{3} \log M$, and so on.

296. **Advantages of logarithms.**—From §§ 293-295 it will be seen that by working with the logarithms of numbers instead of the numbers themselves, the process of

Multiplication	is replaced	by that of	addition,
Division	,,	,,	subtraction,
Involution	,,	,,	multiplication,
Evolution	,,	,,	division.

Ex. 1.—Find the index of the power to which 10 must be raised so as to make 15, given

$$10^{\cdot 4771} = 3, \quad 10^{\cdot 6990} = 5,$$

$$15 = 3 \times 5 = 10^{\cdot 4771} \times 10^{\cdot 6990} = 10^{\cdot 4771 + \cdot 6990} = 10^{1 \cdot 1761},$$

$\therefore$ the required index is $1 \cdot 1761$.

15 expressed as a power of 10 is equal to $10^{1 \cdot 1761}$.

Ex. 2.—Find $\log 15$, given

$$\log 3 = \cdot 4771, \ \log 5 = \cdot 6990,$$

$$\log 15 = \log (3 \times 5) = \log 3 + \log 5 \ (\S\ 293) = 1 \cdot 1761.$$

Exx. 1 and 2 are really the same; Ex. 1 is expressed in the language of indices, Ex. 2 in that of logarithms.

Ex. 3.—Find $\log 504$, given $\log 2 = \cdot 3010$, $\log 3 = \cdot 4771$, $\log 7 = \cdot 8451$.

Now $$504 = 2^3 . 3^2 . 7.$$

Hence by §§ 293, 295, we have

$$\begin{aligned} \log 504 &= 3 \log 2 + 2 \log 3 + \log 7 \\ &= \cdot 9030 + \cdot 9542 + \cdot 8451 \\ &= 2 \cdot 7023 \text{ (see § 305).} \end{aligned}$$

Ex. 4.—Find $\log \sqrt[5]{35}$, given $\log 2$ and $\log 7$ (see Ex. 3).

By § 295 $\log \sqrt[5]{35} = \frac{1}{5} \log 35,$

$$35 = 10 \times 7 \div 2,$$

$$\therefore \log \sqrt[5]{35} = \tfrac{1}{5}(\log 10 + \log 7 - \log 2)$$
$$= \tfrac{1}{5}(1{\cdot}8451 - {\cdot}3010)$$
$$= \tfrac{1}{5} \times 1{\cdot}5441 = {\cdot}3088.$$

EXAMPLES CLXIII.

The following logarithms are given:—

$$\log 2 = {\cdot}3010, \quad \log 11 = 1{\cdot}0414,$$
$$\log 3 = {\cdot}4771, \quad \log 13 = 1{\cdot}1139.$$
$$\log 7 = {\cdot}8451,$$

1. Find the logarithms to base 10 of the following numbers:—

6, 21, 78, 143, 462, 1001.

2. Calculate the logarithms to base 10 of $\frac{70}{11}$, $\frac{33}{7}$, $47\frac{2}{3}$, $20\frac{3}{7}$, $166\frac{5}{6}$.

3. Find the logarithms to base 10 of 8, 24, 27, 49, 96, 169, 1331.

4. Find $\log \sqrt{2}$, $\log \sqrt[3]{3}$, $\log 21^{\frac{1}{3}}$, $\log 143^{\frac{1}{3}}$.

5. Find $\log 5$, $\log 25$, $\log 125$.

[N.B.—$\log(2 + 3)$ is *not* the same as $\log 2 + \log 3$, this last being equal to $\log 6$. $\log 5 = \log \frac{10}{2}$, etc.]

297. Characteristic and Mantissa.—The logarithms of all integral powers of 10 are of course integral; for any number between 1 and 10 the logarithm is a decimal fraction; for all other numbers from 10 upwards the logarithms are mixed numbers, the fractional part being always expressed decimally. For all numbers between 0 and 1 the logarithms are negative (cf. § 283).

E.g. if $\log 3721 = 3{\cdot}5706$, we have

$$\log {\cdot}03721 = \log \tfrac{3721}{100000} = \log 3721 - \log 10^5 = 3{\cdot}5706 - 5$$
$$= -1{\cdot}4294.$$

It is convenient, however, that the *decimal* part in all logarithms should be positive. Here this is effected by adding 1 to the decimal part and subtracting 1 from the integral part, a process which does not alter the value of the logarithm but only its form. We now get

$$\log {\cdot}03721 = -1{\cdot}4294 = (-1 - 1) + (1 - {\cdot}4294)$$
$$= -2 + {\cdot}5706 = \bar{2}{\cdot}5706.$$

The result is written in this last form to indicate that the minus sign refers only to the integral part 2 and *not* to the decimal part, for $-2{\cdot}5706$ would mean $-2-{\cdot}5706$.

The decimal part (made positive, if necessary) of a logarithm is called the **mantissa.** The integral part of a logarithm, *after the decimal part has been made positive*, is called the **characteristic.** Thus in the above ·5706 is the mantissa, and $\bar{2}$ or -2 is the characteristic.

298. **Advantage of base 10.**—Since our system of numbers is a decimal one, it follows that if we know the logarithm of any number, we can at once write down the logarithm of any other number with the same consecutive figures whatever be their respective local values (see § 5), for all such numbers can be formed from one another simply by moving the decimal point, *i.e.* by multiplying or dividing by some integral power of 10.

E.g. $\log 3{\cdot}721 = {\cdot}5706$. Also $3721 = 3{\cdot}721 \times 10^3$,

$\therefore \log 3721 = \log 3{\cdot}721 + \log 10^3 = 3{\cdot}5706$ (§§ 293, 295).

So $\log 372{\cdot}1 = \log (3{\cdot}721 \times 10^2) = 2{\cdot}5706$,

$\log {\cdot}3721 = \log (3{\cdot}721 \times 10^{-1}) = \bar{1}{\cdot}5706$ (§ 297),

$\log {\cdot}03721 = \log (3{\cdot}721 \times 10^{-2}) = \bar{2}{\cdot}5706$ (§ 297).

The logarithms of 3721, 3712, 21·37, 7·321 are all different, for, although they contain the same figures, these figures are not in the same order.

So the logarithms of 3·721, 3·0721, 3·7021 are all different, for, although they all contain the figures 3, 7, 2, 1, the order is not strictly consecutive, owing to the presence of the 0 *between* two of the figures.

299. We may regard a number as being of **standard form** when it contains *one* figure to the *left* of the decimal point, *e.g.* 3·721, all other numbers consisting of the same consecutive figures being obtained from it by multiplying it by some positive or negative integral power of 10.

E.g. $3721 = 3{\cdot}721 \times 10^3$, the decimal point in the standard form being moved 3 places to the *right* for the *positive* index 3 (§ 130).

$\cdot 03721 = 3{\cdot}721 \times 10^{-2}$, the decimal point in the standard form being moved 2 places to the *left* for the *negative* index 2 (§ 132).

The logarithms of all numbers of the standard form have 0 for characteristic, since all such numbers lie between 1 and 10, *i.e.* between 10^0 and 10^1.

Ex. 5.—Simplify :— $2{\cdot}0316 + \bar{1}{\cdot}4168 - \bar{3}{\cdot}7040$.

$$\begin{aligned}\text{Exp.} &= (+2 + {\cdot}0316) + (-1 + {\cdot}4168) - (-3 + {\cdot}7040)\\ &= (2 - 1 + 3) + ({\cdot}0316 + {\cdot}4168 - {\cdot}7040) \quad \text{.....(A)}\\ &= 4 - {\cdot}2556\\ &= 3{\cdot}7444.\end{aligned}$$

The working can be shortened if we notice by a rough calculation that the decimal resulting from the second bracket in line (A) is negative. We then arrange thus :—

$$\begin{aligned}\text{Exp.} &= 3 + (1{\cdot}0316 + {\cdot}4168 - {\cdot}7040)\\ &= 3{\cdot}7444.\end{aligned}$$

Ex. 6.—Multiply $\bar{1}{\cdot}4713$ by 6.

$$\text{Product} = (-1 + {\cdot}4713) \times 6 = -6 + 2{\cdot}8278 = \bar{4}{\cdot}8278.$$

Ex. 7.—Divide $\bar{1}{\cdot}4713$ by 6.

Here the negative characteristic is not exactly divisible by 6, and we must make it divisible by 6 by adding -5 to the characteristic and $+5$ to the mantissa. Thus

$$\begin{aligned}\bar{1}{\cdot}4713 \div 6 &= (-6 + 5{\cdot}4713) \div 6 = -1 + {\cdot}9119 \text{ (nearly)}\\ &= \bar{1}{\cdot}9119.\end{aligned}$$

Ex. 8.—Given $\log 2 = {\cdot}3010$, $\log 3 = {\cdot}4771$, $\log 7 = {\cdot}8451$, find $\log \frac{4}{21}$.

$$\begin{aligned}\log \tfrac{4}{21} &= \log 4 - \log 21 = \log 2^2 - \log (3 \times 7) = 2 \log 2 - \log 3 - \log 7\\ &= {\cdot}6020 - {\cdot}4771 - {\cdot}8451\\ &= {\cdot}6020 - 1{\cdot}3222 = -1 + 1{\cdot}6020 - 1{\cdot}3222\\ &= -1 + {\cdot}2798 = \bar{1}{\cdot}2798.\end{aligned}$$

Ex. 9.—Find $\log \sqrt[5]{{\cdot}00\dot{4}\dot{8}}$, given $\log 2$, $\log 3$ (see Ex. 8), and $\log 11 = 1{\cdot}0414$.

Now $\qquad \log \sqrt[5]{{\cdot}00\dot{4}\dot{8}} = \log ({\cdot}00\dot{4}\dot{8})^{\frac{1}{5}} = \frac{1}{5} \log {\cdot}00\dot{4}\dot{8}.$

Also $\qquad {\cdot}00\dot{4}\dot{8} = \dfrac{48}{9900} = \dfrac{16}{3300} = \dfrac{2^4}{3 . 11 . 10^2},$

therefore, by §§ 293–295.

$$\begin{aligned}\log {\cdot}00\dot{4}\dot{8} &= 4 \log 2 - \log 3 - \log 11 - 2 \log 10\\ &= 1{\cdot}2040 - {\cdot}4771 - 1{\cdot}0414 - 2\\ &= 1{\cdot}2040 - 3{\cdot}5185 = -3 + 4{\cdot}2040 - 3{\cdot}5185\\ &= \bar{3}{\cdot}6855,\end{aligned}$$

$$\therefore \log \sqrt[5]{{\cdot}00\dot{4}\dot{8}} = \tfrac{1}{5} \text{ of } (-5 + 2{\cdot}6855) = \bar{1}{\cdot}5371 \text{ (see Ex. 7).}$$

EXAMPLES CLXIV.

1. Express as the product of a number of standard form (see § 299) and some power of 10, positive or negative:—14·3, 904·51, 7246·1, ·73, ·062, ·000581.

2. Given log 2·647 = 0·4227, find the logs of 264·7, ·2647, 26·470, 0·002647.

From the data can we write down the value of log 2·0347?

3. Simplify the following, making the decimal part in the result positive and, where necessary, correct to the fourth place:—

(1) $4{\cdot}7 + \bar{1}{\cdot}3$. (2) $1{\cdot}623 + \bar{1}{\cdot}712$.

(3) $2{\cdot}6481 - \bar{3}{\cdot}5280$. (4) $2{\cdot}7103 - \bar{1}{\cdot}9073$.

(5) $\bar{1}{\cdot}7251 - {\cdot}8263$. (6) $\bar{2}{\cdot}462 \times 5$.

(7) $\bar{1}{\cdot}7 \times 10$. (8) $\bar{3}{\cdot}716 \div 3$.

(9) $\bar{4}{\cdot}4731 \div 3$. (10) $\bar{4}{\cdot}2417 \div 6$.

(11) $\bar{5}{\cdot}27 \div 7$. (12) $\bar{6}{\cdot}2 \div 11$.

4. Given log 2 = ·3010, log 3 = ·4771, log 7 = ·8451, find the logarithms of—

(1) $\frac{1}{9}$, (2) $\frac{8}{27}$, (3) $\frac{7}{48}$, (4) $\sqrt{\frac{2}{3}}$, (5) $\sqrt[3]{\frac{14}{81}}$, (6) $\sqrt[5]{\frac{1}{7}}$.

300. **To determine the characteristic of the logarithm of any number by inspection.**—Case i. Take any number greater than unity, say 7321. Since 7321 lies between 1000 and 10000, *i.e.* between 10^3 and 10^4, ∴ log 7321 lies between 3 and 4, *i.e.* log 3721 = 3 + some decimal. The characteristic is 3, *i.e.* one less than the number of digits in 7321. So 826·53 lies between 10^2 and 10^3; ∴ log 826·53 has 2 for characteristic, *i.e.* one less than the number of digits in the integral part 826.

Case ii. Take any number less than unity, say ·021. Since ·021 lies between ·1 and ·01, *i.e.* between 10^{-1} and 10^{-2}, ∴ log ·021 lies between −1 and −2,

i.e. log ·021 = −1 − some decimal (call it f) $= \bar{2} + (1 - f)$.

The characteristic is $\bar{2}$, *i.e.* negative, and numerically one greater than the number of ciphers immediately following the decimal point. So ·0021 lies between 10^{-2} and 10^{-3}; ∴ log ·0021 has $\bar{3}$ for characteristic.

We therefore have the following rules for the characteristic :—

Rule i.—*The characteristic of the logarithm of any number greater than unity is one less than the number of digits in the integral part of the number.*

Rule ii.—*The characteristic of the logarithm of any number less than unity is negative and numerically one greater than the number of ciphers before the first significant figure of the number, when expressed as a decimal.*

These rules should be known also conversely, so that from a knowledge of the characteristic it may be possible to infer the position of the decimal point in the corresponding number.

E.g. (1) If the logarithm of a number is 2·8401, we infer (by the converse of Rule i.) that the number has $2 + 1$ or 3 digits in its integral part. It may contain any number of decimal figures.

(2) If the logarithm of a number is $\bar{2}$·8401, we infer (by the converse of Rule ii.) that the number has $2 - 1$ or 1 cipher immediately following the decimal point, *i.e.* the first significant figure is in the second decimal place.

301. **Alternative method** for determining the characteristic by reference to the number of standard form (§ 299). Take the number 3721. This may be put as $3{\cdot}721 \times 10^3$;

$$\therefore \log 3721 = \log 3{\cdot}721 + 3$$
$$= 0 + \text{some decimal } (f) + 3 = 3 + f,$$

$\therefore$ the characteristic is 3, *i.e.* the index of the power 10^3.

So $$\cdot 03721 = 3{\cdot}721 \times 10^{-2},$$
$$\therefore \log \cdot 03721 = \log 3{\cdot}721 - 2 = 0 + f - 2 = \bar{2} + f,$$

$\therefore$ the characteristic is -2, *i.e.* the index of the power 10^{-2}.

Hence we have the following rule for all cases :—

The characteristic of the logarithm of a given number is the index (positive or negative) of that power of 10 *by which the number of standard form must be* multiplied (*not divided*) *to produce the given number.*

302. **Tables of Four-figure Logarithms.**—By means of certain series explained in text-books on Algebra (see *Tutorial Algebra*, Chap. XXIX.) the logarithm of any number can be found to any degree of accuracy. For most practical purposes their calculation to 4 decimal places is sufficient; they are then known as four-figure logarithms. Their values

can be seen from the tables at the end of this chapter for all numbers, in which the significant figures range from 1 to 9999. The tables give only the mantissae, and the decimal point is omitted, as this can be supplied by the reader, as also the necessary characteristic from the rules in §§ 300, 301. It is best to regard the tables as giving the logarithms of numbers of the standard form (§ 299), *i.e.* numbers whose first (or left-hand) figure is in the units place.

Hence, in reading a table, the decimal point is to be placed between the first and second figures of the number and before the first figure of the logarithm.

Tables of logarithms are used—

(1) To find the logarithm of a given number (§ 303).

(2) To find the number whose logarithm is given (§ 304).

303. Suppose we wish to find from the tables the logarithm of 7·623, or of ·007623, or of 762·3, or of 7623000. Since the *significant* figures in all four cases are the same, we shall find the *mantissa* of the logarithm of each in the same place. Looking down the first or left-hand column of the first and second pages we come to the figures 76. Passing horizontally to the right, we come to figures in the fourth column at the head of which stands the figure 2. And we read, therefore, that the mantissa of the logarithm corresponding to *any* number whose first three significant figures are 762 is ·8820. In order to allow for the effect of the fourth significant figure 3, we seek among the last nine columns for that at the head of which stands the figure 3, and in the horizontal line corresponding to the figures 76, we find the number 2. This must be added to the last place in the mantissa found below. Our work will then read thus:—

Mantissa for first three significant figures...	762	=	·8820
Difference to be allowed for fourth figure ...	3	=	2
Mantissa for all numbers whose significant figures are	7623	=	·8822

Hence the logarithms we seek are as follows:—

$$\log 7{\cdot}623 = 0{\cdot}8822$$
$$\log {\cdot}007623 = \bar{3}{\cdot}8822$$
$$\log 762{\cdot}3 = 2{\cdot}8822$$
$$\log 7623000 = 6{\cdot}8822$$

For the numbers 2, 3, 4, ... 9, look in the left-hand column for 20, 30, ... 90.

Ex. 10.—Using the tables, find the logarithms of

8, 1·5, 34, 4·72, 8·623, 8296, 0·46, ·004712, 1·065.

Ans. ·9031, ·1761, 1·5315, ·6739, ·9357, 3·9189, $\bar{1}$·6628, $\bar{3}$·6732, ·0273.

304. **Antilogarithms.**—The converse process of that explained in the previous section is frequently necessary, viz. the determination of a number when its logarithm is known. From this converse point of view, the number is known as the *Antilogarithm.* Just as we tabulate the logarithms when the numbers are given, so we may tabulate the numbers (antilogarithms) when the logarithms are given. On the third and fourth pages of the tables at the end of this chapter they are tabulated for all logarithms whose mantissae range from ·0000 to ·9999. The mantissae only are given, for these are sufficient to determine the significant figures of the number sought, the characteristics, coexisting with the mantissae, determining the positions of the decimal point, and the number of non-significant figures to be assigned.

The reading of these tables is precisely the same as of the logarithm tables on the first and second pages.

Thus suppose we wish to find the numbers whose logarithms are ·8822, $\bar{3}$·8822, 2·8822, 6·8822.

Since the mantissa is the same for each, we shall find the significant figures of the numbers sought in the same columns. Passing down the first column we come to ·88; passing horizontally to the right in the fourth column, headed by the figure 2, we find the number 7621.

Hence we conclude that corresponding to the mantissa ·882 we have the significant figures 7621. To allow for the fourth figure 2 in the mantissa, we add to the number 7621, the number we find in the column headed 2 of the last nine columns of the table. Here we find the number 4; and we gather, therefore, that the significant figures corresponding to a mantissa ·8822 are 7625. So that our results are:—

antilog. of	·8822	is	7·62 5,
,,	,, $\bar{3}$·8822	is	·00762 5,
,,	,, 2·8822	is	762·5,
,,	,, 6·8822	is	762500 0.

305. It will be noticed that in §§ 303, 304 there is a discrepancy in the fourth significant figure of the number. We found that the logarithm of 7·623 was ·8822; but on reversing the process we found corresponding to the mantissa ·8822 the number 7·625.

A similar difference occurred in Ex. 3, § 296, where log 504 was calculated to be 2·7023, whereas in the tables it is given as 2·7024.

This discrepancy arises from the fact that the fourth figures both in the logarithm and antilogarithm tables are only approximately correct (see § 149). And so, whilst generally four-figure tables will give sufficiently accurate results, it may in some cases be necessary to use tables calculated to five or more figures.

Ex. 11.—Using the tables, find the antilogarithms of

·25, ·254, ·2547, 2·8499, $\bar{2}$·6421.

Ans. 1·778, 1·795, 1·798, 707·8, ·04386.

Ex. 12.—By means of the tables calculate ·3639 ÷ 17.

Denoting the quotient by x, we have

$$\log x = \log \cdot 3639 - \log 17,$$

$$\log 3\cdot 63 = \cdot 5599,$$

difference for fourth figure $9 = 11$,

$$\therefore \log 3\cdot 639 = \cdot 5610,$$

$$\therefore \log \cdot 3\cdot 639 = \bar{1}\cdot 5610,$$

Also $\log 17 = 1\cdot 2304$.

$\therefore$ by subtraction $\log x = \bar{2}\cdot 3306$.

To find x we look in the antilogarithm tables.

Antilog. of $\cdot 330 = 2\cdot 138$,

difference for fourth figure $6 = 3$,

$\therefore$ antilog. of $\cdot 3306 = 2\cdot 141$,

$\therefore$ antilog. of $\bar{2}\cdot 3306 = \cdot 02\,141$,

i.e. $x = \cdot 02\,141$.

By ordinary division $\cdot 3639 \div 17 = \cdot 021406$ nearly.

EXAMPLES CLXV.

(The following examples are intended to give practice in the use of the tables.)

Simplify the following expressions by means of logarithms, checking results by ordinary multiplication or division to four significant figures:

1. $76{\cdot}1 \times 2$.	**2.** $28{\cdot}47 \times 3$.	**3.** ${\cdot}21 \times {\cdot}17$.
4. $1{\cdot}316 \times 7$.	**5.** 78×20.	**6.** $2{\cdot}164 \div 2$.
7. $3{\cdot}171 \div 3$.	**8.** $96640 \div 16$.	**9.** $4763 \div 91$.
10. ${\cdot}08644 \div 23$.	**11.** $3{\cdot}724 \div 2{\cdot}5$.	**12.** ${\cdot}1451 \div 18$.

306. Interpolation.—If a number has not more than four significant figures, we can at once write down its logarithm from the tables, for these contain the mantissae for all numbers from 1 to 9999. When the number whose logarithm we require is not in the tables, we must make use of the principle of **proportional parts**:—

If a number be increased by a very small fraction of itself, the increase in the logarithm of the number is approximately proportional to the increase in the number.

The method of applying this principle is known as **interpolation**, and will be most easily understood from the following example.

Ex. 13.—Find $\log 3{\cdot}1416$.

This number contains five significant figures, and hence will not be found in four-figure tables. We take two numbers near to one another between which $3{\cdot}1416$ lies, viz. $3{\cdot}142$ and $3{\cdot}141$. From the tables, we get

$$\begin{aligned} \log 3{\cdot}142 &= {\cdot}4972 \\ \log 3{\cdot}141 &= {\cdot}4970 \\ \text{difference for } {\cdot}001 &\text{ is } {\cdot}0002 \\ \therefore \text{ difference for } {\cdot}0006 &\text{ is } {\cdot}00012 \end{aligned}$$

In practice we omit the ciphers and say mentally that required difference in the fourth and fifth places is 12.

$$\therefore \log 3{\cdot}1416 = {\cdot}4970 + {\cdot}00012 = {\cdot}49712.$$

More accurate tables give the result as ${\cdot}4971509$.

A similar method might be used to determine the antilogarithm corresponding to a mantissa which is not found exactly in the tables. The fourth significant figure cannot, however, be relied upon, and more accurate results would be got by using seven-figure or even five-figure tables.

Ex. 14.—Find the value of $\dfrac{4{\cdot}682 \times 3{\cdot}065}{{\cdot}0173 \times 16{\cdot}12}$.

In the working of such examples a very common error is to put the decimal point of the answer in the wrong position. This may sometimes be avoided by obtaining a rough approximation to the answer by substituting for the given numbers others nearly equal to them. Such a rough approximation would not usually give the exact figures of the answer, but might serve as a guide for the position of the decimal point.

For instance, this example may be roughly put as

$$\frac{5 \times 3}{\cdot 02 \times 16} = \frac{15}{\cdot 32} = 47 \text{ nearly},$$

from which we conclude that the correct result will have *two* integral figures.

$\log 4\cdot 682 = \cdot 6704$	$\log \cdot 0173 = \bar{2}\cdot 2380$
$\log 3\cdot 065 = \cdot 4864$	$\log 16\cdot 12 = 1\cdot 2073$
$\therefore$ log of numerator $= 1\cdot 1568$	$\therefore$ log of denom. $= \bar{1}\cdot 4453$

Writing x for the given expression, we have

$$\log x = \text{log of numerator} - \text{log of denominator}$$
$$= 1\cdot 1568 \qquad - \bar{1}\cdot 4453 = 1\cdot 7115.$$

From the antilogarithm tables we see that corresponding to the mantissa $\cdot 7115$ we have the significant figures 5146. But the characteristic is 1; hence (§ 300) the antilogarithm contains *two* integral figures (which agrees with the result obtained above from the rough approximation).

Hence $$x = 51\cdot 46.$$

Ex. 15.—Find the value of $\dfrac{(\cdot 3125)^2}{(\cdot 4627)^{\frac{1}{3}}}$ ($= x$ say),

$$\log (\cdot 3125)^2 = 2 \log \cdot 3125 = 2 \times \bar{1}\cdot 4949 = \bar{2}\cdot 9898,$$
$$\log (\cdot 4627)^{\frac{1}{3}} = \tfrac{1}{3} \log \cdot 4627 = \tfrac{1}{3} \times (\bar{1}\cdot 6653)$$
$$= \tfrac{1}{3} (\bar{3} + 2\cdot 6653) = \bar{1}\cdot 8884,$$
$$\therefore \log x = \text{log of numerator} - \text{log of denominator}$$
$$= \bar{2}\cdot 9898 - \bar{1}\cdot 8884$$
$$= \bar{1}\cdot 1014,$$
$$\therefore x = \cdot 1263 \text{ (from antilogarithm tables).}$$

Rough approximation:—

$$\text{Numerator} = (\cdot 3)^2 \text{ nearly} = \cdot 09,$$
$$\text{Denominator} = (\cdot 512)^{\frac{1}{3}} \text{ (nearly)} = \cdot 8,$$
$$\therefore x = \frac{\cdot 09}{\cdot 8} = \cdot 11 \text{ nearly},$$

showing that the first significant figure is in the first decimal place, which agrees with the result actually obtained.

EXAMPLES CLXVI.

(The results should be obtained to four significant figures, but the last figure cannot always be relied upon.)

1. Find by logarithms the squares of: —2·17, 17·32, ·625, $\frac{7}{8}$, ·0265, ·142857 (reduce to vulgar fraction).

2. Find by logarithms the square roots of the numbers in Ex. 1.

3. Evaluate:—

(1) $(\cdot 73)^3$, (2) $(1\cdot 745)^4$, (3) $\sqrt[3]{\cdot 002873}$, (4) $(8\cdot 216)^{\frac{1}{5}}$,
(5) $(\cdot 0007349)^{\frac{1}{4}}$, (6) $\sqrt[10]{(\cdot 0761)^3}$, (7) $(1\cdot 431)^{\frac{2}{3}}$, (8) $(1\cdot 05)^{-9}$.

Simplify:—

4. $\dfrac{7\cdot 23 \times 1\cdot 62}{4\cdot 16}$. **5.** $\dfrac{8\cdot 617 \times \cdot 0461}{\cdot 2734}$. **6.** $\dfrac{\cdot 2618}{\cdot 3923}$.

7. $\dfrac{4\cdot 278}{(\cdot 254)^2}$. **8.** $(\frac{1}{2})^{\frac{7}{3}}$. **9.** $\sqrt{\dfrac{\cdot 0234}{\cdot 00897}}$.

10. $\dfrac{\cdot 2693 \times 1\cdot 364}{7\cdot 217 \times \cdot 0163}$. **11.** $\dfrac{(2\cdot 41)^2 \times 2\cdot 61 \times \cdot 3178}{(1\cdot 374)^3}$.

12. $(2\cdot 7)^{\frac{1}{2}} \times (6\cdot 1)^{\frac{1}{3}}$. **13.** $\left(\dfrac{14\cdot 72 \times 2\cdot 68}{\cdot 4983}\right)^{-\frac{1}{2}}$.

14. $\dfrac{\sqrt[3]{19\cdot 41} \times 4\cdot 62 \times (1\cdot 783)^{\frac{2}{3}}}{41\cdot 6 \times \sqrt[7]{7\cdot 67} \times (1\cdot 4)^2}$.

15. 1·231 of £2 14s. 7d. (answer to nearest penny).

16. $7\cdot 3146 \times 1\cdot 8123$.

Ex. 16.—Find the amount on £450 for 2 years at 4% compound interest.

Using the formula of § 202, viz.

$$\text{amount} = \text{principal} \times \left(1 + \frac{\text{rate }\%}{100}\right)^n,$$

we get that the required amount $= £450 \times (1\cdot 04)^2$,

$$\begin{aligned} \therefore \text{ log. of ans.} &= \log 450 + 2 \log 1\cdot 04 \\ &= 2\cdot 6532 + \cdot 0340 \\ &= 2\cdot 6872, \\ \therefore \text{ ans.} &= £486\cdot 6 = £486\ 12\text{s.} \end{aligned}$$

By ordinary working we find the amount to be

£486 14s. 5d. (nearly).

EXAMPLES CLXVII.

1. Given $\log 2 = \cdot 3010300$ and $\log 13 = 1\cdot 1139434$,
find $\log 65$ and $\log \cdot 0065$.

2. Given $\log 2 = \cdot 3010300$, $\log 3 = \cdot 4771213$, $\log 7 = \cdot 8450980$,
find $\log 5250$, $\log \cdot 004\dot{6}$, $\log (1/\sqrt[5]{35})$.

3. Calculate $\log 2^{36}$ and $\log 3^{100}$, and hence deduce the number of digits in (1) 2^{36}, (2) 3^{100} (see Ex. 2).

4. Find (1) $\log 10632$, having given

$$\log 1063 = 3{\cdot}0265333 \text{ and } \log 1{\cdot}064 = {\cdot}0269416;$$

and (2) $\log 324{\cdot}365$, having given

$$\log 32434 = 4{\cdot}511000 \text{ and } \log 32440 = 4{\cdot}511081.$$

5. Given $\log 3{\cdot}8862 = {\cdot}5895251$ and $\log 3{\cdot}8863 = {\cdot}5895363$, calculate approximately $\log {\cdot}03886245$.

6. Find $\log \sqrt[5]{{\cdot}00054 \times 3{\cdot}6}$ (see Ex. 2).

7. Find (without the use of logarithm tables) the value of

(1) $(\log \sqrt{27} + \log 8 - \log \sqrt{1000}) \div \log 1{\cdot}2$,

(2) $\frac{1}{6}\sqrt{\left\{\dfrac{3 \log 1728}{1 + \frac{1}{2}\log {\cdot}36 + \frac{1}{3}\log 8}\right\}}$.

8. Find (to the nearest £) the amount at compound interest on

(1) £4500 for 2 years at $4\frac{1}{4}$%,

(2) £425 for 3 years at 4%,

(3) £2150 for 3 years at 5%,

(4) (to the nearest penny) £1 for 20 years at 5%.

9. Find (to the nearest penny) the value of

(1) $£10 \times \{1 - (1{\cdot}05)^{-20}\}$,

(2) $£10 \times \{(1{\cdot}05)^{-13} - (1{\cdot}05)^{-20}\}$.

[N.B.—Work each part inside the bracket separately and subtract one result from the other.]

10. Find (to within a shilling) the cost of 2874 articles at 8s. $4\frac{3}{4}$d. each. (Reduce 8s. $4\frac{3}{4}$d. to decimal of a shilling.)

11. Given 1 metre = 39·37 inches, express 1 sq. metre in sq. yds.

12. In a right-angled triangle, the sides containing the right angle measure 92·3 and 71·6 feet. Find its area in square feet.

13. Find in sq. cms. the area of a circle whose radius is 45 cms.,

$$(\pi = 3\tfrac{1}{7}).$$

14. Calculate the value of $\frac{4}{3}\pi r^3$, if $\pi = 3\frac{1}{7}$ and $r = 1{\cdot}7$.

15. Given that one cubic foot of water weighs 62·5 lbs., find (correct to one decimal place) the number of ounces in one cubic inch of mercury, which is 13·6 times as heavy as water.

16. Find (to the nearest £) the principal which will amount to £2172 in 4 years at 4% compound interest.

17. A formula for the area of a triangle is

$$\sqrt{s(s-a)(s-b)(s-c)},$$

where a, b, c are the lengths of the three sides and s is the semi-perimeter. Calculate the areas of the triangles (to four significant figures) in which

(1) $a = 273$, $b = 176$, $c = 157$; (2) $a = 29$, $b = 81$, $c = 73$;
(3) $a = 87$, $b = 51$, $c = 93$.

18. Find the value of $R^{-9} - R^{-15}$ when $R = 1{\cdot}04$. (N.B.—Work each part separately and then subtract one result from the other.)

19. Find the area in acres of a rectangular field whose length is 807 yards and breadth 751 yards.

20. Find the edge of a cube whose volume is 7·58 cubic feet.

21. Reduce (1) 1 kilometre to miles, (2) 1 mile to kilometres, given 1 metre = 39·37 inches.

22. One fourth part of the earth's circumference measures 10^7 metres. Express the whole circumference in miles to four significant figures. (See Ex. 21.)

23. Calculate the volume of a room whose dimensions are 8·23, 7·61, 5·25 metres.

24. Find the radius of a circle whose area is one acre.

($\pi = 3\frac{1}{7}$. Answer in yards.)

25. Given that 1 gram = 15·43 grains, 1 lb. (Avoir.) = 7000 grains, 1 pint of pure water weighs $1\frac{1}{4}$ lbs., calculate the number of pints in 1 litre.

26. Find a mean proportional to 7·612 and 18·31; also a fourth proportional to 8·21, 7·34, and 16·20.

27. Quote 3s. $6\frac{1}{2}$d. per yard in francs per metre, having given £1 = 25·23 francs, and 1 metre = 39·37 inches.

28. Find the simple interest for 7 months on £391 16s. at $3\frac{3}{4}$% per annum correct to the nearest penny.

29. Find $(\cdot 3724)^{-2\cdot 43}$.

30. Find t from the formula $s = \frac{1}{2}gt^2$, given $s = 2731$, $g = 981$.

31. Find x, given $2^x = \cdot 034$.

32. If the population of a town containing at present 6000 inhabitants increases annually at the rate of 3%, what will be the total increase at the end of 3 years?

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						4	9	13	17	21	25	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	37
11	0414	0453	0492	0531	0569						4	8	12	15	19	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	19	22	26	30	33
12	0792	0828	0864	0899	0934	0969					3	7	11	14	18	21	25	28	32
							1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	7	10	13	16	20	23	26	30
						1303	1335	1367	1399	1430	3	7	10	12	16	19	22	25	29
14	1461	1492	1523	1553							3	6	9	12	15	18	21	24	28
					1584	1614	1644	1673	1703	1732	3	6	9	12	15	17	20	23	26
15	1761	1790	1818	1847	1875	1903					3	6	9	11	14	17	20	23	26
							1931	1959	1987	2014	3	5	8	11	14	16	19	22	25
16	2041	2068	2095	2122	2148						3	5	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	15	18	21	23
17	2304	2330	2355	2380	2405	2430					3	5	8	10	13	15	18	20	23
							2455	2480	2504	2529	2	5	7	10	12	15	17	19	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	16	19	21
						2672	2695	2718	2742	2765	2	5	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

APPENDIX II.

SPECIMEN OF FACTOR TABLE. (GLAISHER'S *Factor Table for Sixth Million.*)

	598 98	599 01	04	07	10	13	16	19	22	25	28
01	—	13	—	1019	569	619	7	—	19	—	383
07	19	331	1277	37	11	7	47	—	13	2017	59
11	—	19	7	—	13	—	379	439	17	7	11
13	41	53	11	—	7	29	103	73	23	269	1871
17	—	7	—	43	23	139	41	1399	7	17	769
19	11	—	23	7	241	43	71	1619	157	13	7
23	7	167	—	—	19	13	11	7	—	43	17
29	—	—	97	—	11	73	7	263	79	—	47
31	17	7	53	—	—	431	19	11	7	29	13
37	7	17	—	—	13	11	2029	7	—	—	1621
41	11	41	—	23	7	67	—	977	29	877	829
43	—	23	17	11	41	—	7	—	137	19	193
47	97	—	59	7	—	941	911	13	—	11	7
49	227	11	223	17	797	7	31	—	1021	907	83
53	—	13	7	373	2243	761	—	11	1423	7	37
59	79	7	43	—	—	11	—	31	7	—	—
61	83	31	—	7	19	17	13	—	11	1289	7
67	13	601	7	71	53	—	11	—	—	7	43
71	—	11	31	—	—	29	7	—	—	13	337
73	—	7	—	677	11	37	491	13	7	227	2297
77	109	—	17	13	37	7	—	—	19	991	11
79	7	13	11	—	593	—	107	7	17	31	29
83	19	—	71	17	7	1471	463	23	11	2437	13
89	—	23	—	7	13	—	11	—	—	29	7
91	1303	—	13	29	31	7	199	—	—	11	17

The above table shows the lowest prime factors of the numbers 5989801 to 5989891, 5990101 to 5990191, &c. Thus the lowest prime factor of 5992583 is 2437. Apart from its interest as a specimen of the tables which have now been constructed for 9000000 numbers, the above will serve as a collection of prime numbers. Every number shown except in the top and left margins of the table is a prime, *e.g.* 2437, 31, 991, &c. Also every number which corresponds to the places filled by a dash (—) is similarly prime, *e.g.* 5990191. The table shows that, from 5992800 to 5992858, there is not a single prime. Multiples of 3 are, of course, omitted.

APPENDIX III.

PRIME NUMBERS TO 1571 (N), AND NUMBER OF PLACES (n) IN THE CIRCULATING PERIOD OF $\frac{1}{N}$. (SHANKS, *Proc. R. S.*, Feb. 1874, p. 203.)

N	n	N	n	N	n	N	n	N	n	N	n
3	1	193	192	439	219	701	700	991	495	1283	641
7	6	197	98	443	221	709	708	997	166	1289	92
11	2	199	99	449	32	719	359	1009	252	1291	1290
13	6	211	30	457	152	727	726	1013	253	1297	1296
17	16	223	222	461	460	733	61	1019	1018	1301	1300
19	18	227	113	463	154	739	246	1021	1020	1303	1302
23	22	229	228	467	233	743	742	1031	103	1307	653
29	28	233	232	479	239	751	125	1033	1032	1319	659
31	15	239	7	487	486	757	27	1039	519	1321	55
37	3	241	30	491	490	761	380	1049	524	1327	1326
41	5	251	50	499	498	769	192	1051	1050	1361	680
43	21	257	256	503	502	773	193	1061	212	1367	1366
47	46	263	262	509	508	787	393	1063	1062	1373	686
53	13	269	268	521	52	797	199	1069	1068	1381	1380
59	58	271	5	523	261	809	202	1087	1086	1399	699
61	60	277	69	541	540	811	810	1091	1090	1409	32
67	33	281	28	547	91	821	820	1093	273	1423	158
71	35	283	141	557	278	823	822	1097	1096	1427	713
73	8	293	146	563	281	827	413	1103	1102	1429	1428
79	13	307	153	569	284	829	276	1109	1108	1433	1432
83	41	311	155	571	570	839	419	1117	558	1439	719
89	44	313	312	577	576	853	213	1123	561	1447	1446
97	96	317	79	587	293	857	856	1129	564	1451	290
101	4	331	110	593	592	859	26	1151	575	1453	726
103	34	337	336	599	299	863	862	1153	1152	1459	162
107	53	347	173	601	300	877	438	1163	581	1471	735
109	108	349	116	607	202	881	440	1171	1170	1481	740
113	112	353	32	613	51	883	441	1181	1180	1483	247
127	42	359	179	617	88	887	886	1187	593	1487	1486
131	130	367	366	619	618	907	151	1193	1192	1489	248
137	8	373	186	631	315	911	455	1201	200	1493	373
139	46	379	378	641	32	919	459	1213	202	1499	214
149	148	383	382	643	107	929	464	1217	1216	1511	755
151	75	389	388	647	646	937	936	1223	1222	1523	761
157	78	397	99	653	326	941	940	1229	1228	1531	1530
163	81	401	200	659	658	947	473	1231	41	1543	1542
167	166	409	204	661	220	953	952	1237	206	1549	1548
173	43	419	418	673	224	967	322	1249	208	1553	1552
179	178	421	140	677	338	971	970	1259	1258	1559	779
181	180	431	215	683	341	977	976	1277	638	1567	1566
191	95	433	432	691	230	983	982	1279	639	1571	1570

APPENDIX IV.

Circulating decimals having any desired number of places may be easily constructed by the following table, which is abridged from that calculated by Mr. W. Shanks, and communicated to the Royal Society in 1874. The table gives one or more denominators for fractions which, with any numerator prime to the denominator, will run to the required number of places.

No.	Denominator.	No.	Denominator.	No.	Denominator.
1.	3, 9	14.	909091	32.	353, 449, 641, 1409
2.	11	15.	31	33.	67
3.	27, 37	16.	17	34.	103, 4013
4.	101			35.	71
5.	41, 271	18.	19, 52579	41.	83, 1231
6.	7, 13	20.	3541, 27961	42.	49, 127, 2689
7.	239, 4649	21.	43, 1933	43.	173
8.	73, 137	22.	121, 23, 4093, 8779	44.	89
9.	81, 333667	26.	859	46.	47, 139, 2531
10.	9091	27.	243, 757	50.	251, 5051
11.	21649, 513239	28.	29, 281	51.	613
12.	9901	29.	3191, 16763	52.	521
13.	53, 79	30.	211, 241, 2161	53.	107

It must be noted that there are always (except for No. 1) other denominators than those given; in particular, that the product of two or more of the values given for a particular number, or of one of these numbers with 3, or with 9, or with 27 or 37 if the number of places is divisible by 3, or with 11 if the number of places is even, will provide a denominator giving rise to the same number of places.

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